

Stationary isothermic surfaces and some characterizations of the hyperplane *

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1 Introduction

This is based on the author's recent work with R. Magnanini [MS2, MS3]. Let Ω be a domain in $\mathbb{R}^N$ with $N \geq 3$, and let $u = u(x, t)$ be the unique bounded solution of the following problem for the heat equation:

$$\partial_t u = \Delta u \quad \text{in } \Omega \times (0, +\infty), \quad (1.1)$$

$$u = 1 \quad \text{on } \partial\Omega \times (0, +\infty), \quad (1.2)$$

$$u = 0 \quad \text{on } \Omega \times \{0\}. \quad (1.3)$$

The problem we consider is to characterize the boundary $\partial\Omega$ such that the solution u has a stationary isothermic surface, say Γ . A hypersurface Γ in Ω is said to be a *stationary isothermic surface* of u if at each time t the solution u remains constant on Γ (a constant depending on t). Examples we easily notice are isoparametric hypersurfaces. Namely, Γ and $\partial\Omega$ are either parallel hyperplanes, concentric spheres, or concentric spherical cylinders. This complete classification of isoparametric hypersurfaces was given by Levi-Civita [LC] and Segre [Seg].

Almost complete characterizations of the sphere have already been obtained by [MS1, MS2] with the help of Aleksandrov's sphere theorem [Alek]. In this note,

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we consider some characterizations of the hyperplane. Assume that Ω satisfies the uniform exterior sphere condition and Ω is given by

$$\Omega = \{ x = (x', x_N) \in \mathbb{R}^N : x_N > \varphi(x') \}, \quad (1.4)$$

where $\varphi = \varphi(x')$ ($x' \in \mathbb{R}^{N-1}$) is a continuous function on $\mathbb{R}^{N-1}$. We recall that Ω satisfies the *uniform exterior sphere condition* if there exists a number $r_0 > 0$ such that for every $\xi \in \partial\Omega$ there exists a ball $B_{r_0}(y)$ satisfying $\overline{B_{r_0}(y)} \cap \overline{\Omega} = \{\xi\}$, where $B_{r_0}(y)$ denotes an open ball centered at $y \in \mathbb{R}^N$ and with radius $r_0 > 0$. Then we have

Theorem 1.1 ([MS3]) *Assume that there exists a stationary isothermic surface $\Gamma \subset \Omega$. Then, under one of the following conditions (i), (ii), and (iii), $\partial\Omega$ must be a hyperplane.*

- (i) $N = 3$.
- (ii) $N \geq 4$ and φ is globally Lipschitz continuous on $\mathbb{R}^{N-1}$.
- (iii) $N \geq 4$ and there exists a non-empty open subset A of $\partial\Omega$ such that on A either $H_{\partial\Omega} \geq 0$ or $\kappa_j \leq 0$ for all $j = 1, \dots, N-1$, where $H_{\partial\Omega}$ and $\kappa_1, \dots, \kappa_{N-1}$ are the mean curvature of $\partial\Omega$ and the principal curvatures of $\partial\Omega$, respectively, with respect to the upward normal vector to $\partial\Omega$.

Remark. When $N = 2$, this problem is easy. Since the curvature of the curve $\partial\Omega$ is constant from (2.3) in Lemma 2.1 in Section 2 of this note, we see that $\partial\Omega$ must be a straight line.

2 Outline of the proof of Theorem 1.1

In this section we give an outline of the proof. For the details, see [MS2, MS3]. Let $d = d(x)$ be the distance function defined by

$$d(x) = \text{dist}(x, \partial\Omega), \quad x \in \Omega. \quad (2.1)$$

We start with a lemma.

