

MONODROMY OF p -ADIC SOLUTIONS OF PICARD-FUCHS EQUATIONS *

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Picard-Fuchs equations are differential equations coming from (algebraic) geometry. Classically their solutions can be written as *period integrals* for families of varieties. In this note we want to look at *p -adic solutions* of the same differential equations. In p -adic analysis we can not use period integrals to describe these solutions.

Katz-Oda construction of the Gauss-Manin connection

First recall the purely algebraic construction of the differential equations due to Katz and Oda. Let $S = \text{Spec} A$ an affine scheme which is smooth over an open part of $\text{Spec} \mathbb{Z}$. Let $f : X \rightarrow S$ be a projective smooth morphism. The Koszul filtration on the absolute De Rham complex $\Omega_X^\bullet$ is defined by

$$K^{i\bullet} := \text{image}(f^* \Omega_S^i \otimes \Omega_X^{\bullet-i} \rightarrow \Omega_X^\bullet).$$

Then $K^{0\bullet}/K^{1\bullet} \simeq \Omega_{X/S}^\bullet$, $K^{1\bullet}/K^{2\bullet} \simeq f^* \Omega_S^1 \otimes \Omega_{X/S}^{\bullet-1}$.
The Gauss-Manin connection

$$\nabla : \mathcal{H}^m(X, \Omega_{X/S}^\bullet) \rightarrow \Omega_S^1 \otimes \mathcal{H}^m(X, \Omega_{X/S}^\bullet)$$

is the boundary map in the hypercohomology sequence associated with the exact sequence of complexes

$$0 \rightarrow K^{1\bullet}/K^{2\bullet} \rightarrow K^{0\bullet}/K^{2\bullet} \rightarrow K^{0\bullet}/K^{1\bullet} \rightarrow 0$$

*details for this note are presented in

J. Stienstra, *The generalized De Rham-Witt complex and congruence differential equations*, in: Arithmetic Algebraic Geometry; Progress in Math. 89; Birkhäuser 1991

J. Stienstra, M. van der Put, B. van der Marel, *On p -adic monodromy*, to appear in Math. Zeitschrift 1991

From this we see in particular

$$\text{image}(\mathcal{H}^m(X, \Omega_X^\bullet) \rightarrow \mathcal{H}^m(X, \Omega_{X/S}^\bullet)) \subset \ker \nabla$$

Let Diff_S denote the algebra of differential operators on A relative to $\mathbb{Z}$ and let Diff'_S be the subalgebra generated by the derivations of A . Then the Gauss-Manin connection defines a Lie algebra homomorphism

$$\nabla : \text{Der} A \rightarrow \text{End}_{\mathbb{Z}}(\mathcal{H}^*(X, \Omega_{X/S}^\bullet))$$

$$\nabla(D) = (D \otimes 1) \circ \nabla$$

which extends to an algebra homomorphism

$$\nabla : \text{Diff}'_S \rightarrow \text{End}_{\mathbb{Z}}(\mathcal{H}^*(X, \Omega_{X/S}^\bullet))$$

In other words: the Gauss-Manin connection makes $\text{End}_{\mathbb{Z}}(\mathcal{H}^*(X, \Omega_{X/S}^\bullet))$ a module over Diff'_S . Linear relations in this module are Picard-Fuchs differential equations.

For our treatment of p -adic solutions of we use the **generalized De Rham-Witt complex** $\mathcal{W}\Omega_X^\bullet$. This complex can be constructed for every scheme X on which 2 is invertible. It is a Zariski sheaf of anti-commutative differential graded algebras with the following structures and properties:

- all degrees ≥ 0 .
 $\mathcal{W}\Omega_X^0 = \mathcal{W}\mathcal{O}_X$ is the sheaf of generalized Witt vectors on X
- For all $N \geq 1$ there is a graded algebra endomorphism F_N on $\mathcal{W}\Omega_X^\bullet$ (F for Frobenius). These satisfy

$$\begin{aligned} F_N F_M &= F_{NM} \quad \forall N, M \\ dF_N &= NF_N d \quad \forall N \end{aligned}$$

where d = differential of $\mathcal{W}\Omega_X^\bullet$

- Let $\tilde{\Omega}_X^\bullet := \bigoplus_{i \geq 0} \Omega_X^i / (i! \text{-torsion in } \Omega_X^i)$ where $\Omega_X^\bullet$ is the De Rham complex on X rel. $\mathbb{Z}$. Then there exists a homomorphism of sheaves of differential graded algebras

$$\pi : \mathcal{W}\Omega_X^\bullet \rightarrow \tilde{\Omega}_X^\bullet;$$

such that $\pi : \mathcal{W}\mathcal{O}_X \rightarrow \mathcal{O}_X$ gives the first Witt vector coordinate.

