

Multi-peak positive solutions for nonlinear Schrödinger equations with critical frequency

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0. Introduction

In this report, we consider the following nonlinear Schrödinger equations:

$$\begin{aligned} -\epsilon^2 \Delta u + V(x)u &= |u|^{p-1}u \quad \text{in } \mathbf{R}^N, \\ u &> 0 \quad \text{in } \mathbf{R}^N, \\ u &\in H^1(\mathbf{R}^N), \end{aligned} \tag{*}_\epsilon$$

where $\epsilon > 0$ is small parameter and p satisfies $1 < p < \infty$ ($N = 1, 2$), $1 < p < \frac{N+2}{N-2}$ ($N \geq 3$). We are interested in the existence of solutions of $(*)_\epsilon$ for small $\epsilon > 0$ and their behavior as $\epsilon \rightarrow 0$. We assume the potential $V(x)$ satisfies the following assumptions:

(V.1) $V(x) \in C^1(\mathbf{R}^N, \mathbf{R})$ and $V(x) \geq 0$ for all $x \in \mathbf{R}^N$.

(V.2) $0 < \liminf_{|x| \rightarrow \infty} V(x) \leq \sup_{x \in \mathbf{R}^N} V(x) < \infty$.

Under the above assumptions, the solutions of $(*)_\epsilon$ are characterized as critical points of

$$\Psi_\epsilon(u) = \int_{\mathbf{R}^N} \frac{1}{2} (\epsilon^2 |\nabla u|^2 + V(x)u^2) - \frac{1}{p+1} u_+^{p+1} dx \in C^2(H^1(\mathbf{R}^N), \mathbf{R}).$$

When $V(x)$ satisfies

$$\inf_{x \in \mathbf{R}^N} V(x) > 0, \tag{0.1}$$

$(*)_\epsilon$ has a family of single-peak solutions $u_\epsilon(x)$ concentrating around a local minimum of $V(x)$ for small $\epsilon > 0$ in the following sense: let x_ϵ be a unique maximum of $u_\epsilon(x)$. Then x_ϵ approaches to a local minimum x_0 of $V(x)$ as $\epsilon \rightarrow 0$ and

$$w_\epsilon(y) = u_\epsilon(\epsilon y + x_\epsilon) \tag{0.2}$$

converges to a least energy solution $w_0(x)$ of the following “limit equation”

$$-\Delta w + V(x_0)w = w^p, \quad w > 0 \quad \text{in } \mathbf{R}^N, \quad w \in H^1(\mathbf{R}^N). \tag{0.3}$$

In particular, $u_\epsilon(x)$ satisfies

$$\lim_{\epsilon \rightarrow 0} \|u_\epsilon\|_{L^\infty(\mathbf{R}^N)} = |w_0(0)| > 0, \quad (0.4)$$

$$\lim_{\epsilon \rightarrow 0} \epsilon^{-N} \Psi_\epsilon(u_\epsilon) = \left(\frac{1}{2} - \frac{1}{p+1}\right) b_{V(x_0), \mathbf{R}^N} > 0. \quad (0.5)$$

Here $b_{V(x_0), \mathbf{R}^N}$ is defined by

$$b_{V(x_0), \mathbf{R}^N} = \inf_{u \in H^1(\mathbf{R}^N) \setminus \{0\}} \left(\frac{\int_{\mathbf{R}^N} |\nabla u|^2 + V(x_0) u^2 \, dx}{\|u_+\|_{L^{p+1}(\mathbf{R}^N)}^2} \right)^{\frac{p+1}{p-1}}$$

and $\left(\frac{1}{2} - \frac{1}{p+1}\right) b_{V(x_0), \mathbf{R}^N}$ is the lowest non-trivial critical value of the functional corresponding to (0.3). See [FW, O1, O2, DF] for related results.

Recently, Byeon and Wang [BW1, BW2] have started to study the case

$$\inf_{x \in \mathbf{R}^N} V(x) = 0. \quad (0.6)$$

They showed that $(*)_\epsilon$ also has single-peak solutions concentrating around an isolated component A of $\{x \in \mathbf{R}^N; V(x) = 0\}$. The features of their solutions are completely different from the solutions under the condition (0.1), which depend on the behavior of $V(x)$ around A . More precisely, their solutions $u_\epsilon(x)$ satisfy

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \|u_\epsilon\|_{L^\infty(\mathbf{R}^N)} &= 0, \\ \lim_{\epsilon \rightarrow 0} \epsilon^{-N} \Psi_\epsilon(u_\epsilon) &= 0. \end{aligned}$$

These are in contrast with (0.4)–(0.5). We can also see $\liminf_{\epsilon \rightarrow 0} \epsilon^{-\frac{2}{p-1}} \|u_\epsilon\|_{L^\infty(\mathbf{R}^N)} \in (0, \infty]$ and $\liminf_{\epsilon \rightarrow 0} \epsilon^{-\frac{2(p+1)}{p-1}} \Psi_\epsilon(u_\epsilon) \in (0, \infty]$. (We remark that $p < \frac{N+2}{N-2}$ implies $\frac{2(p+1)}{p-1} > N$.) Under additional conditions on the behavior of $V(x)$ near A , we can introduce a rescaled function — which is different from (0.2) — to observe the behavior of $u_\epsilon(x)$ around A . More precisely, under one of the conditions (L.2)–(L.4) (see p.5 below), we define $g(\epsilon) > 0$ as in the table 0.1 below and set

$$w_\epsilon(y) = \left(\frac{g(\epsilon)}{\epsilon}\right)^{\frac{2}{p-1}} u_\epsilon(g(\epsilon)y + x_\epsilon)$$

for a suitable point x_ϵ around A , then $w_\epsilon(y)$ approaches to the least energy solution of

$$-\Delta w + V_0(y)w = w^p, \quad w > 0 \quad \text{in } \Omega_0, \quad w \in H_0^1(\Omega_0). \quad (L)_{V_0, \Omega_0}$$

Here $\Omega_0 \subset \mathbf{R}^N$ and $V_0(x) \in C(\Omega_0, \mathbf{R})$ are also given in the table 0.1 below. In particular, $u_\epsilon(x)$ satisfies

$$\lim_{\epsilon \rightarrow 0} \left(\frac{g(\epsilon)}{\epsilon} \right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N} \Psi_\epsilon(u_\epsilon) \in (0, \infty)$$

and the decaying rate of $\Psi_\epsilon(u_\epsilon)$ depends on the behavior of $V(x)$ around A .

In this report, we study the multi-peak solutions of $(*)_\epsilon$ combining various types of peaks. We assume that $\Lambda_1, \dots, \Lambda_k \subset \mathbf{R}^N$ satisfy

(V.3) $\Lambda_1, \dots, \Lambda_k \subset \mathbf{R}^N$ are bounded open sets satisfying $\Lambda_i \cap \Lambda_j = \emptyset$ ($i \neq j$) and

$$m_i = \inf_{x \in \Lambda_i} V(x) < \inf_{x \in \partial \Lambda_i} V(x) < \infty \quad (i = 1, \dots, k).$$

Here we remark that m_i may be 0. In what follows, we write

$$A_i = \{x \in \Lambda_i; V(x) = m_i\} \quad (i = 1, \dots, k).$$

Under the assumption (V.3), we will show that $(*)_\epsilon$ has k -peak solutions which have 1-peak in each Λ_i and it concentrates around a local minimum of $V(x)$ in Λ_i . We remark that the behavior of the desired solution $u_\epsilon(x)$ in Λ_i depends on the behavior of $V(x)$ around A_i and just single scaling (0.2) is not enough to describe the behavior of $u_\epsilon(x)$. Moreover the localized critical level $\Psi_{\epsilon, \Lambda_i}(u_\epsilon)$ — which is defined as (1.1) below — approaches to 0 with different rate with respect to ϵ . This makes our problems difficult.

It seems that the existence of multi-peak solutions joining solutions which have different scales is not well studied. In our knowledge, there is only one work by Byeon and Oshita [BO] in which they constructed multi-peak solutions by a Lyapunov-Schmidt reduction method under the assumptions of non-degeneracy of solutions of limit problems.

Recently, in [S], we constructed the multi-peak solutions under the assumptions (V.1)–(V.3). We used a variational approach and it does not need the assumptions of non-degeneracy of solutions of limit problems. In the following sections, we will introduce main results and an outline of the proof in [S]. We also study about the asymptotic behavior of solutions and consider other examples than [S].

1. Main Results

Firstly, we introduce main theorems in [S].

(a) Existence of multi-peak solutions

In this section, we consider the existence of multi-peak solutions of $(*)_\epsilon$. To state theorems, we set

$$\Psi_{\epsilon, \Lambda_i}(u) = \int_{\Lambda_i} \frac{1}{2} (\epsilon^2 |\nabla u|^2 + V(x) u^2) - \frac{1}{p+1} u_+^{p+1} dx. \quad (1.1)$$

and define

$$c_{\epsilon, \Lambda_i} = \inf_{u \in H_0^1(\Lambda_i) \setminus \{0\}} \left(\frac{\int_{\Lambda_i} \epsilon^2 |\nabla u|^2 + V(x) u^2 dx}{\|u_+\|_{L^{p+1}(\Lambda_i)}^2} \right)^{\frac{p+1}{p-1}}. \quad (1.2)$$

Under the assumptions (V.1)–(V.3), we can observe

$$\lim_{\epsilon \rightarrow 0} \epsilon^{-N} c_{\epsilon, \Lambda_i} < \infty, \quad \liminf_{\epsilon \rightarrow 0} \epsilon^{-\frac{2(p+1)}{p-1}} c_{\epsilon, \Lambda_i} \in (0, \infty].$$

We remark that $\left(\frac{1}{2} - \frac{1}{p+1}\right) c_{\epsilon, \Lambda_i}$ is the lowest non-trivial critical value of the functional which corresponds to the equation

$$-\epsilon^2 \Delta u + V(x)u = u^p, \quad u > 0 \quad \text{in } \Lambda_i, \quad u \in H_0^1(\Lambda_i). \quad (1.3)$$

Our first theorem is

Theorem 1.1 ([S]). *Assume that (V.1)–(V.2) hold and $\Lambda_1, \dots, \Lambda_k \subset \mathbf{R}^N$ satisfy (V.3). Then there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$, there exists a positive solution $u_\epsilon(x)$ of $(*)_\epsilon$ which satisfies*

$$\Psi_{\epsilon, \Lambda_i}(u_\epsilon) = \left(\frac{1}{2} - \frac{1}{p+1}\right) c_{\epsilon, \Lambda_i} + o(\epsilon^{\frac{2(p+1)}{p-1}}) \quad (i = 1, \dots, k), \quad (1.4)$$

$$\Psi_{\epsilon, \mathbf{R}^N \setminus (\cup_{i=1}^k \Lambda_i)}(u_\epsilon) = o(\epsilon^{\frac{2(p+1)}{p-1}}). \quad (1.5)$$

Moreover for some constants $C, c > 0$, $u_\epsilon(x)$ satisfies

$$u_\epsilon(x) \leq C \exp\left(-\frac{c \operatorname{dist}(x, \cup_{i=1}^k \Lambda_i)}{\epsilon}\right) \quad \text{for } x \in \mathbf{R}^N \setminus (\cup_{i=1}^k \Lambda_i). \quad (1.6)$$

