

# Visual Zariski-van Kampen Theorem

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## 1. Introduction

A theorem of Zariski-van Kampen tells us how to compute the fundamental group  $\pi_1(\mathbb{P}^2 - C, p_0)$ , where  $\mathbb{P}^2 = \mathbb{P}^2(\mathbb{C})$  is the complex projective plane,  $C$  is a plane algebraic curve and  $\mathbb{P}^2 - C$  is the complement. ( $p_0$  is a reference point.)

However, it is not easy in general to carry out the computation. This is because we can not catch in general the global behavior of the braid monodromy.

In this note, we give a simple method for the computations of fundamental groups for a special type of curves  $C$ , that is to say, for real curves with some properties. Our method

can be seen more or less in some other literatures. See e.g. Kaneko [5] and Hironaka [4].

## 2. Notations

We use the following notations:

(i) Product of permutations:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

(ii) Product of paths:

$$\begin{array}{c} \curvearrowright \\ \alpha \end{array} \begin{array}{c} \curvearrowright \\ \beta \end{array} = \alpha\beta$$

(iii) Braids:

$$\begin{array}{cccc} \begin{array}{c} \diagup \quad \diagdown \\ | \quad | \end{array} & \begin{array}{c} \diagdown \quad \diagup \\ | \quad | \end{array} & \begin{array}{c} | \quad | \\ \diagup \quad \diagdown \end{array} & \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \\ \sigma_1 & \sigma_2 & \sigma_3 & \sigma_3 \sigma_2 \sigma_1 \end{array}$$

The Artin braid group  $B_d$  of  $d$  strings can be expressed as

$$B_d = \langle \sigma_1, \dots, \sigma_{d-1} \mid \begin{array}{l} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \\ (1 \leq i \leq d-2) \\ \sigma_i \sigma_j = \sigma_j \sigma_i \quad (|i-j| \geq 2) \end{array} \rangle$$

(that is,  $B_d$  is generated by  $\sigma_1, \dots, \sigma_d$  and have the generating relations  $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ , etc. )

(iv) Fundamental group of  $\mathbb{C} - \{d \text{ points}\}$

Let  $p_1, \dots, p_d$  be  $d$  points in  $\mathbb{C}$ . Take a reference point  $p_0$  in  $\mathbb{C} - \{p_1, \dots, p_d\}$ . Then

$$\pi_1(\mathbb{C} - \{p_1, \dots, p_d\}, p_0) = \langle \gamma_1, \dots, \gamma_d \rangle,$$

the free group of rank  $d$  generated by  $\gamma_1, \dots, \gamma_d$ , where  $\gamma_j$  ( $1 \leq j \leq d$ ) are lassos (meridians) as in Figure 1.

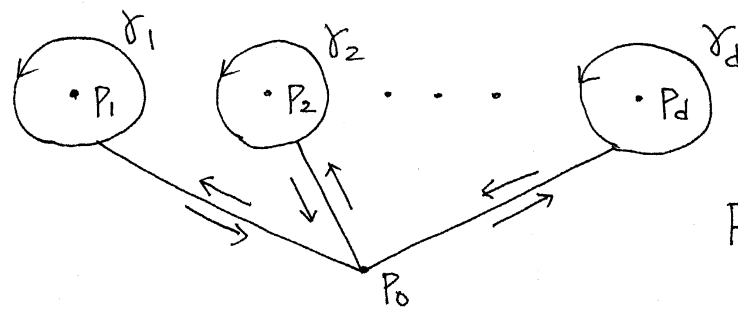


Figure 1

The Artin braid group  $B_d$  acts on the free group  $F_d = \langle \gamma_1, \dots, \gamma_d \rangle$  as follows:

$$\sigma_i(\gamma_i) = \gamma_i^{-1} \gamma_{i+1} \gamma_i, \quad \sigma_i(\gamma_{i+1}) = \gamma_i, \quad (1 \leq i \leq d-2)$$

$$\sigma_i(\gamma_j) = \gamma_j \quad (j \neq i, i+1).$$

$B_d$  acts faithfully on  $F_d$  (see Birman [1]).