$$\bullet \forall a \in \mathcal{O}_X \quad \exists \underline{a} \in \underline{\mathcal{W}\mathcal{O}}_X \text{ s.t. } \pi \underline{a} = a$$

$$F_N \underline{a} = \underline{a}^N \quad \forall N, \quad \underline{a} \cdot \underline{b} = \underline{ab} \quad \forall a, b$$

Because of $d F_N = N F_N d$ we have a homomorphism of differential graded algebras

$$F_N : \bigoplus_i \underline{\mathcal{W}\Omega}_X^i[-i] \rightarrow \underline{\mathcal{W}\Omega}_X^\bullet / N$$

equal to F_N in each degree. This fits into the following commutative diagrams

$$\begin{array}{ccc} \bigoplus_i H^{m-i}(X, \underline{\mathcal{W}\Omega}_X^i) & \xrightarrow{F_N} & \mathcal{H}^m(X, \underline{\mathcal{W}\Omega}_X^\bullet / N) \\ \downarrow & & \downarrow \pi \\ \downarrow \tau_N & & \mathcal{H}^m(X, \Omega_X^\bullet / N) \\ \downarrow & \swarrow & \downarrow 0 \\ \mathcal{H}^m(X, \Omega_{X/S}^\bullet / N) & \xrightarrow{\nabla} & \Omega_S^1 \otimes \mathcal{H}^m(X, \Omega_{X/S}^\bullet / N) \end{array}$$

$$\begin{array}{ccc} H^m(X, \underline{\mathcal{W}\mathcal{O}}_X) & \xrightarrow{F_N} & \mathcal{H}^m(X, \underline{\mathcal{W}\Omega}_X^\bullet / N) \\ F_N \downarrow & & \downarrow \\ H^m(X, \underline{\mathcal{W}\mathcal{O}}_X) & & \mathcal{H}^m(X, \Omega_X^\bullet / N) \\ \pi \downarrow & & \downarrow \\ H^m(X, \mathcal{O}_X) & & \downarrow \\ \downarrow & & \downarrow \\ H^m(X, \mathcal{O}_X) / N & \leftarrow & \mathcal{H}^m(X, \Omega_{X/S}^\bullet / N) \\ & & \downarrow \nabla \\ & & \Omega_S^1 \otimes \mathcal{H}^m(X, \Omega_{X/S}^\bullet / N) \end{array}$$

Assume:

$S = \text{Spec} A$ smooth over open part of $\text{Spec} \mathbb{Z}[\frac{1}{2}]$

$f : X \rightarrow S$ projective smooth morphism, relative dimension r

all $H^j(X, \Omega_{X/S}^i)$ are free A -modules, $H^r(X, \Omega_{X/S}^r) \simeq A$.

Then $\pi : H^m(X, \underline{\mathcal{W}\mathcal{O}}_X) \rightarrow H^m(X, \mathcal{O}_X)$ is surjective. Choose:

$\{\omega_1, \dots, \omega_h\}$ basis of $H^m(X, \mathcal{O}_X)$

$\{\tilde{\omega}_1, \dots, \tilde{\omega}_h\}$ dual basis of $H^{r-m}(X, \Omega_{X/S}^r)$

$\tilde{\omega}_1, \dots, \tilde{\omega}_h \in H^m(X, \underline{W}\mathcal{O}_X)$ s.t. $\pi \tilde{\omega}_i = \omega_i$

Define for $N \in \mathbb{N}$ the $h \times h$ -matrix B_N over A by

$$\pi F_N \underline{\tilde{\omega}} = B_N \underline{\omega}$$

where $\underline{\omega}$ = column vector with components $\omega_1, \dots, \omega_h$; similarly for $\underline{\tilde{\omega}}$.