Lemma 2.1 *The following assertions hold:*

- (1) $\Gamma = \{ (x', \psi(x')) \in \mathbb{R}^N : x' \in \mathbb{R}^{N-1} \}$ for some real analytic function $\psi = \psi(x')$ ($x' \in \mathbb{R}^{N-1}$);
- (2) There exists a number $R > 0$ such that $d(x) = R$ for every $x \in \Gamma$;
- (3) φ is real analytic and the mapping: $\partial\Omega \ni \xi \mapsto x(\xi) \equiv \xi + R\nu(\xi) \in \Gamma$ is a diffeomorphism, where $\nu(\xi)$ denotes the upward unit normal vector to $\partial\Omega$ at $\xi \in \partial\Omega$, that is, $\partial\Omega$ and Γ are parallel hypersurfaces with distance R ;
- (4) the following inequality holds:

$$-\frac{1}{r_0} \leq \kappa_j(\xi) < \frac{1}{R} \quad (j = 1, \dots, N-1) \quad \text{for every } \xi \in \partial\Omega, \quad (2.2)$$

where $r_0 > 0$ is the radius of the uniform exterior sphere condition for Ω ;

- (5) there exists a number $c > 0$ satisfying

$$\prod_{j=1}^{N-1} \left(\frac{1}{R} - \kappa_j(\xi) \right) = c \quad \text{for every } \xi \in \partial\Omega. \quad (2.3)$$

Proof. The strong maximum principle implies that $\frac{\partial u}{\partial x_N} < 0$, and (1) holds. Since Γ is stationary isothermic, (2) follows from a result of Varadhan [Va]:

$$-\frac{1}{\sqrt{s}} \log W(x, s) \rightarrow d(x) \quad \text{as } s \rightarrow \infty,$$

where $W(x, s) = s \int_0^\infty u(x, t) e^{-st} dt$ for $s > 0$. The inequality $-\frac{1}{r_0} \leq \kappa_j(\xi)$ in (2.2) follows from the uniform exterior sphere condition for Ω . See Lemma 2.2 of [MS2] together with Lemma 3.1 of [MS1] for the remainder. $\square$

Let us proceed to the proof of Theorem 1.1. Set

$$\Gamma^* = \left\{ x \in \Omega : d(x) = \frac{R}{2} \right\}. \quad (2.4)$$

Denote by κ_j^* and $\hat{\kappa}_j$ ($j = 1, \dots, N-1$) the principal curvatures of Γ^* and Γ , respectively, with respect to the upward unit normal vectors. Then, the mean curvatures H_{Γ^*} and H_Γ of Γ^* and Γ are given by

$$H_{\Gamma^*} = \frac{1}{N-1} \sum_{j=1}^{N-1} \kappa_j^* \quad \text{and} \quad H_\Gamma = \frac{1}{N-1} \sum_{j=1}^{N-1} \hat{\kappa}_j,$$

respectively. These principal curvatures have the following relationship:

$$\kappa_j = \frac{\kappa_j^*}{1 + \frac{R}{2}\kappa_j^*} = \frac{\hat{\kappa}_j}{1 + R\hat{\kappa}_j} \quad (j = 1, \dots, N-1). \quad (2.5)$$

Let $\mu = cR^{N-1}$. Then, it follows from (2.3) and (2.5) that

$$\prod_{j=1}^{N-1} (1 - R\kappa_j) = \mu, \quad \prod_{j=1}^{N-1} (1 + R\hat{\kappa}_j) = \frac{1}{\mu}, \quad \text{and} \quad \prod_{j=1}^{N-1} \frac{1 - \frac{R}{2}\kappa_j^*}{1 + \frac{R}{2}\kappa_j^*} = \mu. \quad (2.6)$$

We distinguish three cases:

$$(I) \mu > 1, \quad (II) \mu < 1, \quad \text{and} \quad (III) \mu = 1.$$

Let us consider case (I) first. By the arithmetic-geometric mean inequality and the first equation of (2.6) we have

$$1 - RH_{\partial\Omega} = \frac{1}{N-1} \sum_{j=1}^{N-1} (1 - R\kappa_j) \geq \left\{ \prod_{j=1}^{N-1} (1 - R\kappa_j) \right\}^{\frac{1}{N-1}} = \mu^{\frac{1}{N-1}} > 1.$$

This shows that

$$H_{\partial\Omega} \leq -\frac{1}{R} \left(\mu^{\frac{1}{N-1}} - 1 \right) < 0. \quad (2.7)$$