We remark that under the assumptions of Theorem 1.1, for any non-empty subset $\{i_1, \dots, i_\ell\} \subset \{1, \dots, k\}$, $\Lambda_{i_1}, \dots, \Lambda_{i_\ell}$ also satisfy (V.3). Thus we have

Corollary 1.2 ([S]). *There exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$, there exists a positive solution $u_\epsilon(x)$ of $(*)_\epsilon$ such that*

$$\Psi_{\epsilon, \Lambda_{i_j}}(u_\epsilon) = \left(\frac{1}{2} - \frac{1}{p+1}\right) c_{\epsilon, \Lambda_{i_j}} + o(\epsilon^{\frac{2(p+1)}{p-1}}) \quad (j = 1, \dots, \ell), \quad (1.7)$$

$$\Psi_{\epsilon, \mathbf{R}^N \setminus (\cup_{j=1}^\ell \Lambda_{i_j})}(u_\epsilon) = o(\epsilon^{\frac{2(p+1)}{p-1}}). \quad (1.8)$$

Moreover for some constants $C, c > 0$, $u_\epsilon(x)$ satisfies

$$u_\epsilon(x) \leq C \exp\left(-\frac{c \operatorname{dist}(x, \cup_{j=1}^\ell \Lambda_{i_j})}{\epsilon}\right) \quad \text{for } x \in \mathbf{R}^N \setminus (\cup_{j=1}^\ell \Lambda_{i_j}).$$

Epecially, for small $\epsilon > 0$, $()_\epsilon$ has at least $2^k - 1$ positive solutions.*

(b) Asymptotic behavior of solutions

In this section, we consider the asymptotic behavior of solutions of $(*)_\epsilon$ obtained in Theorem 1.1 and Corollary 1.2. For some $i \in \{1, \dots, k\}$, we assume that Λ_i satisfies one of the following assumptions:

$$(L.1) \quad \inf_{x \in \Lambda_i} V(x) = m_i > 0$$

$$(L.2) \quad V(x) \text{ has a unique local minimum } x_i \text{ in } \Lambda_i \text{ and } V(x) \text{ is represented as}$$

$$V(x) = P(x - x_i) + Q(x - x_i).$$

Here $P(x)$ is a m -homogeneous positive function for some $m > 0$, that is,

$$\begin{aligned} P(x) &> 0 \quad \text{for all } x \in \mathbf{R}^N \setminus \{0\}, \\ P(tx) &= t^m P(x) \quad \text{for all } (t, x) \in [0, \infty) \times \mathbf{R}^N, \end{aligned}$$

$$\text{and } \lim_{|x| \rightarrow 0} |x|^{-m} Q(x) = 0.$$

$$(L.3) \quad V(x) \text{ has a unique local minimum } x_i \text{ in } \Lambda_i \text{ and } V(x) \text{ is represented as}$$

$$V(x) = \exp\{-r(x - x_i)^\ell - Q(x - x_i)\}.$$

Here $\ell > 0$ and $r(x) : \mathbf{R}^N \setminus \{0\} \rightarrow (0, \infty)$ is a positive continuous function such that

$$\begin{aligned} r(x) &> 0 \quad \text{for all } x \in \mathbf{R}^N \setminus \{0\}, \\ r(tx) &= \frac{1}{t} r(x) \quad \text{for all } (t, x) \in (0, \infty) \times (\mathbf{R}^N \setminus \{0\}). \end{aligned}$$

We also assume $\Omega \equiv \{x \in \mathbf{R}^N; r(x) > 1\}$ is strictly star-shaped with respect to 0 and $\lim_{|x| \rightarrow 0} |x|^\ell Q(x) = 0$.

$$(L.4) \quad \Omega \equiv \text{int}\{x \in \Lambda_i; V(x) = 0\} \text{ is a non-empty connected bounded set with smooth boundary.}$$

In [BW1], (L.2) is called finite case, (L.3) is infinite case and (L.4) is flat case. Now we have

Theorem 1.3 ([S]). *Suppose that the assumptions of Theorem 1.1 is satisfied and let $u_\epsilon(x)$ be a positive solution obtained in Theorem 1.1. Assume also, for some $i \in \{1, \dots, k\}$, Λ_i satisfies one of the assumptions (L.1)–(L.4). Set $g(\epsilon) > 0$, $\Omega_0 \subset \mathbf{R}^N$ and $V_0(x) \in C(\Omega_0, \mathbf{R}^N)$ as in the table 0.1 below.*

(i) c_{ϵ, Λ_i} given in (1.2) satisfies

$$\lim_{\epsilon \rightarrow 0} \left(\frac{g(\epsilon)}{\epsilon} \right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N} c_{\epsilon, \Lambda_i} = b_{V_0, \Omega_0}.$$

Here, b_{V_0, Ω_0} is defined by

$$b_{V_0, \Omega_0} = \inf_{u \in H_0^1(\Omega_0) \setminus \{0\}} \left(\frac{\int_{\Omega_0} |\nabla u|^2 + V_0(x) u^2 dx}{\|u_+\|_{L^{p+1}(\Omega_0)}^2} \right)^{\frac{p+1}{p-1}}. \quad (1.9)$$

(ii) After extracting a subsequence $\epsilon_n \rightarrow 0$, there exists a sequence $x_n \in \Lambda_i$ and a least energy solution $w_0(x)$ of $(L)_{V_0, \Omega_0}$ such that the rescaled function

$$w_{\epsilon_n}(y) = \left(\frac{g(\epsilon_n)}{\epsilon_n} \right)^{\frac{2}{p-1}} u_{\epsilon_n}(g(\epsilon_n)y + x_n) \quad (1.10)$$

converges to $w_0(x)$ in the following sense

$$x_n \rightarrow x_i \in A_i, \quad (1.11)$$

$$\|w_{\epsilon_n} - w_0\|_{H^1(O_{i,n})} \rightarrow 0. \quad (1.12)$$

Here we set $O_{i,n} = \{y \in \mathbf{R}^N; g(\epsilon_n)y + x_n \in \Lambda_i\}$.

	(L.1)	(L.2)	(L.3)	(L.4)
$g(\epsilon)$	ϵ	$\epsilon^{\frac{2}{m+2}}$	$(\log \epsilon^{-2})^{-\frac{1}{t}}$	1
$V_0(x)$	m	$P(x)$	0	0
Ω_0	$\mathbf{R}^N$	$\mathbf{R}^N$	Ω	Ω

table 0.1

Remark 1.4. Byeon and Oshita [BO] constructed multi-peak solutions by a Lyapunov-Schmidt reduction method; More precisely, they showed the existence of multi-peak solutions under the following situations:

- $V(x) \in C(\mathbf{R}^N, \mathbf{R})$ and $V(x)$ satisfies $V(x) \geq 0$ for all $x \in \mathbf{R}^N$ and (V.2)–(V.3).
- All Λ_i ($i = 1, \dots, k$) satisfy one of the assumptions (L.2)–(L.4) and assumptions of the non-degeneracy of least energy solutions of limit equations.

Under the additional conditions $V(x) \in C^4(\mathbf{R}^N, \mathbf{R})$ and $u \rightarrow |u|^{p-1}u \in C^4(\mathbf{R})$, they also showed the existence of multi-peak solutions $u_\epsilon(x)$ which join peaks concentrating local minima satisfying (L.2)–(L.4) and topologically non-trivial critical points x_i of $V(x)$ in Λ_i with $V(x_i) > 0$. We remark that they also study the situation $\lim_{|x| \rightarrow \infty} V(x) = \infty$.

(c) Other limit equations

In this section, we consider about more general assumptions than (L.2)–(L.3). Firstly, we give some examples.

Examples.

- (i) Suppose that $N = 1$ and $\Lambda = (t_1, t_2)$ for $t_1 < 0 < t_2$. We also assume that for $a > b > 1$,

$$V(x) = \begin{cases} |x|^a & \text{if } x \in (t_1, 0], \\ |x|^b & \text{if } x \in (0, t_2). \end{cases}$$

Then, setting $g(\epsilon) = \epsilon^{\frac{2}{a+b}}$, $\Omega_0 = (0, \infty)$ and $V_0(x) = |x|^a$, after extracting subsequence $\epsilon_n \rightarrow 0$, there exists a least energy solution $w_0(x)$ of $(L)_{V_0, \Omega_0}$ such that the rescaled function $w_{\epsilon_n}(y)$, which defined as (1.10), converges to $w_0(x)$.

- (ii) Suppose that $N = 2$ and $0 \in \Lambda$. We also assume that for $a > b > 1$,

$$V(x_1, x_2) = |x_1|^a + |x_2|^b \quad \text{for } (x_1, x_2) \in \Lambda.$$

Then, setting $g(\epsilon) = \epsilon^{\frac{2}{a+b}}$, $\Omega_0 = \mathbf{R}^2$ and $V_0(x_1, x_2) = |x_2|^b$, after extracting subsequence $\epsilon_n \rightarrow 0$, there exists a least energy solution $w_0(x)$ of $(L)_{V_0, \Omega_0}$ such that the rescaled function $w_{\epsilon_n}(y)$, which defined as (1.10), converges to $w_0(x)$.

In what follows, we give a condition that there exist limit problems. This condition contains (L.2)–(L.3) and two examples above. We fix a $i_0 \in \{1, \dots, k\}$ and we write Λ_{i_0} by Λ . Without loss of generality, we assume that $0 \in \Lambda$ and 0 is a local minimum of $V(x)$. We set

$$\begin{aligned} V_\epsilon(x) &= \left(\frac{g(\epsilon)}{\epsilon} \right)^2 V(g(\epsilon)x), \\ \Lambda_{g(\epsilon)} &= \{y \in \mathbf{R}^N; g(\epsilon)y \in \Lambda\}. \end{aligned}$$

Now, we consider the following assumption.

- (L*) 0 is a unique local minimum of $V(x)$ in Λ . Moreover, there exist $g(\epsilon) > 0$ such that $g(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$ and setting

$$\begin{aligned} V_0(x) &= \lim_{\epsilon \rightarrow 0} V_\epsilon(x) \in [0, \infty], \\ \Omega_0 &= \text{int}\{x \in \mathbf{R}^N \mid V_0(x) < \infty\}, \end{aligned}$$

it holds the following:

(i) For any compact set $D \subset\subset \Omega_0$,

$$\lim_{\epsilon \rightarrow 0} \sup_{x \in D} |V_\epsilon(x) - V_0(x)| = 0. \quad (1.13)$$

(ii) For any closed set $E \supset\supset \Omega_0$,

$$\lim_{\epsilon \rightarrow 0} \inf_{x \in \Lambda_{g(\epsilon)} \setminus E} V_\epsilon(x) = \infty. \quad (1.14)$$

(iii) If $\Omega_0 = \mathbf{R}^N$, then for any $\delta \in (0, 1)$ there exists $\epsilon_0 = \epsilon_0(\delta) > 0$ such that for $\epsilon \in (0, \epsilon_0)$,

$$V_\epsilon(x) \geq (1 - \delta)V_0(x) \quad \text{for all } x \in \Lambda_{g(\epsilon)}, \quad (1.15)$$

Moreover, for $k \in \mathbf{N}$ with $1 \leq k \leq N$, $V_0(x)$ satisfies

$$V_0(x_1, \dots, x_k, x_{k+1}, \dots, x_N) = V_0(x_1, \dots, x_k, 0, \dots, 0) \quad \text{for all } x \in \mathbf{R}^N \quad (1.16)$$

$$\lim_{|x_1| + \dots + |x_k| \rightarrow \infty} V_0(x_1, \dots, x_k, 0, \dots, 0) = \infty. \quad (1.17)$$

If $\Omega_0 \neq \mathbf{R}^N$, then $\partial\Omega_0$ is smooth and $V_0(x)$ satisfies

$$\lim_{|x| \rightarrow \infty} V_0(x) = \infty \quad (1.18)$$

We remark that $(L)_{V_0, \Omega_0}$ has a least energy solution under the condition (1.16)–(1.17) or (1.18). Here we have

Theorem 1.5. *Suppose that the assumptions of Theorem 1.1 is satisfied and let $u_\epsilon(x)$ be a positive solution obtained in Theorem 1.1. Assume also Λ satisfies the assumptions (L^*) .*

(i) c_{ϵ, Λ_i} given in (1.2) satisfies

$$\lim_{\epsilon \rightarrow 0} \left(\frac{g(\epsilon)}{\epsilon} \right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N} c_{\epsilon, \Lambda_i} = b_{V_0, \Omega_0}.$$

Here, b_{V_0, Ω_0} is defined by (1.9).