$B_p \bmod p$ for prime p is known as the *Hasse-Witt matrix* of ...

Theorem. Suppose $P_1, \dots, P_h \in \text{Diff}'_S$ are such that

$$\nabla(P_1) \tilde{\omega}_1 + \dots + \nabla(P_h) \tilde{\omega}_h = 0 \quad \text{in } H^{2r-m}(X, \Omega_{X/S}^\bullet)$$

Then one has the following **congruence differential equation**

$$P_1 B_{N,i1} + \dots + P_h B_{N,ih} \equiv 0 \bmod N$$

for all $N \in \mathbb{N}$, for $i = 1, \dots, h$.

Idea of proof: for every derivation D on A

$$\langle \tau_N \tilde{\omega}_i, \tilde{\omega}_j \rangle \equiv B_{N,ij} \bmod N$$

$$\nabla(D)(\tau_N \tilde{\omega}_i) = 0$$

$$D \langle \tau_N \tilde{\omega}_i, \tilde{\omega}_j \rangle = \langle \tau_N \tilde{\omega}_i, \nabla(D)(\tilde{\omega}_j) \rangle.$$

Hypergeometric curves

Let $0 < \mathbf{a}, \mathbf{b}, \mathbf{c} < \mathbf{n}$ be integers with $\gcd(\mathbf{n}, \mathbf{a}, \mathbf{b}, \mathbf{c}) = 1$. Let $X = X_{\mathbf{n}; \mathbf{a}, \mathbf{b}, \mathbf{c}}$ be the smooth projective model, over $A := \mathbb{Z}[\mu_{\mathbf{n}}][\lambda, (\mathbf{n}\lambda(1-\lambda))^{-1}]$, of

$$y^{\mathbf{n}} = x^{\mathbf{a}}(x-1)^{\mathbf{b}}(x-\lambda)^{\mathbf{c}}.$$

The cohomology $H^1(X, \mathcal{O}_X)$ can be calculated as Čech cohomology with respect to covering of X $X_1 = \{x \neq \infty\}$, $X_2 = \{x \neq 0\}$. For a detailed description we need:

$$\alpha = \mathbf{a}/\mathbf{n}, \quad \beta = \mathbf{b}/\mathbf{n}, \quad \gamma = \mathbf{c}/\mathbf{n},$$

$$\langle l \rangle = -[-\langle l\alpha \rangle - \langle l\beta \rangle - \langle l\gamma \rangle] \in \{0, 1, 2, 3\}$$

$$\mathcal{J} := \{(l, j) \in (\mathbb{Z}/n\mathbb{Z}) \times \mathbb{Z} \mid 0 < j < \langle l \rangle\};$$

$[\cdot]$ and $\langle \cdot \rangle$ are the usual integral and fractional part functions.

For $(l, j) \in \mathcal{J}$ define

$$\begin{aligned} v_l &= y^{\tilde{l}} x^{-[\tilde{l}\alpha]} (x-1)^{-[\tilde{l}\beta]} (x-\lambda)^{-[\tilde{l}\gamma]} \\ \omega_{(l,j)} &= \text{coho class of Čech 1-cocycle } x^{-j} v_l \\ \tilde{\omega}_{(l,j)} &= n^{-1} x^{j-1} v_l^{-1} dx \\ &= n^{-1} x^{j-1-\langle \tilde{l}\alpha \rangle} (x-1)^{-\langle \tilde{l}\beta \rangle} (x-\lambda)^{-\langle \tilde{l}\gamma \rangle} dx \end{aligned}$$

with $\tilde{l} \in \mathbb{N}$, $l \equiv \tilde{l} \pmod{n}$. Then

$$\begin{aligned} \{\omega_{(l,j)}\}_{(l,j) \in \mathcal{J}} &= \text{basis of } H^1(X, \mathcal{O}_X) \\ \{\tilde{\omega}_{(l,j)}\}_{(l,j) \in \mathcal{J}} &= \text{dual basis for } H^0(X, \Omega_{X/S}^1) \end{aligned}$$