Remark If  $P_1, \dots, P_d$  are points in  $\mathbb{P}^1$  (the complex projective line), then

$$\pi_1(\mathbb{P}^1 - \{P_1, \dots, P_d\}, P_0) = \langle \gamma_1, \dots, \gamma_d \mid \gamma_d \cdots \gamma_1 = 1 \rangle$$

### 3. Statement of Zariski-van Kampen Theorem

Let  $(X_0 : X_1 : X_2)$  be a homogeneous coordinate system in  $\mathbb{P}^2$ . Put  $z = X_1/X_0$ ,  $w = X_2/X_0$ , the affine coordinate system. Then  $L_\infty = \{X_0 = 0\}$  is the line at infinity. Let  $C$  be an algebraic curve in  $\mathbb{P}^2$ . Taking a suitable coordinate system, we may assume that  $C$  is defined in  $\mathbb{C}^2 = \mathbb{P}^2 - L_\infty$  by the following equation:

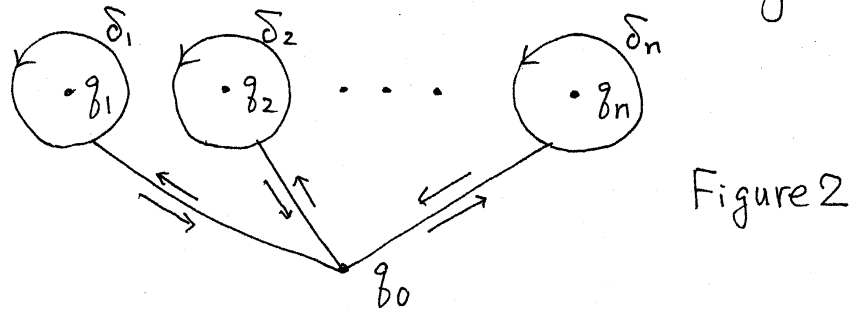
$$C: f(z, w) = w^d + a_1(z)w^{d-1} + \dots + a_d(z) = 0, \quad \dots (1)$$

where  $a_j(z)$  are polynomials of  $z$ .

The discriminant  $D(a_1(z), \dots, a_d(z))$  of the equation (1) with respect to  $w$  is a polynomial of  $z$ . Let

$$\{\gamma_1, \dots, \gamma_n\} \quad \dots (2)$$

be the set of zeros of the discriminant. Let  $z_0$  be a reference point in  $\mathbb{C} - \{z_1, \dots, z_n\}$ . Let  $\delta_1, \dots, \delta_n$  be lassos as in Figure 2:



For a point  $z \in \mathbb{C} - \{z_1, \dots, z_n\}$ , let

$$w_1(z), \dots, w_d(z) \quad \dots (3)$$

be the solutions of the equation (1). We fix

$$p_1 = w_1(z_0), \dots, p_d = w_d(z_0) \quad \dots (4)$$

in this order. Take a reference point  $p_0 \in \mathbb{C} - \{p_1, \dots, p_d\}$ . Let  $\gamma_1, \dots, \gamma_d$  be the lassos as in Figure 1.

Now, when  $z$  moves along a loop  $\delta$  in  $\mathbb{C} - \{z_1, \dots, z_n\}$  starting from  $z_0$ , the points in (3) move and give a braid  $\theta(\delta)$  of  $d$  strings.

$$\theta : \pi_1(\mathbb{C} - \{z_1, \dots, z_n\}, z_0) \longrightarrow B_d$$

is then a homomorphism, called the braid monodromy representation.

Theorem 1 (Zariski - van Kampen)

$$\pi_1(\mathbb{C}^2 - C, p_0) = \langle \gamma_1, \dots, \gamma_d \mid \theta(\delta_j) \gamma_i = \gamma_i, \\ (1 \leq i \leq d, 1 \leq j \leq n) \rangle.$$

Theorem 2 (Zariski - van Kampen) If  $C$  does not pass through the point  $\infty = (0:0:1)$ , then

$$\pi_1(\mathbb{P}^2 - C, p_0) = \langle \gamma_1, \dots, \gamma_d \mid \theta(\delta_j) \gamma_i = \gamma_i, \\ (1 \leq i \leq d, 1 \leq j \leq n), \quad \gamma_d \cdots \gamma_1 = 1 \rangle.$$

For these theorems, the reader may refer Dimca[2].

Thus if we know the braid monodromy  $\theta(\delta_j)$ , then the fundamental group can be calculated. But it is not easy to know it in general.

#### 4. Strongly real curves

Let  $C$  be an algebraic curve in  $\mathbb{P}^2$  given by the equation (1). Consider the following conditions:

(i)  $C$  is a real curve, that is, every polynomial  $a_i(z)$  ( $1 \leq i \leq d$ ) has real coefficients.

(ii) Every point  $q_j$  ( $1 \leq j \leq n$ ) in (2) is a real point.

(iii) Every ramification point of  $\pi$  in every  $\pi^{-1}(q_j)$  ( $1 \leq j \leq n$ ) is a real point, where

$$\pi: (z, w) \in C \longmapsto z \in \mathbb{C}$$

is the projection.