(ii) After extracting a subsequence $\epsilon_n \rightarrow 0$, there exists a sequence $x_n \in \Lambda_i$ and a least energy solution $w_0(x)$ of $(L)_{V_0, \Omega_0}$ such that the rescaled function

$$w_{\epsilon_n}(y) = \left(\frac{g(\epsilon_n)}{\epsilon_n} \right)^{\frac{2}{p-1}} u_{\epsilon_n}(g(\epsilon_n)y + x_n) \quad (1.19)$$

converges to $w_0(x)$ in the following sense

$$x_n \rightarrow 0, \quad (1.20)$$

$$\|w_{\epsilon_n} - w_0\|_{H^1(O_{i,n})} \rightarrow 0. \quad (1.21)$$

Here we set $O_{i,n} = \{y \in \mathbf{R}^N; g(\epsilon_n)y + x_n \in \Lambda_i\}$.

In next sections, we will give a outline of the proof of main theorems.

2. Variational formulation

In this section, we give our variational formulation of $(*)_\epsilon$, which used in [S]. We will reduce to a variational problem defined on infinite dimensional torus $\Sigma_{\epsilon, \Lambda_1} \times \cdots \times \Sigma_{\epsilon, \Lambda_k}$.

(a) Preliminary

We use the following notation: for an open subset $D \subset \mathbf{R}^N$,

$$\begin{aligned} \langle u, v \rangle_{\epsilon, D} &= \int_D \epsilon^2 \nabla u \cdot \nabla v + V(x)uv \, dx \quad \text{for } u, v \in H^1(D), \\ \|u\|_{\epsilon, D}^2 &= \langle u, u \rangle_{\epsilon, D} \quad \text{for } u \in H^1(D), \\ \|f\|_{\epsilon, D}^* &= \sup_{u \in H^1(D), \|u\|_{\epsilon, D} \leq 1} |f(u)| \quad \text{for } f \in (H^1(D))^*. \end{aligned}$$

For an open subset $D \subset \mathbf{R}^N$ and $W(x) \in C(D, \mathbf{R})$, we also set

$$\begin{aligned} E_{W,D} &= \{v \in H^1(D); \int_D W(x)v^2 \, dx < \infty\}, \\ \langle u, v \rangle_{W,D} &= \int_D \nabla u \cdot \nabla v + W(x)uv \, dx \quad \text{for } u, v \in E_{W,D}, \\ \|u\|_{W,D}^2 &= \langle u, u \rangle_{W,D} \quad \text{for } u \in E_{W,D}. \end{aligned}$$

In what follows, we assume that Λ_i has smooth boundary. We set $\Lambda_* = \bigcup_{i=1}^k \Lambda_i$. By the following proposition, for subsets $D = \Lambda_1, \dots, \Lambda_k$ or $\mathbf{R}^N \setminus \Lambda_*$, norm $\|\cdot\|_{\epsilon, D}$ is equivalent to $\|\cdot\|_{H^1(D)}$.

Proposition 2.1. *There exists $C_1 > 0$ independent of $\epsilon \in (0, 1)$ such that for subsets $D = \Lambda_1, \dots, \Lambda_k$ or $\mathbf{R}^N \setminus \Lambda_*$,*

$$\|u\|_{L^2(D)}^2 \leq \frac{C_1}{\epsilon^2} \|u\|_{\epsilon, D}^2 \quad \text{for } u \in H^1(D). \quad (2.1)$$

Proof. It follows from the Poincaré inequality. (See Proposition 1.1 of [S].) ■

From Proposition 2.1, we can easily show the following lemma.

Lemma 2.2. *There exists $\nu_0 > 0$ independent of $\epsilon \in (0, 1)$ such that for subsets $D = \Lambda_1, \dots, \Lambda_k$ or $\mathbf{R}^N \setminus \Lambda_*$,*

$$\frac{1}{2} \|u\|_{\epsilon, D}^2 \leq \|u\|_{\epsilon, D}^2 - 2\nu_0 \epsilon^2 \|u\|_{L^2(D)}^2 \quad \text{for } u \in H^1(D). \quad (2.2)$$

Proof. Setting $\nu_0 = \frac{1}{4C_1}$ for the constant C_1 appeared in (2.1), we get inequality (2.2). ■

In what follows, we choose Λ'_i such that

$$A_i \subset\subset \Lambda'_i \subset\subset \Lambda_i,$$

and set $\Lambda'_* = \bigcup_{i=1}^k \Lambda'_i$. We also use the following lemma.

Lemma 2.3. *There exists a bounded linear operator $P : H^1(\Lambda_* \setminus \Lambda'_*) \rightarrow H^1(\mathbf{R}^N \setminus \Lambda'_*)$ such that for some $C_2 > 0$ independent of $\epsilon \in (0, 1)$,*

$$\begin{aligned} (Pu)(x) &= u(x) \quad \text{for } x \in \Lambda_* \setminus \Lambda'_* \text{ and } u \in H^1(\Lambda_* \setminus \Lambda'_*), \\ \|Pu\|_{\epsilon, \mathbf{R}^N \setminus \Lambda'_*} &\leq C_2 \|u\|_{\epsilon, \Lambda_* \setminus \Lambda'_*} \quad \text{for } u \in H^1(\Lambda_* \setminus \Lambda'_*). \end{aligned} \quad (2.3)$$

Proof. By a standard way, there exists a bounded linear operator $P : H^1(\Lambda_* \setminus \Lambda'_*) \rightarrow H^1(\mathbf{R}^N \setminus \Lambda'_*)$ such that for some $C > 0$,

$$\begin{aligned} (Pu)(x) &= u(x) \quad \text{for } x \in \Lambda_* \setminus \Lambda'_* \text{ and } u \in H^1(\Lambda_* \setminus \Lambda'_*), \\ \|Pu\|_{L^2(\mathbf{R}^N \setminus \Lambda'_*)} &\leq C \|u\|_{L^2(\Lambda_* \setminus \Lambda'_*)} \quad \text{for } u \in H^1(\Lambda_* \setminus \Lambda'_*), \\ \|Pu\|_{H^1(\mathbf{R}^N \setminus \Lambda'_*)} &\leq C \|u\|_{H^1(\Lambda_* \setminus \Lambda'_*)} \quad \text{for } u \in H^1(\Lambda_* \setminus \Lambda'_*). \end{aligned}$$

Thus, noting $\inf_{x \in \Lambda_* \setminus \Lambda'_*} V(x) > 0$, we have

$$\begin{aligned} \|Pu\|_{\epsilon, \mathbf{R}^N \setminus \Lambda'_*}^2 &\leq \epsilon^2 \|Pu\|_{H^1(\mathbf{R}^N \setminus \Lambda'_*)}^2 + \left(\sup_{x \in \mathbf{R}^N} V(x) \right) \|Pu\|_{L^2(\mathbf{R}^N \setminus \Lambda'_*)}^2 \\ &\leq C^2 \epsilon^2 \|u\|_{H^1(\Lambda_* \setminus \Lambda'_*)}^2 + C^2 \left(\sup_{x \in \mathbf{R}^N} V(x) \right) \|u\|_{L^2(\Lambda_* \setminus \Lambda'_*)}^2 \\ &\leq C^2 \int_{\Lambda_* \setminus \Lambda'_*} \epsilon^2 |\nabla u|^2 dx + C^2 \frac{\epsilon^2 + \sup_{x \in \mathbf{R}^N} V(x)}{\inf_{x \in \Lambda_* \setminus \Lambda'_*} V(x)} \int_{\Lambda_* \setminus \Lambda'_*} V(x) u^2 dx \\ &\leq C^2 \left(1 + \frac{1 + \sup_{x \in \mathbf{R}^N} V(x)}{\inf_{x \in \Lambda_* \setminus \Lambda'_*} V(x)} \right) \|u\|_{\epsilon, \Lambda_* \setminus \Lambda'_*}^2. \end{aligned}$$

This is nothing but (2.3). ■

(b) **Functional setting**

Firstly, to find critical points of $\Psi_\epsilon(u)$, we modify the nonlinearity $|u|^{p-1}u$. We use a local mountain pass approach introduced by del Pino and Felmer [DF]. We choose $f_\epsilon \in C^1([0, \infty), \mathbf{R})$ such that for some $0 < \ell_1 < \ell_2$ and $\alpha - 2 > \frac{2(p+1)}{p-1} - N > 0$

$$f_\epsilon(\xi) = \begin{cases} \xi^p & \text{for } \xi \in (0, \epsilon^{\frac{\alpha}{p-1}} \ell_1), \\ \epsilon^\alpha \nu_0 \xi & \text{for } \xi \in (\epsilon^{\frac{\alpha}{p-1}} \ell_2, \infty), \end{cases} \quad 0 \leq f'_\epsilon(\xi) \leq 2\epsilon^\alpha \nu_0 \quad \text{for } \xi \in [0, \infty). \quad (2.4)$$

Here ν_0 is given in Lemma 2.2. We set

$$g_\epsilon(x, \xi) = \begin{cases} \xi^p & \text{if } x \in \Lambda_* \text{ and } \xi \geq 0, \\ f_\epsilon(\xi) & \text{if } x \notin \Lambda_* \text{ and } \xi \geq 0, \\ 0 & \text{if } \xi \leq 0, \end{cases} \quad (2.5)$$

$$G_\epsilon(x, \xi) = \int_0^\xi g_\epsilon(x, \tau) d\tau, \quad F_\epsilon(\xi) = \int_0^\xi f_\epsilon(\tau) d\tau.$$

Now, we define a functional $\Phi_\epsilon(u)$ by

$$\Phi_\epsilon(u) = \frac{1}{2} \|u\|_{\epsilon, \mathbf{R}^N}^2 - \int_{\mathbf{R}^N} G_\epsilon(x, u) dx \in C^2(H^1(\mathbf{R}^N), \mathbf{R}). \quad (2.6)$$

We remark that $\Phi_\epsilon(u)$ satisfies (PS)-condition. (cf.[DT]) We also note that if $u_\epsilon(x)$ is a critical point of $\Phi_\epsilon(u)$ satisfying

$$0 \leq u_\epsilon(x) \leq \epsilon^{\frac{\alpha}{p-1}} \ell_1 \quad \text{for } x \in \mathbf{R}^N \setminus \Lambda_*, \quad (2.7)$$

then $u_\epsilon(x)$ is a critical point of $\Psi_\epsilon(u)$. Thus, in what follows, we will find critical points of $\Phi_\epsilon(u)$ which satisfy (2.7).