Lift $\omega_{(l,j)}$ to $\tilde{\omega}_{(l,j)}$ in $H^1(X, \underline{\mathcal{W}\mathcal{O}}_X)$ as follows. $\underline{x^{-j} v_l}$ is section of $\underline{\mathcal{W}\mathcal{O}}_X$ over $X_1 \cap X_2$. The Čech cocycle condition is trivially satisfied! Take

$$\tilde{\omega}_{(l,j)} = \text{cohomology class of the Čech 1-cocycle } \underline{x^{-j} v_l}.$$

Then

$$\pi F_N \tilde{\omega}_{(l,j)} = \text{cohomology class of the Čech 1-cocycle } (x^{-j} v_l)^N$$

Recall the definition $\pi F_N \tilde{\omega} = B_N \omega$. Thus, indexing the rows and columns of B_N with elements of $\mathcal{J}$, one finds

$$B_{N,(l,j),(l',j')} = 0$$

if $l' \neq lN$, whereas for $l' = lN$

$$B_{N,(l,j),(l',j')} = (-1)^L \sum_k \binom{[N \langle l\beta \rangle]}{L-k} \binom{[N \langle l\gamma \rangle]}{k} \lambda^k$$

here $L = j' - jN + [N \langle l\alpha \rangle] + [N \langle l\beta \rangle] + [N \langle l\gamma \rangle]$.

Then one easily checks the following congruence differential equation

$$\nabla(P_{(l',j')}) B_{N,(l,j),(l',j')} \equiv 0 \pmod{NA}$$

where $P_{(\mu, j')}$ is the hypergeometric differential operator, with $\Theta = \lambda \frac{d}{d\lambda}$,

$$\Theta(\Theta - j' + \langle l'\alpha \rangle + \langle l'\gamma \rangle) - \\ - \lambda(\Theta + \langle l'\gamma \rangle)(\Theta - j' + \langle l'\alpha \rangle + \langle l'\beta \rangle + \langle l'\gamma \rangle)$$

We now turn to **p-adic solutions**, p prime > 2 . Our method is based on the commutativity of the diagram

$$\begin{array}{ccc} H^m(X, \mathcal{W}\mathcal{O}_X) & \xrightarrow{F_p} & H^m(X, \mathcal{W}\mathcal{O}_X) \\ \downarrow & & \downarrow \\ \downarrow F_{p^{r+1}} & & \downarrow F_{p^r} \\ \downarrow & & \downarrow \\ \mathcal{H}^m(X, \mathcal{W}\Omega_X^\bullet/p^{r+1}) & \rightarrow & \mathcal{H}^m(X, \mathcal{W}\Omega_X^\bullet/p^r) \\ \downarrow & & \downarrow \\ \mathcal{H}^m(X, \Omega_X^\bullet/p^{r+1}) & \rightarrow & \mathcal{H}^m(X, \Omega_X^\bullet/p^r) \\ \downarrow & & \downarrow \\ \mathcal{H}^m(X, \Omega_{X/S}^\bullet/p^{r+1}) & \rightarrow & \mathcal{H}^m(X, \Omega_{X/S}^\bullet/p^r) \\ \downarrow \nabla & & \downarrow \nabla \\ \Omega_S^1 \otimes \mathcal{H}^m(X, \Omega_{X/S}^\bullet/p^{r+1}) & \rightarrow & \Omega_S^1 \otimes \mathcal{H}^m(X, \Omega_{X/S}^\bullet/p^r) \end{array}$$

In the limit for $r \rightarrow \infty$ it gives

$$\lim_{\leftarrow F_p} H^m(X, \mathcal{W}\mathcal{O}_X) \rightarrow (\mathcal{H}^m(X, \Omega_{X/S}^\bullet) \otimes \mathbb{Z}_p)^\nabla$$

and thus we try to find p -adic solutions of Picard-Fuchs equations by "lifting against Frobenius". This amounts to solving algebraic equations!

Vectors fixed by Frobenius

Assume $\det B_p \notin pA$. Let

$$A^0 = A[(\det B_p)^{-1}], \quad A_0 = A^0/pA^0, \quad A^\wedge = \varprojlim_n A^0/p^n A^0.$$

A_0 is a direct product of domains. Fix one such component and let R be its inverse image in $A^\wedge$. Then R is complete and separated in the p -adic topology and $\det B_p$ is invertible in R .