Since

$$(N-1)H_{\partial\Omega} = \operatorname{div} \left(\frac{\nabla\varphi}{\sqrt{1 + |\nabla\varphi|^2}} \right) \quad \text{in } \mathbb{R}^{N-1},$$

by using the divergence theorem we get a contradiction as in the proof of Theorem 3.3 in [MS2]. In case (II), by the arithmetic-geometric mean inequality and the second equation of (2.6) we have

$$1 + RH_{\Gamma} = \frac{1}{N-1} \sum_{j=1}^{N-1} (1 + R\hat{\kappa}_j) \geq \left\{ \prod_{j=1}^{N-1} (1 + R\hat{\kappa}_j) \right\}^{\frac{1}{N-1}} = \mu^{-\frac{1}{N-1}} > 1.$$

This shows that

$$H_{\Gamma} \geq \frac{1}{R} \left(\mu^{-\frac{1}{N-1}} - 1 \right) > 0, \quad (2.8)$$

which yields a contradiction similarly.

Thus, it remains to consider case (III). By the above arguments we have

$$H_{\partial\Omega} \leq 0 \leq H_{\Gamma}. \quad (2.9)$$

Let us consider case (i) of Theorem 1.1 first. Since $N = 3$ and $\mu = 1$, it follows from the third equation of (2.6) that

$$2H_{\Gamma^*} = \kappa_1^* + \kappa_2^* = 0.$$

We observe that Γ^* is a graph of a function on $\mathbb{R}^2$. Therefore, by the Bernstein's theorem for the minimal surface equation, Γ^* must be a hyperplane. This gives the conclusion desired. (See [GT, Giu] for the Bernstein's theorem.)

Secondly, we consider case (iii) of Theorem 1.1. We have

$$1 - RH_{\partial\Omega} = \frac{1}{N-1} \sum_{j=1}^{N-1} (1 - R\kappa_j) \geq \left\{ \prod_{j=1}^{N-1} (1 - R\kappa_j) \right\}^{\frac{1}{N-1}} = 1.$$

Hence, condition (iii) implies that

$$\kappa_j \equiv 0 \text{ on } A \text{ (} j = 1, \dots, N-1 \text{)}.$$

Then by the analyticity of $\partial\Omega$ we get

$$\kappa_j \equiv 0 \text{ on } \partial\Omega \text{ (} j = 1, \dots, N-1 \text{)},$$

which shows that $\partial\Omega$ must be a hyperplane.

Thus it remains to consider case (ii) of Theorem 1.1. In this case, there exists a constant $L \geq 0$ satisfying

$$\sup_{\mathbb{R}^{N-1}} |\nabla\varphi| = L < \infty.$$

Then, it follows from (1) and (3) of Lemma 2.1 that

$$\sup_{\mathbb{R}^{N-1}} |\nabla\psi| = \sup_{\mathbb{R}^{N-1}} |\nabla\varphi| = L < \infty. \quad (2.10)$$

Hence, in view of this and (3) of Lemma 2.1, we can define a number $K^* > 0$ by

$$K^* = \inf\{K > 0 : \psi \leq \varphi + K \text{ in } \mathbb{R}^{N-1}\}. \quad (2.11)$$

Then we have

$$\varphi \leq \psi \leq \varphi + K^* \text{ in } \mathbb{R}^{N-1}. \quad (2.12)$$

We define a real analytic function h on $\mathbb{R}^{N-1}$ by

$$h(x') = \varphi(x') + K^* \text{ for } x' \in \mathbb{R}^{N-1}.$$

Moreover, by writing

$$M(h) = \operatorname{div} \left(\frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right) \text{ and } M(\psi) = \operatorname{div} \left(\frac{\nabla \psi}{\sqrt{1 + |\nabla \psi|^2}} \right),$$

from (2.9) and (2.12) we have

$$M(h) \leq 0 \leq M(\psi) \text{ and } \psi \leq h \text{ in } \mathbb{R}^{N-1}. \quad (2.13)$$

Hence, the method of sub- and super-solutions with the help of (2.10) yields that there exists $v \in C^\infty(\mathbb{R}^{N-1})$ satisfying

$$M(v) = 0 \text{ and } \psi \leq v \leq h \text{ in } \mathbb{R}^{N-1}, \text{ and } \sup_{\mathbb{R}^{N-1}} |\nabla v| < \infty.$$

(See [MS3] for the details.) Therefore, Moser's theorem [Mo], Corollary, p. 591, implies that v is affine. We set $\eta = \nabla v \in \mathbb{R}^{N-1}$.