Next, we reduce our problem to a problem on $H^1(\Lambda_*)$. For given $u \in H^1(\Lambda_*)$, we consider the following minimizing problem:

$$I_\epsilon(u) = \inf_{\varphi \in H_0^1(\mathbf{R}^N \setminus \Lambda_*)} \Phi_\epsilon(Pu + \varphi). \quad (2.8)$$

Remark 2.4. Letting $\varphi_\epsilon(u)$ be a minimizer of (2.8), then $Pu + \varphi_\epsilon(u)$ is a minimizer of $\inf_{v \in E_u} \Phi_\epsilon(v)$, where $E_u = \{v \in H^1(\mathbf{R}^N) ; v = u \text{ on } \Lambda_*\}$. Thus $w(x) = (Pu + \varphi_\epsilon(u))|_{\mathbf{R}^N \setminus \Lambda_*}(x)$ satisfies the following boundary value problem:

$$-\epsilon^2 \Delta w + V(x)w = f_\epsilon(w) \quad \text{in } \mathbf{R}^N \setminus \Lambda_*, \quad w = u \quad \text{on } \partial\Lambda_*. \quad (2.9)$$

For the minimizing problem (2.8), we have the following proposition.

Proposition 2.5. For any $u \in H^1(\Lambda_*)$ and $\epsilon \in (0, 1)$, the minimizing problem (2.8) has a unique minimizer $\varphi_\epsilon(u)$ which satisfies

- (i) $u \mapsto \varphi_\epsilon(u) : H^1(\Lambda_*) \rightarrow H_0^1(\mathbf{R}^N \setminus \Lambda_*)$ is of class C^1 .
- (ii) $I_\epsilon(u) = \Phi_\epsilon(Pu + \varphi_\epsilon(u)) : H^1(\Lambda_*) \rightarrow \mathbf{R}$ is of class C^2 .
- (iii) $u \in H^1(\Lambda_*)$ is a critical point of $I_\epsilon(u)$ if and only if $Pu + \varphi_\epsilon(u)$ is a critical point of $\Phi_\epsilon(u)$.
- (iv) $I_\epsilon(u)$ satisfies (PS)-condition.

Proof. We outline the proof. (See Proposition 1.6 of [S].) We consider the functional $J(\varphi) = \Phi_\epsilon(Pu + \varphi) : H_0^1(\mathbf{R}^N \setminus \Lambda_*) \rightarrow \mathbf{R}$. Then $J(\varphi)$ is strongly convex and coercive, that is, $J(\varphi)$ satisfies

$$J''(\varphi)(h, h) \geq \frac{1}{2} \|h\|_{\epsilon, \mathbf{R}^N \setminus \Lambda_*}^2 \quad \text{for any } \varphi, h \in H_0^1(\mathbf{R}^N \setminus \Lambda_*). \quad (2.10)$$

From (2.10), (2.8) has a unique minimizer $\varphi_\epsilon(u)$. (i) follows from the implicit function theorem. (ii)–(iv) follows from the fact that

$$\begin{aligned} \Phi'_\epsilon(Pu + \varphi_\epsilon(u))h &= 0 \quad \text{for all } h \in H_0^1(\mathbf{R}^N \setminus \Lambda_*), \\ \varphi'_\epsilon(u)\zeta &\in H_0^1(\mathbf{R}^N \setminus \Lambda_*) \quad \text{for all } \zeta \in H^1(\Lambda_*), \end{aligned} \quad (2.11)$$

that is,

$$I'_\epsilon(u)\zeta = \Phi'_\epsilon(Pu + \varphi_\epsilon(u))(P\zeta) \quad \text{for all } \zeta \in H^1(\Lambda_*). \quad \blacksquare$$

Setting $Q_\epsilon(u) = (Pu + \varphi_\epsilon(u))|_{\mathbf{R}^N \setminus \Lambda_*}$, by Remark 2.4, $Q_\epsilon(u)$ satisfies (2.9). In what follows, we identify $H^1(\Lambda_*)$ and $H^1(\Lambda_1) \oplus \cdots \oplus H^1(\Lambda_k)$. Moreover, for functions $u_i \in H^1(\Lambda_i)$ ($i = 1, \dots, k$), we write $u = (u_1, \dots, u_k)$ if $u \in H^1(\Lambda_*)$ satisfies $u_i = u|_{\Lambda_i}$ ($i = 1, \dots, k$). When $u = (u_1, \dots, u_k)$, we also write

$$\begin{aligned} I_\epsilon(u_1, \dots, u_k) &\quad \text{for } I_\epsilon(u), \\ Q_\epsilon(u_1, \dots, u_k) &\quad \text{for } Q_\epsilon(u). \end{aligned} \quad (2.12)$$

Moreover, we can write

$$\begin{aligned} I_\epsilon(u_1, \dots, u_k) &= \sum_{i=1}^k \left(\frac{1}{2} \|u_i\|_{\epsilon, \Lambda_i}^2 - \frac{1}{p+1} \|u_i\|_{L^{p+1}(\Lambda_i)}^{p+1} \right) \\ &\quad + \frac{1}{2} \|Q_\epsilon(u_1, \dots, u_k)\|_{\epsilon, \mathbf{R}^N \setminus \Lambda_*}^2 - \int_{\mathbf{R}^N \setminus \Lambda_*} F_\epsilon(Q_\epsilon(u_1, \dots, u_k)) dx. \end{aligned}$$

For $h_\epsilon(x) \in L^\infty(\mathbf{R}^N \setminus \Lambda_*)$ satisfying $\|h_\epsilon\|_{L^\infty(\mathbf{R}^N \setminus \Lambda_*)} \leq 2\epsilon^\alpha \nu_0$, we consider the following linear boundary value problem:

$$-\epsilon^2 \Delta v + V(x)v = h_\epsilon(x)v \quad \text{in } \mathbf{R}^N \setminus \Lambda_*, \quad v = u \quad \text{on } \Lambda_* \setminus \Lambda'_*. \quad (2.13)$$

In a similar way to Proposition 2.5, we can easily observe (2.13) has a unique solution. We denote the unique solution of (2.13) by $Q_{h_\epsilon, \epsilon}(u_1, \dots, u_k)(x)$. Then we can write as $Q_\epsilon(u_1, \dots, u_k)(x) = Q_{f_\epsilon(Q_\epsilon(u_1, \dots, u_k))/Q_\epsilon(u_1, \dots, u_k), \epsilon}(u_1, \dots, u_k)(x)$. Here, we define the auxiliary functionals

$$I_{\epsilon, \Lambda_i}(u_i) = \frac{1}{2} \|u_i\|_{\epsilon, \Lambda_i}^2 - \frac{1}{p+1} \|u_i\|_{L^{p+1}(\Lambda_i)}^{p+1} + \frac{1}{2} \|Q_{0, \epsilon}(0, \dots, u_i, \dots, 0)\|_{\epsilon, \mathbf{R}^N \setminus \Lambda_*}^2 \\ \in C^2(H^1(\Lambda_i), \mathbf{R}) \quad (i = 1, \dots, k).$$

The following proposition gives estimates of difference between $I_\epsilon(u_1, \dots, u_k)$ and auxiliary functional $\sum_{i=1}^k I_{\epsilon, \Lambda_i}(u_i)$ and it plays an important role in our proof.

Proposition 2.6. *There exists $C_3 > 0$ such that for $\epsilon \in (0, 1)$ and $(u_1, \dots, u_k) \in H^1(\Lambda_1) \oplus \dots \oplus H^1(\Lambda_k)$,*

$$|I_\epsilon(u_1, \dots, u_k) - \sum_{i=1}^k I_{\epsilon, \Lambda_i}(u_i)| \leq \epsilon^{\alpha-2} C_3 \sum_{i=1}^k \|u_i\|_{\epsilon, \Lambda_i}^2, \quad (2.14)$$

To prove Proposition 2.6, we show the following lemma.

Lemma 2.7. *There exists $C_4 > 0$ such that for $\epsilon \in (0, 1)$ and $(u_1, \dots, u_k) \in H^1(\Lambda_1) \oplus \dots \oplus H^1(\Lambda_k)$,*

$$\|Q_{h_\epsilon, \epsilon}(u_1, \dots, u_k)\|_{\epsilon, \mathbf{R}^N \setminus \Lambda_*} \leq C_4 \|u\|_{\epsilon, \Lambda_* \setminus \Lambda'_*}, \quad (2.15)$$

$$\|Q_\epsilon(u_1, \dots, u_k)\|_{\epsilon, \mathbf{R}^N \setminus \Lambda_*} \leq C_4 \|u\|_{\epsilon, \Lambda_* \setminus \Lambda'_*}, \quad (2.16)$$

$$\|Q_\epsilon(u_1, \dots, u_k) - Q_{0, \epsilon}(u_1, \dots, u_k)\|_{\epsilon, \mathbf{R}^N \setminus \Lambda_*} \leq \epsilon^{\alpha-2} C_4 \|u\|_{\epsilon, \Lambda_* \setminus \Lambda'_*}. \quad (2.17)$$

Proof. First of all, we set $v(x) = Q_{0, \epsilon}(u_1, \dots, u_k)(x)$ and $w(x) = Q_\epsilon(u_1, \dots, u_k)(x)$. From Remark 2.4, we remark that w is a unique minimizer of the minimizing problem $K_\epsilon(w) = \inf_{u \in E_u} K_\epsilon(u)$, where $K_\epsilon(u) = \int_{\mathbf{R}^N \setminus \Lambda_*} \frac{1}{2} (\epsilon^2 |\nabla u|^2 + V(x)u^2) - F_\epsilon(u) dx$. Thus by Lemma 2.2 and Lemma 2.3, we have

$$\frac{1}{4} \|w\|_{\epsilon, \mathbf{R}^N \setminus \Lambda_*}^2 \leq K_\epsilon(w) \leq K_\epsilon(Pu) \leq \|Pu\|_{\epsilon, \mathbf{R}^N \setminus \Lambda_*}^2 \leq C_2^2 \|u\|_{\epsilon, \Lambda_* \setminus \Lambda'_*}^2. \quad (2.18)$$

Thus (2.16) follows from (2.18). We can show (2.15) in a similar way. Next we show (2.17). By variational characterizations for $w(x)$ and $v(x)$ $w - v$ satisfies

$$\langle w - v, \varphi \rangle_{\epsilon, \mathbf{R}^N \setminus \Lambda_*} = \int_{\mathbf{R}^N \setminus \Lambda_*} f_\epsilon(w) \varphi dx \quad \text{for all } \varphi \in H_0^1(\mathbf{R}^N \setminus \Lambda_*). \quad (2.19)$$