Let P be the set of primes $\neq p$. For every scheme Y such that every $l \in P$ is invertible in $\mathcal{O}_Y^*$ one can use the idempotent operator $E_p :=$

$\prod_{l \in P} (1 - l^{-1} V_l F_l)$ on $\underline{\mathcal{W}\mathcal{O}_Y}$ to split off the sheaf of *p-typical Witt vectors* on Y .

$$\mathcal{W}\mathcal{O}_Y = E_p \underline{\mathcal{W}\mathcal{O}_Y}$$

There exists a $\mathbb{Z}_p$ -algebra endomorphism σ of R such that

$$\sigma(x) \equiv x^p \pmod{pR} \quad \forall x \in R$$

There are many such σ . Given a choice for σ there is a unique homomorphism of rings

$$\lambda : R \rightarrow \mathcal{W}(R)$$

such that $\pi F_p^n \lambda = \sigma^n \quad \forall n \in \mathbb{N}$; here $\mathcal{W}(R)$ is the ring of *p-typical Witt vectors* over R and $\pi : \mathcal{W}(R) \rightarrow R$ is the projection onto first coordinate

Notations:

$$\sigma(x) = x^\sigma, \quad F = F_p;$$

for a matrix $M = (m_{ij})$

$$M^{(p^r)} = (m_{ij}^{p^r}), \quad M^{\sigma^r} = (m_{ij}^{\sigma^r}), \quad \lambda(M) = (\lambda(m_{ij})), \quad \underline{\underline{M}} = (\underline{\underline{m_{ij}}});$$

for A -algebra $A' \quad X \otimes A = X \times_S \text{Spec} A'$.

Theorem

$\exists H \in GL_h(R)$ s.t. $B_{p^{r+1}} \equiv B_p^\sigma H \pmod{p^{r+1}} \quad \forall r \geq 0$.

$\exists \hat{\omega}_1, \dots, \hat{\omega}_h \in H^m(X \otimes R, \mathcal{W}\mathcal{O}_{X \otimes R})$ s.t. $F \underline{\hat{\omega}} = \lambda(H) \underline{\hat{\omega}}$ and $\pi \hat{\omega}_i = \omega_i$,
 $\underline{\hat{\omega}}$ = column vector $(\hat{\omega}_1, \dots, \hat{\omega}_h)^t$.

Fix an algebraically closed field $\Omega \supset R/pR$ and define

$$(R/pR)^{\acute{e}t} := \varinjlim_{B \in \mathcal{B}} B.$$

where $\mathcal{B}$ is the set of finite étale extensions of R/pR in Ω . For every $B \in \mathcal{B}$ there is a unique finite étale $\tilde{B}$ over R such that $B = \tilde{B}/p\tilde{B}$. We define

$$R^{\acute{e}t} := \text{the } p\text{-adic completion of } \varinjlim_{B \in \mathcal{B}} \tilde{B}.$$

$(R/pR)^{ét}$ is an infinite étale extension of R/pR and $R^{ét}/pR^{ét} = (R/pR)^{ét}$. The *algebraic fundamental group* $\pi_1(\text{Spec}(R/pR), \Omega)$ is by definition the Galois group of $(R/pR)^{ét}/(R/pR)$. It acts on $R^{ét}$. σ induces an endomorphism σ of $R^{ét}$.

$$(R^{ét})^\sigma = \mathbb{Z}_p, \quad (R^{ét})^{\pi_1} = R.$$

Proposition $\exists C \in GL_h(R^{ét})$ s.t. $C^\sigma H = C$.

idea of proof: The system of equations

$$\begin{aligned} C_0^{(p)} H - C_0 &= 0, \quad \delta \cdot \det C_0 - 1 = 0, \\ C_{i+1}^{(p)} H - C_{i+1} + p^{-1}[C_i^\sigma - C_i^{(p)}] H &= 0 \quad (i \geq 0) \end{aligned}$$

can inductively be solved with $h \times h$ -matrices C_i over $R^{ét}$. Then $C := \sum_i p^i C_i$ is a solution.