On the other hand, by the definition of K^* in (2.11), there exists a sequence $\{z_n\}$ in $\mathbb{R}^{N-1}$ satisfying

$$\lim_{n \rightarrow \infty} (h(z_n) - \psi(z_n)) = 0. \quad (2.14)$$

Define a sequence of functions $\{\varphi_n\}$ by

$$\varphi_n(x') = h(x' + z_n) - h(z_n) \quad (= \varphi(x' + z_n) - \varphi(z_n)).$$

From (2.2) and (2.10) we see that all the second derivatives of φ are bounded in $\mathbb{R}^{N-1}$. Hence we can conclude that there exists a subsequence $\{\varphi_{n'}\}$ of $\{\varphi_n\}$ and a function $\varphi_\infty \in C^1(\mathbb{R}^{N-1})$ such that $\varphi_{n'} \rightarrow \varphi_\infty$ in $C^1(\mathbb{R}^{N-1})$ as $n' \rightarrow \infty$. Since $M(\varphi_n) \leq 0$ in $\mathbb{R}^{N-1}$, we have that $M(\varphi_\infty) \leq 0$ in $\mathbb{R}^{N-1}$ in the weak sense. Also, since $0 \leq h(x' + z_{n'}) - v(x' + z_{n'})$ in $\mathbb{R}^{N-1}$, with the help of (2.14), letting $n' \rightarrow \infty$ yields that

$$0 \leq \varphi_\infty(x') - \eta \cdot x' \text{ in } \mathbb{R}^{N-1}.$$

Consequently, we have

$$M(\varphi_\infty) \leq 0 = M(\eta \cdot x') \text{ and } \varphi_\infty \geq \eta \cdot x' \text{ in } \mathbb{R}^{N-1}, \text{ and } \varphi_\infty(0) = 0 = \eta \cdot 0. \quad (2.15)$$

Hence, the strong comparison principle implies that $\varphi_\infty(x') \equiv \eta \cdot x'$ in $\mathbb{R}^{N-1}$. Here we have used Theorem 10.7 together with Theorem 8.19 in [GT]. Therefore we conclude that

$$\varphi(x' + z_n) - (v(x' + z_n) - K^*) \rightarrow 0 \text{ in } C^1(\mathbb{R}^{N-1}). \quad (2.16)$$

Similarly, we can obtain

$$v(x' + z_n) - \psi(x' + z_n) \rightarrow 0 \text{ in } C^1(\mathbb{R}^{N-1}). \quad (2.17)$$

Therefore, it follows from (3) of Lemma 2.1, (2.16), and (2.17) that the distance between two hyperplanes determined by two affine functions v and $v - K^*$ must be R . Hence, since $v - K^* \leq \varphi \leq \psi \leq v$ in $\mathbb{R}^{N-1}$, we conclude that

$$\psi \equiv v \text{ and } \varphi \equiv v - K^* \text{ in } \mathbb{R}^{N-1},$$

which shows that $\partial\Omega$ is a hyperplane. $\square$

3 Concluding remarks

Let us explain the relationship between Theorem 1.1 and Theorems 3.2, 3.3, and 3.4 in [MS2]. When $\mu = 1$, we have

$$1 + RH_\Gamma = \frac{1}{N-1} \sum_{j=1}^{N-1} (1 + R\hat{\kappa}_j) \geq \left\{ \prod_{j=1}^{N-1} (1 + R\hat{\kappa}_j) \right\}^{\frac{1}{N-1}} = 1.$$

Therefore, the assumption of Theorem 3.2 that $H_\Gamma \leq 0$ implies that $\hat{\kappa}_j \equiv 0$ ($j = 1, \dots, N-1$). This shows that Γ is a hyperplane, and hence $\partial\Omega$ must be a hyperplane. Thus, Theorem 3.2 is contained in Theorem 1.1 with its proof. In the case where Ω is given by (1.4), Theorem 3.3 is contained in Theorem 1.1 with condition (iii). Since Theorem 3.4 does not assume the uniform exterior sphere condition for Ω , Theorem 3.4 is independent of Theorem 1.1.

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