Since $w - v \in H_0^1(\mathbf{R}^N \setminus \Lambda_*)$, setting $\varphi = w - v$ in (2.19), we have

$$\begin{aligned} \|w - v\|_{\epsilon, \mathbf{R}^N \setminus \Lambda_*}^2 &= \int_{\mathbf{R}^N \setminus \Lambda_*} f_\epsilon(w)(w - v) \, dx \\ &\leq \epsilon^\alpha \nu_0 \|w\|_{L^2(\mathbf{R}^N \setminus \Lambda_*)} \|w - v\|_{L^2(\mathbf{R}^N \setminus \Lambda_*)} \\ &\leq \epsilon^{\alpha-2} \nu_0 C_1 \|w\|_{\epsilon, \mathbf{R}^N \setminus \Lambda_*} \|w - v\|_{\epsilon, \mathbf{R}^N \setminus \Lambda_*}. \end{aligned} \quad (2.20)$$

Here we used Proposition 2.1. By (2.20), we get

$$\|w - v\|_{\epsilon, \mathbf{R}^N \setminus \Lambda_*} \leq \epsilon^{\alpha-2} \nu_0 C_1 \|w\|_{\epsilon, \mathbf{R}^N \setminus \Lambda_*}. \quad (2.21)$$

From (2.16) and (2.21), for some $C_4 > 0$ which is independent of $u = (u_1, \dots, u_k)$ and $\epsilon \in (0, 1)$, we obtain

$$\|w - v\|_{\epsilon, \mathbf{R}^N \setminus \Lambda_*} \leq \epsilon^{\alpha-2} C_4 \|u\|_{\epsilon, \Lambda_* \setminus \Lambda'_*}. \quad \blacksquare$$

Proof of Proposition 2.6. From (2.12), we have

$$\begin{aligned} I_\epsilon(u_1, \dots, u_k) &- \sum_{i=1}^k I_{\epsilon, \Lambda_i}(u_i) \\ &= \frac{1}{2} \left(\|Q_\epsilon(u_1, \dots, u_k)\|_{\epsilon, \mathbf{R}^N \setminus \Lambda_*}^2 - \sum_{i=1}^k \|Q_{0,\epsilon}(0, \dots, u_i, \dots, 0)\|_{\epsilon, \mathbf{R}^N \setminus \Lambda_*}^2 \right) \\ &\quad - \int_{\mathbf{R}^N \setminus \Lambda_*} F_\epsilon(Q_\epsilon(u_1, \dots, u_k)) \, dx \\ &= \frac{1}{2} (I) + (II). \end{aligned}$$

Since $Q_{0,\epsilon}(u_1, \dots, u_k)$ is a solution of the linear equation, we have $Q_{0,\epsilon}(u_1, \dots, u_k) = \sum_{i=1}^k Q_{0,\epsilon}(0, \dots, u_i, \dots, 0)$. Thus,

$$\begin{aligned} (I) &= \|Q_\epsilon(u_1, \dots, u_k)\|_{\epsilon, \mathbf{R}^N \setminus \Lambda_*}^2 - \|Q_{0,\epsilon}(u_1, \dots, u_k)\|_{\epsilon, \mathbf{R}^N \setminus \Lambda_*}^2 \\ &\quad + 2 \sum_{1 \leq i < j \leq k} \langle Q_{0,\epsilon}(0, \dots, u_i, \dots, 0), Q_{0,\epsilon}(0, \dots, u_j, \dots, 0) \rangle_{\epsilon, \mathbf{R}^N \setminus \Lambda_*}. \end{aligned} \quad (2.22)$$

Now we remark that, for $i \neq j$, the following estimate holds:

$$|\langle Q_{0,\epsilon}(0, \dots, u_i, \dots, 0), Q_{0,\epsilon}(0, \dots, u_j, \dots, 0) \rangle_{\epsilon, \mathbf{R}^N \setminus \Lambda_*}| \leq C e^{-\frac{a}{\epsilon}} \|u_i\|_{\epsilon, \Lambda_i} \|u_j\|_{\epsilon, \Lambda_j}, \quad (2.23)$$

where constants $C, a > 0$ are independent of $(u_1, \dots, u_k) \in H^1(\Lambda_1) \oplus \dots \oplus H^1(\Lambda_k)$ and $\epsilon \in (0, 1)$. It is showed in [ST]. The key of the proof is the subsolution estimate in [Si]. (Also see [GT].) From Lemma 2.7, (2.22) and (2.23), it follows that

$$|(I)| \leq 2\epsilon^{\alpha-2} C_4^2 \|u\|_{\epsilon, \Lambda_*}^2 + 2C e^{-\frac{a}{\epsilon}} 2 \sum_{1 \leq i < j \leq k} \|u_i\|_{\epsilon, \Lambda_i} \|u_j\|_{\epsilon, \Lambda_j} \quad (2.24)$$

On the other hand, since $F(\xi) \leq \frac{1}{2}\epsilon^\alpha \nu_0 \xi^2$, we have

$$\begin{aligned} |(II)| &\leq \frac{1}{2}\epsilon^\alpha \nu_0 \|Q_\epsilon(u_1, \dots, u_k)\|_{L^2(\mathbf{R}^N \setminus \Lambda_*)}^2 \\ &\leq \frac{1}{2}\epsilon^{\alpha-2} \nu_0 C_1 \|Q_\epsilon(u_1, \dots, u_k)\|_{\epsilon, \mathbf{R}^N \setminus \Lambda_*}^2 \\ &\leq \frac{1}{2}\epsilon^{\alpha-2} \nu_0 C_1 C_4^2 \|u\|_{\epsilon, \Lambda_*}^2. \end{aligned} \quad (2.25)$$

Here we use Proposition 2.1 and Lemma 2.7. We get (2.14) from (2.24) and (2.25). $\blacksquare$

We use the following notation: for $u_i, v_i \in H^1(\Lambda_i)$ ($i = 1, \dots, k$),

$$\begin{aligned} \langle u_i, v_i \rangle_{\epsilon, \Lambda_i, \#} &= \langle u_i, v_i \rangle_{\epsilon, \Lambda_i} \\ &\quad + \langle Q_{0,\epsilon}(0, \dots, u_i, \dots, 0), Q_{0,\epsilon}(0, \dots, v_i, \dots, 0) \rangle_{\epsilon, \mathbf{R}^N \setminus \Lambda_*} \quad (i = 1, \dots, k), \\ \|u_i\|_{\epsilon, \Lambda_i, \#}^2 &= \langle u_i, u_i \rangle_{\epsilon, \Lambda_i, \#} \quad (i = 1, \dots, k). \end{aligned}$$

By Lemma 2.7, we easily get

$$\|u_i\|_{\epsilon, \Lambda_i} \leq \|u_i\|_{\epsilon, \Lambda_i, \#} \leq (1 + C_4) \|u_i\|_{\epsilon, \Lambda_i} \quad \text{for all } u_i \in H^1(\Lambda_i).$$

Thus $\|\cdot\|_{\epsilon, \Lambda_i, \#}$ is equivalent to $\|\cdot\|_{\epsilon, \Lambda_i}$ for each i . With this notation, $I_{\epsilon, \Lambda_i}(u)$ can be written as

$$I_{\epsilon, \Lambda_i}(u_i) = \frac{1}{2} \|u_i\|_{\epsilon, \Lambda_i, \#}^2 - \frac{1}{p+1} \|u_i\|_{L^{p+1}(\Lambda_i)}^{p+1} : H^1(\Lambda_i) \rightarrow \mathbf{R}. \quad (i = 1, \dots, k)$$

We can easily see that $I_{\epsilon, \Lambda_i}(u_i)$ ($i = 1, \dots, k$) has a mountain pass geometry and satisfies $(PS)_c$ -condition for all $c \in \mathbf{R}$ in a standard way.

(c) Reduction to a problem on $\Sigma_{\epsilon, \Lambda_1} \times \dots \times \Sigma_{\epsilon, \Lambda_k}$

In this section, we reduce our problem to a variational problem on an infinite dimensional torus $\Sigma_{\epsilon, \Lambda_1} \times \dots \times \Sigma_{\epsilon, \Lambda_k}$, where

$$\Sigma_{\epsilon, \Lambda_i} = \{v_i \in H^1(\Lambda_i); \|v_i\|_{\epsilon, \Lambda_i, \#}^2 = d_{\epsilon, \Lambda_i}\} \quad (i = 1, \dots, k). \quad (2.26)$$

Here we define d_{ϵ, Λ_i} by

$$d_{\epsilon, \Lambda_i} = \inf_{v_i \in H^1(\Lambda_i) \setminus \{0\}} \left(\frac{\|v_i\|_{\epsilon, \Lambda_i, \#}}{\|v_i\|_{L^{p+1}(\Lambda_i)}} \right)^{\frac{2(p+1)}{p-1}} \quad (i = 1, \dots, k). \quad (2.27)$$

Then d_{ϵ, Λ_i} satisfies the following decay estimates.

- Lemma 2.8.** (i) If $\inf_{x \in \Lambda_i} V(x) = m_i > 0$, then $\lim_{\epsilon \rightarrow 0} \epsilon^{-N} d_{\epsilon, \Lambda_i} = b_{m_i, \mathbf{R}^N} > 0$.
(ii) If $\inf_{x \in \Lambda_i} V(x) = 0$, then $\lim_{\epsilon \rightarrow 0} \epsilon^{-N} d_{\epsilon, \Lambda_i} = 0$ and $\liminf_{\epsilon \rightarrow 0} \epsilon^{-\frac{2(p+1)}{p-1}} d_{\epsilon, \Lambda_i} \in (0, \infty]$.

Proof. Lemma 2.8 can be shown by a similar way of Proposition 4.2 below. ■

We consider the auxiliary problems constrained on sphere $\Sigma_{\epsilon, \Lambda_i}$:

$$J_{\epsilon, \Lambda_i}(v_i) = \sup_{t > 0} I_{\epsilon, \Lambda_i}(tv_i) = \left(\frac{1}{2} - \frac{1}{p+1} \right) \left(\frac{d_{\epsilon, \Lambda_i}}{\|v_i\|_{L^{p+1}(\Lambda_i)}^2} \right)^{\frac{p+1}{p-1}} : \Sigma_{\epsilon, \Lambda_i} \rightarrow \mathbf{R} \quad (i = 1, \dots, k).$$

For $v_i \in \Sigma_{\epsilon, \Lambda_i}$, we can see that $t \mapsto I_{\epsilon, \Lambda_i}(tv_i) : [0, \infty) \rightarrow \mathbf{R}$ takes a global maximum at

$$t = t_{\epsilon, \Lambda_i}(v_i) = d_{\epsilon, \Lambda_i}^{\frac{1}{p-1}} \|v_i\|_{L^{p+1}(\Lambda_i)}^{-\frac{p+1}{p-1}} \quad (2.28)$$

and

$$J_{\epsilon, \Lambda_i}(v_i) = I_{\epsilon, \Lambda_i}(t_{\epsilon, \Lambda_i}(v_i)v_i) = \left(\frac{1}{2} - \frac{1}{p+1} \right) \left(\frac{d_{\epsilon, \Lambda_i}}{\|v_i\|_{L^{p+1}(\Lambda_i)}^2} \right)^{\frac{p+1}{p-1}}.$$

We choose $r > 0$ such that $1 - p(1+r)^{-(p-1)} < 0$ and set

$$N_{\epsilon, \Lambda_i} = \{v_i \in \Sigma_{\epsilon, \Lambda_i}; \|v_i\|_{L^{p+1}(\Lambda_i)}^{p+1} \geq (1+r)^{-\frac{p-1}{2}} d_{\epsilon, \Lambda_i}\} \quad (i = 1, \dots, k). \quad (2.29)$$

N_{ϵ, Λ_i} is a neighborhood of least energy critical points of $J_{\epsilon, \Lambda_i}(v_i)$. In fact, we can easily get the following lemma.