$R \hookrightarrow R^{ét}$ induces $H^m(X \otimes R, \mathcal{W}\mathcal{O}_{X \otimes R}) \hookrightarrow H^m(X \otimes R^{ét}, \mathcal{W}\mathcal{O}_{X \otimes R^{ét}})$. Define

$$\xi_1, \dots, \xi_h \in H^m(X \otimes R^{ét}, \mathcal{W}\mathcal{O}_{X \otimes R^{ét}})$$

by

$$\underline{\xi} = \lambda(C) \underline{\hat{\omega}}.$$

Then

$$F \underline{\xi} = \underline{\xi}, \quad \pi \underline{\xi} = C \underline{\omega}.$$

Proposition

$H^m(X \otimes R^{ét}, \mathcal{W}\mathcal{O}_{X \otimes R^{ét}})$ is a free $\mathcal{W}(R^{ét})$ -module with bases $\{\xi_1, \dots, \xi_h\}$ and $\{\hat{\omega}_1, \dots, \hat{\omega}_h\}$

$H^m(X \otimes R, \mathcal{W}\mathcal{O}_{X \otimes R})$ is a free $\mathcal{W}(R)$ -module with basis $\{\hat{\omega}_1, \dots, \hat{\omega}_h\}$.

$\pi : H^m(X \otimes R^{ét}, \mathcal{W}\mathcal{O}_{X \otimes R^{ét}}) \rightarrow H^m(X \otimes R^{ét}, \mathcal{O}_{X \otimes R^{ét}})$ restricts to an isomorphism $\pi : \Lambda \simeq \pi \Lambda$ on

$$\Lambda := \ker(F - 1 \text{ on } H^m(X \otimes R^{ét}, \mathcal{W}\mathcal{O}_{X \otimes R^{ét}})).$$

Write Λ resp. ξ instead of $\pi\Lambda$ resp. $\pi\xi$.

Theorem. Λ is a free $\mathbb{Z}_p$ -module with basis $\{\xi_1, \dots, \xi_h\}$.

$$H^m(X, \mathcal{O}_X) \otimes_A R^{\acute{e}t} = \Lambda \otimes_{\mathbb{Z}_p} R^{\acute{e}t}$$

$$\underline{\xi} = C\underline{\omega}, \quad \nabla \underline{\xi} = 0$$

Thus the rows of C satisfy the same differential equations as $\{\tilde{\omega}_1, \dots, \tilde{\omega}_h\}$.

$\pi_1 := \pi_1(\text{Spec}(R/pR), \Omega)$ acts on $R^{\acute{e}t}$. By functoriality this induces an action of π_1 on $H^m(X, \mathcal{O}_X) \otimes_A R^{\acute{e}t}$ and on $H^m(X \otimes R^{\acute{e}t}, \mathcal{W}\mathcal{O}_{X \otimes R^{\acute{e}t}})$. Since F and π are π_1 equivariant we obtain the **p -adic monodromy representation**:

$$\mathcal{M} : \pi_1(\text{Spec}(R/pR), \Omega) \rightarrow \text{Aut}_{\mathbb{Z}_p}(\Lambda)$$

$$\mathcal{M}(\tau)\underline{\xi} = C^\tau C^{-1} \underline{\xi} \quad \text{for } \tau \in \pi_1.$$

$$\underline{\xi} = \text{column vector } (\xi_1, \dots, \xi_h)^t$$

The **p -adic monodromy group** $\mathcal{M}(\pi_1)$ for the hypergeometric curve

$$y^5 = x(x-1)^2(x-\lambda)^3.$$

is computed in J. Stienstra, M. van der Put, B. van der Marel, *On p -adic monodromy*. It turns out to be conjugate to:

case $p \equiv \pm 1 \pmod{5}$

$$\left\{ \begin{pmatrix} \eta a & & & 0 \\ & \eta^2 b & & \\ & & \eta^{-2} b & \\ 0 & & & \eta^{-1} a \end{pmatrix} \middle| \begin{array}{l} a, b \in \mathbb{Z}_p^*, \\ \eta \in \mu_5 \end{array} \right\}.$$

case $p \equiv \pm 2 \pmod{5}$

$$\left\{ \begin{pmatrix} \eta a & & & 0 \\ & \eta^2 a^\sigma & & \\ & & \eta^{-2} a^\sigma & \\ 0 & & & \eta^{-1} a \end{pmatrix} \middle| \begin{array}{l} a \in \mathcal{W}(\mathbb{F}_{p^2})^*, \\ \eta \in \mu_5 \end{array} \right\}$$