Lemma 2.9. For any $\epsilon \in (0, 1)$,

$$\begin{aligned} \left(\frac{1}{2} - \frac{1}{p+1} \right) d_{\epsilon, \Lambda_i} &= \inf_{v_i \in N_{\epsilon, \Lambda_i}} J_{\epsilon, \Lambda_i}(v_i) \quad (i = 1, \dots, k), \\ (1+r) \left(\frac{1}{2} - \frac{1}{p+1} \right) d_{\epsilon, \Lambda_i} &\leq \inf_{v_i \in \partial N_{\epsilon, \Lambda_i}} J_{\epsilon, \Lambda_i}(v_i) \quad (i = 1, \dots, k). \end{aligned}$$

Proof. By a direct computation, we can easily see that

$$J_{\epsilon, \Lambda_i}(v_i) \leq (1+r) \left(\frac{1}{2} - \frac{1}{p+1} \right) d_{\epsilon, \Lambda_i} \text{ if and only if } \|v_i\|_{L^{p+1}(\Lambda_i)}^{p+1} \geq (1+r)^{-\frac{p-1}{2}} d_{\epsilon, \Lambda_i}. \quad (2.30)$$

Thus Lemma 2.9 follows. ■

For minimizing sequences of J_{ϵ, Λ_i} , we have the following estimates.

Lemma 2.10. *If a sequence $v_{i,\epsilon} \in \Sigma_{\epsilon,\Lambda_i}$ ($\epsilon \rightarrow 0$) satisfies*

$$J_{\epsilon,\Lambda_i}(v_{i,\epsilon}) = \left(\frac{1}{2} - \frac{1}{p+1}\right) d_{\epsilon,\Lambda_i} + o\left(\epsilon^{\frac{2(p+1)}{p-1}}\right), \quad (2.31)$$

then we have

$$\|v_{i,\epsilon}\|_{L^{p+1}(\Lambda_i)}^{p+1} = d_{\epsilon,\Lambda_i} + o\left(\epsilon^{\frac{2(p+1)}{p-1}}\right). \quad (2.32)$$

Proof. From (2.31) and (2.30), we have

$$\|v_{i,\epsilon}\|_{L^{p+1}(\Lambda_i)}^{p+1} = \left(1 + \frac{o\left(\epsilon^{\frac{2(p+1)}{p-1}}\right)}{d_{\epsilon,\Lambda_i}}\right)^{-\frac{p-1}{2}} d_{\epsilon,\Lambda_i}.$$

Using a Taylor expansion, for small $|r|$, we find $(1+r)^{-\frac{p-1}{2}} = 1 + O(r)$. Thus we get

$$\|v_{i,\epsilon}\|_{L^{p+1}(\Lambda_i)}^{p+1} = \left(1 + \frac{o\left(\epsilon^{\frac{2(p+1)}{p-1}}\right)}{d_{\epsilon,\Lambda_i}}\right) d_{\epsilon,\Lambda_i} = d_{\epsilon,\Lambda_i} + o\left(\epsilon^{\frac{2(p+1)}{p-1}}\right). \quad \blacksquare$$

We set a subset N_ϵ as follows:

$$N_\epsilon = \{(v_1, \dots, v_k) \in \Sigma_{\epsilon,\Lambda_1} \times \dots \times \Sigma_{\epsilon,\Lambda_k}; \|v_i\|_{L^{p+1}(\Lambda_i)}^{p+1} \geq (1+r)^{-\frac{p-1}{2}} d_{\epsilon,\Lambda_i} \ (i = 1, \dots, k)\}. \quad (2.33)$$

From Lemma 2.9, we can see $N_\epsilon \neq \emptyset$. We try to find a critical point of $J_\epsilon(v_1, \dots, v_k) : N_\epsilon \rightarrow (0, \infty]$ which is defined by

$$J_\epsilon(v_1, \dots, v_k) = \sup_{s_1, \dots, s_k \geq 0} I_\epsilon(s_1 v_1, \dots, s_k v_k). \quad (2.34)$$

For simplicity, in what follows, we use notation: $\mathbf{v} = (v_1, \dots, v_k) \in N_\epsilon$, $\mathbf{s} = (s_1, \dots, s_k) \in [0, \infty)^k$, $\mathbf{sv} = (s_1 v_1, \dots, s_k v_k)$.

The following proposition is important.

Proposition 2.11. *There exists $\epsilon_1 \in (0, 1)$ such that for all $\epsilon \in (0, \epsilon_1)$ we have*

- (i) *There exist constants $R_2 > R_1 > 0$ independent of ϵ such that for any $\mathbf{v} \in N_\epsilon$, $\mathbf{s} \mapsto I_\epsilon(\mathbf{sv})$ has a unique maximizer $\mathbf{s}_\epsilon(\mathbf{v}) = (s_{1,\epsilon}(\mathbf{v}), \dots, s_{k,\epsilon}(\mathbf{v}))$ in $[R_1, R_2]^k$.*
- (ii) *$\mathbf{v} \mapsto \mathbf{s}_\epsilon(\mathbf{v}) : N_\epsilon \rightarrow \mathbf{R}^k$ is of class of C^1 .*
- (iii) *$J_\epsilon(\mathbf{v}) : N_\epsilon \rightarrow \mathbf{R}$ is of class of C^1 .*
- (iv) *$J_\epsilon(\mathbf{v}) : N_\epsilon \rightarrow \mathbf{R}$ satisfies (PS)-condition.*
- (v) *If $\mathbf{v} \in N_\epsilon$ is a critical point of $J_\epsilon(\mathbf{v})$, then $\mathbf{s}_\epsilon(\mathbf{v})\mathbf{v} \in H^1(\Lambda_1) \oplus \dots \oplus H^1(\Lambda_k)$ is a critical point of $I_\epsilon(u_1, \dots, u_k)$.*

(vi) For all $\mathbf{v} = (v_1, \dots, v_k) \in N_\epsilon$, we have

$$J_\epsilon(\mathbf{v}) = \sum_{i=1}^k J_{\epsilon, \Lambda_i}(v_i) + o(\epsilon^{\frac{2(p+1)}{p-1}}). \quad (2.35)$$

where $o(\epsilon^{\frac{2(p+1)}{p-1}})$ is uniformly for $\mathbf{v} \in N_\epsilon$.

Proof. See Proposition 1.13 and Proposition 1.14 of [S]. ■

3. Outline of the proof of Theorem 1.1.

In this section, we will show Theorem 1.1. We define

$$c_\epsilon = \inf_{\mathbf{v} \in N_\epsilon} J_\epsilon(\mathbf{v}). \quad (3.1)$$

Then we show the following proposition.

Proposition 3.1. For small $\epsilon \in (0, \epsilon_1)$, we have

- (i) $c_\epsilon = \left(\frac{1}{2} - \frac{1}{p+1}\right) \sum_{i=1}^k d_{\epsilon, \Lambda_i} + o(\epsilon^{\frac{2(p+1)}{p-1}})$.
- (ii) $\inf_{\mathbf{v} \in N_\epsilon} J_\epsilon(\mathbf{v}) < \inf_{\mathbf{v} \in \partial N_\epsilon} J_\epsilon(\mathbf{v})$.
- (iii) c_ϵ is a critical value of $J_\epsilon(\mathbf{v})$. Moreover corresponding critical points lie in N_ϵ .

Proof. From (2.35), we recall

$$J_\epsilon(\mathbf{v}) = \sum_{i=1}^k J_{\epsilon, \Lambda_i}(v_i) + o(\epsilon^{\frac{2(p+1)}{p-1}}) \quad \text{for all } \mathbf{v} \in N_\epsilon,$$

where $o(\epsilon^{\frac{2(p+1)}{p-1}})$ is uniformly for $\mathbf{v} \in N_\epsilon$. Thus from Lemma 2.9, it follows

$$\begin{aligned} c_\epsilon &= \inf_{\mathbf{v} \in N_\epsilon} J_\epsilon(\mathbf{v}) = \sum_{i=1}^k \inf_{v_i \in N_{\epsilon, \Lambda_i}} J_{\epsilon, \Lambda_i}(v_i) + o(\epsilon^{\frac{2(p+1)}{p-1}}) \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right) \sum_{i=1}^k d_{\epsilon, \Lambda_i} + o(\epsilon^{\frac{2(p+1)}{p-1}}). \end{aligned} \quad (3.2)$$

Next we show (ii). We note $\partial N_\epsilon = \bigcup_{i=1}^k (N_{\epsilon, \Lambda_1} \times \dots \times \partial N_{\epsilon, \Lambda_i} \times \dots \times N_{\epsilon, \Lambda_k})$. Again, from Lemma 2.9, we get for each j

$$\begin{aligned} \inf_{\mathbf{v} \in N_{\epsilon, \Lambda_1} \times \dots \times \partial N_{\epsilon, \Lambda_j} \times \dots \times N_{\epsilon, \Lambda_k}} J_\epsilon(\mathbf{v}) &= \sum_{i \neq j} \inf_{v_i \in N_{\epsilon, \Lambda_i}} J_{\epsilon, \Lambda_i}(v_i) + \inf_{v_j \in \partial N_{\epsilon, \Lambda_j}} J_{\epsilon, \Lambda_j}(v_j) + o(\epsilon^{\frac{2(p+1)}{p-1}}) \\ &\geq \left(\frac{1}{2} - \frac{1}{p+1}\right) \sum_{i \neq j} d_{\epsilon, \Lambda_i} + (1+r) \left(\frac{1}{2} - \frac{1}{p+1}\right) d_{\epsilon, \Lambda_j} + o(\epsilon^{\frac{2(p+1)}{p-1}}) \\ &= c_\epsilon + r \left(\frac{1}{2} - \frac{1}{p+1}\right) d_{\epsilon, \Lambda_j} + o(\epsilon^{\frac{2(p+1)}{p-1}}). \end{aligned} \quad (3.3)$$

(ii) follows from (3.2)–(3.3). Since $J_\epsilon(\mathbf{v})$ satisfies (PS)-condition, we can see c_ϵ is a critical value of $J_\epsilon(\mathbf{v})$ in a standard way. ■

Corollary 3.2. *Minimizer $\mathbf{v}_\epsilon = (v_{1,\epsilon}, \dots, v_{k,\epsilon})$ of (3.1) satisfies*

$$J_{\epsilon, \Lambda_i}(v_{i,\epsilon}) = \left(\frac{1}{2} - \frac{1}{p+1}\right) d_{\epsilon, \Lambda_i} + o\left(\epsilon^{\frac{2(p+1)}{p-1}}\right) \quad (i = 1, \dots, k).$$

Proof. From (2.35), we have

$$\begin{aligned} \left(\frac{1}{2} - \frac{1}{p+1}\right) d_{\epsilon, \Lambda_i} &\leq J_{\epsilon, \Lambda_i}(v_{i,\epsilon}) = J_\epsilon(\mathbf{v}_\epsilon) - \sum_{\ell \neq i} J_{\epsilon, \Lambda_\ell}(v_{\ell,\epsilon}) + o\left(\epsilon^{\frac{2(p+1)}{p-1}}\right) \\ &\leq c_\epsilon - \left(\frac{1}{2} - \frac{1}{p+1}\right) \sum_{\ell \neq i} d_{\epsilon, \Lambda_\ell} + o\left(\epsilon^{\frac{2(p+1)}{p-1}}\right) \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right) d_{\epsilon, \Lambda_i} + o\left(\epsilon^{\frac{2(p+1)}{p-1}}\right). \end{aligned}$$

Thus we get Corollary 3.2. ■

From the a minimizer of (3.1), we get a critical point of $\Phi_\epsilon(u)$ by the following:

Proposition 3.3. *Let $\mathbf{v}_\epsilon = (v_{1,\epsilon}, \dots, v_{k,\epsilon})$ be a minimizer of (3.1). Then*

$$u_\epsilon(x) = \begin{cases} s_{i,\epsilon}(\mathbf{v}_\epsilon) v_{i,\epsilon}(x) & \text{for } x \in \Lambda_i \quad (i = 1, \dots, k), \\ Q_\epsilon(s_{1,\epsilon}(\mathbf{v}_\epsilon) v_{1,\epsilon}, \dots, s_{k,\epsilon}(\mathbf{v}_\epsilon) v_{k,\epsilon})(x) & \text{for } x \in \mathbf{R}^N \setminus \Lambda_*. \end{cases} \quad (3.4)$$

is a critical point of $\Phi_\epsilon(u)$. Here $(s_{1,\epsilon}(\mathbf{v}), \dots, s_{k,\epsilon}(\mathbf{v}))$ are given in Proposition 2.11.

Proof. It follows from Proposition 2.5, Proposition 2.11 and Proposition 3.1. ■

Here we outline the proof of Theorem 1.1

Outline of the proof of Theorem 1.1. We can show that $u_\epsilon(x)$ defined by (3.4) has an exponential decay on $\mathbf{R}^N \setminus \Lambda'_*$ as $\epsilon \rightarrow 0$. (See [S].) Thus $u_\epsilon(x)$ satisfies (2.7) and $u_\epsilon(x)$ is a critical point of original functional $\Psi_\epsilon(u)$. Moreover, we have the following detailed estimate of $u_\epsilon(x)$ (See [S].):

$$\|u_\epsilon\|_{L^{p+1}(\Lambda_i)}^{p+1} = c_{\epsilon, \Lambda_i} + o\left(\epsilon^{\frac{2(p+1)}{p-1}}\right), \quad (3.5)$$

$$\|u_\epsilon\|_{\epsilon, \Lambda_i}^2 = c_{\epsilon, \Lambda_i} + o\left(\epsilon^{\frac{2(p+1)}{p-1}}\right). \quad (3.6)$$

Thus (1.4)–(1.5) follow from (3.5)–(3.6). We get (1.6) by a standard way. The proof of Theorem 1.1 is completed. ■

4. Asymptotic profile of solutions $u_\epsilon(x)$.

In this section, we will prove Theorem 1.5. By a similar way, we can prove Theorem 1.3. (See Section 3 of [S].) First of all, we note the following.

Remark 4.1. When assumption (L^*) holds, there exists a constant $C_5 > 0$ independent of $\epsilon \in (0, 1)$ such that

$$\|w\|_{H^1(\Lambda_{g(\epsilon)})} \leq C_5 \|w\|_{V_\epsilon, \Lambda_{g(\epsilon)}} \quad \text{for all } w \in H^1(\Lambda_{g(\epsilon)}). \quad (4.1)$$

In fact we can see that, for some $\ell > 0$ and $\delta > 0$ independent of $\epsilon \in (0, 1)$, $V_\epsilon(y)$ satisfies

$$V_\epsilon(y) \geq \begin{cases} 0 & \text{for } y \in (-\ell, \ell)^k \times \mathbf{R}^{N-k} \subset \mathbf{R}^N, \\ \delta & \text{elsewhere.} \end{cases}$$

Thus (4.1) can be shown as Proposition 2.1. (See Lemma 1.2 of [S].)

In the following arguments, $d_{\epsilon, \Lambda}$ defined in (2.27) and b_{V_0, Ω_0} defined in (1.9) will play important roles.

Proposition 4.2. b_{V_0, Ω_0} is achieved by some $w(x) \in H_0^1(\Omega_0)$ and

$$\left(\frac{g(\epsilon)}{\epsilon}\right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N} d_{\epsilon, \Lambda} \rightarrow b_{V_0, \Omega_0} \quad \text{as } \epsilon \rightarrow 0. \quad (4.2)$$

Proof. From Lemma 2.11 of [S], it suffice to show

$$\left(\frac{g(\epsilon)}{\epsilon}\right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N} c_{\epsilon, \Lambda} \rightarrow b_{V_0, \Omega_0} \quad \text{as } \epsilon \rightarrow 0. \quad (4.3)$$

Firstly we show

$$\left(\frac{g(\epsilon)}{\epsilon}\right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N} c_{\epsilon, \Lambda} \leq b_{V_0, \Omega_0} + o(1). \quad (4.4)$$

Suppose that $w(x) \in H_0^1(\Omega)$ achieves $b_{V_0, \Omega}$. We choose a function $\varphi \in C_0^1(\mathbf{R}^N, [0, 1])$ such that

$$\varphi(x) = \begin{cases} 1 & \text{for } x \in \Lambda' \\ 0 & \text{for } x \in \mathbf{R}^N \setminus \Lambda \end{cases} \quad |\nabla \varphi(x)| \leq C \quad \text{for } x \in \mathbf{R}^N,$$

and set

$$v_\epsilon(x) = \varphi(x) \left(\frac{\epsilon}{g(\epsilon)}\right)^{\frac{2}{p-1}} w\left(\frac{x}{g(\epsilon)}\right), \quad \psi_\epsilon(x) = \varphi(g(\epsilon)x).$$

Then by direct computations, we have

$$\begin{aligned} \|v_\epsilon\|_{\epsilon, \Lambda}^2 &= \left(\frac{\epsilon}{g(\epsilon)}\right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^N \left[\int_{\Lambda_{g(\epsilon)}} \psi_\epsilon(x)^2 \{|\nabla w(x)|^2 + V_\epsilon(x)w(x)^2\} dx \right. \\ &\quad \left. + \int_{\Lambda_{g(\epsilon)} \setminus \Lambda'_{g(\epsilon)}} |\nabla \psi_\epsilon(x)|^2 w(x)^2 + 2\nabla \psi_\epsilon(x) \cdot \nabla w(x) \psi_\epsilon(x) w(x) dx \right] \\ &= \left(\frac{\epsilon}{g(\epsilon)}\right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^N ((I) + (II)). \end{aligned}$$

Here we have

$$|(II)| \leq C^2 \|w\|_{L^2(\Lambda_{g(\epsilon)} \setminus \Lambda'_{g(\epsilon)})}^2 + 2C \|\nabla w\|_{L^2(\Lambda_{g(\epsilon)} \setminus \Lambda'_{g(\epsilon)})} \|w\|_{L^2(\Lambda_{g(\epsilon)} \setminus \Lambda'_{g(\epsilon)})} \rightarrow 0$$

as $\epsilon \rightarrow 0$. Thus, from (1.13), we find

$$\|v_\epsilon\|_{\epsilon, \Lambda}^2 = \left(\frac{\epsilon}{g(\epsilon)}\right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^N (\|w\|_{V_0, \Omega_0}^2 + o(1)). \quad (4.5)$$

By a similar way, we get

$$\|v_{\epsilon+}\|_{L^{p+1}(\Lambda)}^{p+1} = \left(\frac{\epsilon}{g(\epsilon)}\right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^N (\|w_+\|_{L^{p+1}(\Omega_0)}^{p+1} + o(1)). \quad (4.6)$$

Thus from definition of $c_{\epsilon, \Lambda}$ and (4.5)–(4.6),

$$\begin{aligned} \left(\frac{g(\epsilon)}{\epsilon}\right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N} c_{\epsilon, \Lambda} &\leq \left(\frac{g(\epsilon)}{\epsilon}\right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N} \left(\frac{1}{2} - \frac{1}{p+1}\right) \left(\frac{\|v_\epsilon\|_{\epsilon, \Lambda}}{\|v_{\epsilon+}\|_{L^{p+1}(\Lambda)}}\right)^{\frac{2(p+1)}{p-1}} \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right) \left(\frac{\|w\|_{V_0, \Omega_0} + o(1)}{\|w_+\|_{L^{p+1}(\Omega_0)} + o(1)}\right)^{\frac{2(p+1)}{p-1}} \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right) \left(\frac{\|w\|_{V_0, \Omega_0}}{\|w_+\|_{L^{p+1}(\Omega_0)}}\right)^{\frac{2(p+1)}{p-1}} + o(1) \\ &= b_{V_0, \Omega_0} + o(1). \end{aligned} \quad (4.7)$$

Next, we show

$$b_{V_0, \Omega_0} \leq \left(\frac{g(\epsilon)}{\epsilon}\right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N} c_{\epsilon, \Lambda} + o(1). \quad (4.8)$$

Let $v_\epsilon(x) \in H_0^1(\Lambda)$ attains $c_{\epsilon, \Lambda}$. We may assume $\|v_\epsilon\|_{\epsilon, \mathbf{R}^N}^2 = \left(\frac{\epsilon}{g(\epsilon)}\right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^N$. Then, from (4.4), we see that

$$\lim_{\epsilon \rightarrow 0} \left(\frac{g(\epsilon)}{\epsilon}\right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N} \|v_{\epsilon+}\|_{L^{p+1}(\Lambda)} > 0. \quad (4.9)$$

We set

$$w_\epsilon(x) = \left(\frac{g(\epsilon)}{\epsilon}\right)^{\frac{2}{p-1}} v_\epsilon(g(\epsilon)x).$$

Then $\|w_\epsilon\|_{V_\epsilon, \mathbf{R}^N} = 1$ and $\lim_{\epsilon \rightarrow 0} \|w_\epsilon\|_{L^{p+1}(\mathbf{R}^N)} > 0$. We need to divide in two cases: $\Omega_0 = \mathbf{R}^N$ or $\Omega_0 \neq \mathbf{R}^N$. Firstly, we consider the case $\Omega = \mathbf{R}^N$. By direct computations, we have

$$\|w_\epsilon\|_{V_0, \mathbf{R}^N}^2 = \left(\frac{g(\epsilon)}{\epsilon}\right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N} \int_{\Lambda} \epsilon^2 |\nabla v_\epsilon|^2 + \left(\frac{\epsilon}{g(\epsilon)}\right)^2 V_0 \left(\frac{x}{g(\epsilon)}\right) v_\epsilon^2(x) dx.$$

Thus, from (1.15), for any $\delta \in (0, 1)$, we obtain

$$\|w_\epsilon\|_{(1-\delta)V_0, \mathbf{R}^N}^2 \leq \left(\frac{g(\epsilon)}{\epsilon}\right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N} \|v_\epsilon\|_{\epsilon, \mathbf{R}^N}^2 + o(1). \quad (4.10)$$

On the other hand, we have

$$\|w_\epsilon\|_{L^{p+1}(\mathbf{R}^N)}^{p+1} = \left(\frac{g(\epsilon)}{\epsilon}\right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N} \|v_\epsilon\|_{L^{p+1}(\Lambda)}^{p+1}. \quad (4.11)$$

From (4.10)–(4.11) we get

$$b_{(1-\delta)V_0, \mathbf{R}^N} \leq \left(\frac{g(\epsilon)}{\epsilon}\right)^{\frac{2(p+1)}{p-1}} g(\epsilon)^{-N} c_{\epsilon, \Lambda} + o(1).$$

(4.8) follows from the fact $b_{(1-\delta)V_0, \mathbf{R}^N} \rightarrow b_{V_0, \mathbf{R}^N}$ as $\delta \rightarrow 0$. Next, we consider the case $\Omega_0 \neq \mathbf{R}^N$. Since $\|w_\epsilon\|_{V_\epsilon, \mathbf{R}^N} = 1$, we can choose a subsequence $\epsilon_n \rightarrow 0$ and $w_0(x) \in H^1(\mathbf{R}^N)$ such that

$$w_{\epsilon_n}(x) \rightarrow w_0(x) \quad \text{weakly in } H^1(\mathbf{R}^N) \text{ and strongly in } L_{loc}^{p+1}(\mathbf{R}^N) \quad (4.12).$$

From (1.18) and (4.12), we see $w_{\epsilon_n}(x) \rightarrow w_0(x)$ strongly in $L^{p+1}(\mathbf{R}^N)$. Moreover, from (1.14), for any closed set $E \supset \supset \Omega_0$ we find

$$\|w_{\epsilon_n}\|_{L^2(\Lambda_{g(\epsilon_n)} \setminus E)}^2 \leq \frac{1}{\inf_{x \in \Lambda_{g(\epsilon_n)} \setminus E} V_\epsilon(x)} \int_{\Lambda_{g(\epsilon_n)} \setminus E} V_\epsilon(x) w_{\epsilon_n}(x)^2 dx \rightarrow 0.$$

In particular, we find $w_0 \in H_0^1(\Omega_0)$ and $w_0 \neq 0$. Thus we have

$$\begin{aligned} b_{V_0, \Omega_0} &\leq \left(\frac{1}{2} - \frac{1}{p+1}\right) \left(\frac{\|w_0\|_{V_0, \Omega_0}}{\|w_0\|_{L^{p+1}(\Omega_0)}}\right)^{\frac{2(p+1)}{p-1}} \\ &\leq \liminf_{\epsilon_n \rightarrow 0} \left(\frac{1}{2} - \frac{1}{p+1}\right) \left(\frac{\|w_{\epsilon_n}\|_{\epsilon_n, \Lambda_{g(\epsilon_n)}}}{\|w_{\epsilon_n}\|_{L^{p+1}(\Lambda_{g(\epsilon_n)})}}\right)^{\frac{2(p+1)}{p-1}} \\ &= \liminf_{\epsilon_n \rightarrow 0} \left(\frac{g(\epsilon_n)}{\epsilon_n}\right)^{\frac{2(p+1)}{p-1}} g(\epsilon_n)^{-N} \left(\frac{1}{2} - \frac{1}{p+1}\right) \left(\frac{\|v_{\epsilon_n}\|_{\epsilon_n, \Lambda}}{\|v_{\epsilon_n}\|_{L^{p+1}(\Lambda)}}\right)^{\frac{2(p+1)}{p-1}} \\ &= \liminf_{\epsilon_n \rightarrow 0} \left(\frac{g(\epsilon_n)}{\epsilon_n}\right)^{\frac{2(p+1)}{p-1}} g(\epsilon_n)^{-N} c_{\epsilon_n, \Lambda}. \end{aligned} \quad (4.13)$$

Since (4.13) is not depend on subsequence, (4.8) holds. From (4.4) and (4.8), we get (4.4) and complete the proof of Proposition 4.2. $\blacksquare$

To prove Theorem 1.5, we define a rescaled functional

$$\tilde{\Phi}_{\Lambda_{g(\epsilon)}}(w) = \frac{1}{2} \|w\|_{V_{\epsilon}, \Lambda_{g(\epsilon)}}^2 - \frac{1}{p+1} \|w_+\|_{L^{p+1}(\Lambda_{g(\epsilon)})}^{p+1} : H^1(\Lambda_{g(\epsilon)}) \rightarrow \mathbf{R}.$$

We also define a functional corresponding to the limit problem $(L)_{V_0, \Omega_0}$ by

$$I_{V_0, \Omega_0}(w) = \frac{1}{2} \|w\|_{V_0, \Omega_0}^2 - \frac{1}{p+1} \|w_+\|_{L^{p+1}(\Omega_0)}^{p+1} : E_{V_0, \Omega_0} \cap H_0^1(\Omega_0) \rightarrow \mathbf{R}.$$

Now we have the following proposition.

Proposition 4.3. Suppose $(w_\epsilon(y)) \subset H^1(\Lambda_{g(\epsilon)})$ satisfies

$$\|w_\epsilon\|_{L^{p+1}(\Lambda_{g(\epsilon)})}^{p+1} = b_{V_0, \Omega_0} + o(1), \quad (4.14)$$

$$\|w_\epsilon\|_{V_{\epsilon}, \Lambda_{g(\epsilon)}}^2 \leq b_{V_0, \Omega_0} + o(1), \quad (4.15)$$

$$\tilde{\Phi}'_{\Lambda_{g(\epsilon)}}(w_\epsilon)\varphi = 0 \quad \text{for all } \varphi \in H_0^1(\Lambda_{g(\epsilon)}). \quad (4.16)$$

If (L^*) holds, then after extracting a subsequence $\epsilon_n \rightarrow 0$, there exist $x_n \in \Lambda_{\epsilon_n}$ and a least energy solution $w_0 \in H_0^1(\Omega_0)$ of $(L)_{V_0, \Omega_0}$ such that

$$\epsilon_n x_n \rightarrow 0,$$

$$\|w_{\epsilon_n} - w_0(\cdot - x_n)\|_{H^1(\Lambda_{\epsilon_n})} \rightarrow 0 \quad \text{as } \epsilon_n \rightarrow 0,$$

$$I_{V_0, \Omega_0}(w_0) = \left(\frac{1}{2} - \frac{1}{p+1}\right) b_{V_0, \Omega_0}, \quad I'_{V_0, \Omega_0}(w_0) = 0.$$

To prove Proposition 4.3, we will use the following lemma.

Lemma 4.4. Suppose $w \in H_0^1(\Omega_0) \setminus \{0\}$ satisfies $I'_{V_0, \Omega_0}(w)w \leq 0$. Then

$$\|w\|_{V_0, \Omega_0}^2 \geq b_{V_0, \Omega_0}.$$

Proof. $I'_{V_0, \Omega_0}(w)w \leq 0$ implies $\|w\|_{V_0, \Omega_0}^2 \leq \|w_+\|_{L^{p+1}(\Omega_0)}^{p+1}$. Thus, from the definition of b_{V_0, Ω_0} , we have

$$b_{V_0, \Omega_0} \leq \left(\frac{\|w\|_{V_0, \Omega_0}}{\|w_+\|_{L^{p+1}(\Omega_0)}} \right)^{\frac{2(p+1)}{p-1}} \leq \|w\|_{V_0, \Omega_0}^2. \quad \blacksquare$$

Proof of Proposition 4.3. For the case $\Omega_0 = \mathbf{R}^N$, we show the proposition. For the case $\Omega_0 \neq \mathbf{R}^N$, we can show it by a similar way. We use concentration compactness argument. From (4.14)–(4.16) and (1.15)–(1.17), we can easily see that there exists $x_\epsilon = (0, \dots, 0, x_{\epsilon, k}, \dots, x_{\epsilon, N}) \in \mathbf{R}^N$ such that for large $R > 0$, $\liminf_{\epsilon \rightarrow 0} \|w_\epsilon\|_{L^{p+1}(B_R(x_\epsilon))} > 0$.

From (4.15)–(4.16), there exist a subsequence $\epsilon_n \rightarrow 0$ and $w_0 \in H^1(\mathbf{R}^N)$ such that for any bounded set $D \subset \mathbf{R}^N$,

$$v_{\epsilon_n}(x) = w_{\epsilon_n}(x + x_{\epsilon_n}) \rightarrow w_0(x) \quad \text{weakly in } H^1(D) \text{ and strongly in } L^{p+1}(D). \quad (4.17)$$

For any $\varphi \in C_0^1(\Omega_0)$ we have

$$\int_{\Lambda_{g(\epsilon_n)} + x_{\epsilon_n}} \nabla v_{\epsilon_n} \cdot \nabla \varphi + V_{\epsilon_n}(x + x_{\epsilon_n}) v_{\epsilon_n} \varphi \, dx - \int_{\Lambda_{g(\epsilon_n)} + x_{\epsilon_n}} v_{\epsilon_n}^p \varphi \, dx = 0.$$

Since $\text{supp } \varphi$ is compact, from (1.13), (1.15)–(1.17), $w_0(x)$ satisfies

$$\int_{\mathbf{R}^N} \nabla w_0 \cdot \nabla \varphi + V_0(x) w_0 \varphi \, dx - \int_{\mathbf{R}^N} w_0^p \varphi \, dx \leq 0,$$

that is, $I'_{V_0, \Omega_0}(w_0) w_0 \leq 0$. From Lemma 4.4 and (4.15), it follows that

$$b_{V_0, \mathbf{R}^N} \leq \|w_0\|_{V_0, \mathbf{R}^N}^2 \leq \liminf_{\epsilon_n \rightarrow 0} \|v_{\epsilon_n}\|_{V_{\epsilon_n}, \Lambda_{g(\epsilon_n)} + x_{\epsilon_n}}^2 \leq b_{V_0, \mathbf{R}^N}. \quad (4.18)$$

By (4.17) and (4.18), we can see

$$\lim_{\epsilon_n \rightarrow 0} \|v_{\epsilon_n} - w_0\|_{V_{\epsilon_n}, \Lambda_{g(\epsilon_n)} + x_{\epsilon_n}}^2 = 0.$$

From (4.1), we also obtain $\lim_{\epsilon_n \rightarrow 0} \|v_{\epsilon_n} - w_0\|_{H^1(\Lambda_{g(\epsilon_n)} + x_{\epsilon_n})} = 0$. Therefore we have $I'_{V_0, \Omega_0}(w_0) = 0$ and

$$\tilde{\Phi}_{\Lambda_{g(\epsilon_n)}}(w_{\epsilon_n}) \rightarrow I_{V_0, \Omega_0}(w_0) = \left(\frac{1}{2} - \frac{1}{p+1}\right) b_{V_0, \Omega_0}.$$

We complete the proof of Proposition 4.3. ■

End of proof of Theorem 1.5. (i) of Theorem 1.5 follows from (4.3). Let $u_\epsilon(x)$ be a critical point of $\Psi_\epsilon(u)$ obtained in Theorem 1.1. We set $w_\epsilon(x) = \left(\frac{g(\epsilon)}{\epsilon}\right)^{\frac{2}{p-1}} u_\epsilon(g(\epsilon)x)$. Then from Proposition 4.2 and (3.5)–(3.6), $w_\epsilon|_{\Lambda_{g(\epsilon)}}(x)$ satisfies (4.14)–(4.16). This implies (ii) of Theorem 1.5. Thus we complete the proof of Theorem 1.5. ■

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