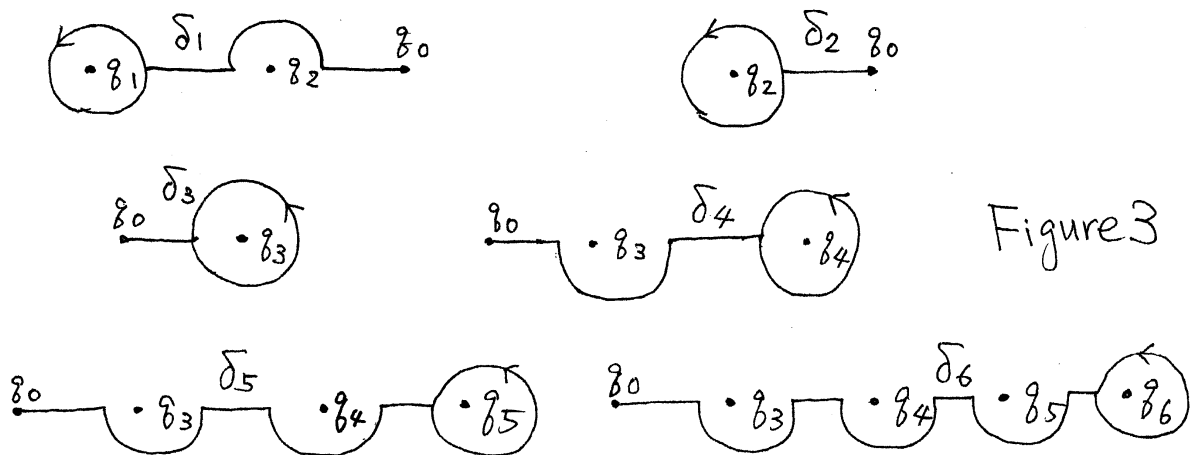
(iv) Every point  $P_i$  ( $1 \leq i \leq d$ ) in (4) is a real point.

We call a curve  $C$  with the conditions (i) ~ (iv) a strongly real curve. For such a curve  $C$ , we will show how to carry out the computations of the fundamental groups.

We use the above notations. We may assume that

$$q_1 < \cdots < q_n \quad \text{and} \quad P_1 < \cdots < P_d.$$

Assume for example that  $n=6$  and  $g_2 < g_0 < g_3$ . We take the lassos  $\delta_j$  ( $1 \leq j \leq 6$ ) as in Figure 3:



(The lassos are on the real line, except around  $g_j$  where they are on a circle or a semicircle with the center  $g_j$ .)

Using these lassos, we can observe the global behavior of the braid monodromy  $\theta(\delta_j)$  ( $1 \leq j \leq n$ ) and the actions on  $\gamma_i$  ( $1 \leq i \leq d$ ).

For example, if the point  $z$  on  $\delta_4$  in Figure 3 moves on the lower-hemicircle with the center  $g_3$ , then  $w_1(z), \dots, w_d(z)$  move so that they determine the half of the local monodromy given by the circle around  $g_3$ . Hence  $\gamma_j$  is changed by the action of the half monodromy, provided that every point of every  $\pi^{-1}(g)$  ( $g_3 - \varepsilon < g < g_3 + \varepsilon$ ) is real. If

some points in  $\pi^{-1}(\mathfrak{g})$  are not real, we must be careful and treat it case by case, as will be shown by examples.

### 5. Examples

We now explain our method using examples.

Example 1 Let  $C$  be the famous configuration of 6 lines as in Figure 4.

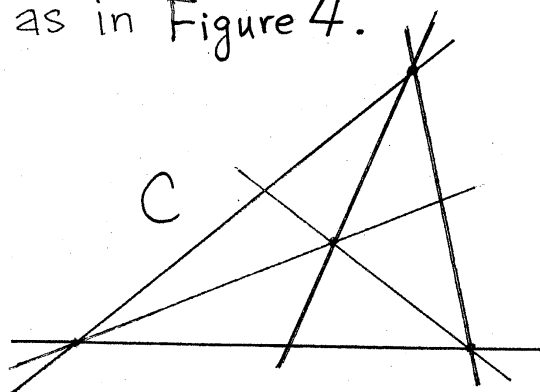


Figure 4

We take a coordinate system  $(z, w)$  as in Figure 5 so that  $C$  is strongly real. We assume that  $\mathfrak{g}_0$  is the origin. We observe Figure 5 (see the next page).

The 3 lines pass through the ramification point  $R_3$ . The half monodromy of  $\theta(\delta_3) = (\sigma_2 \sigma_3 \sigma_2)^2$  is  $\sigma_2 \sigma_3 \sigma_2$ . Thus

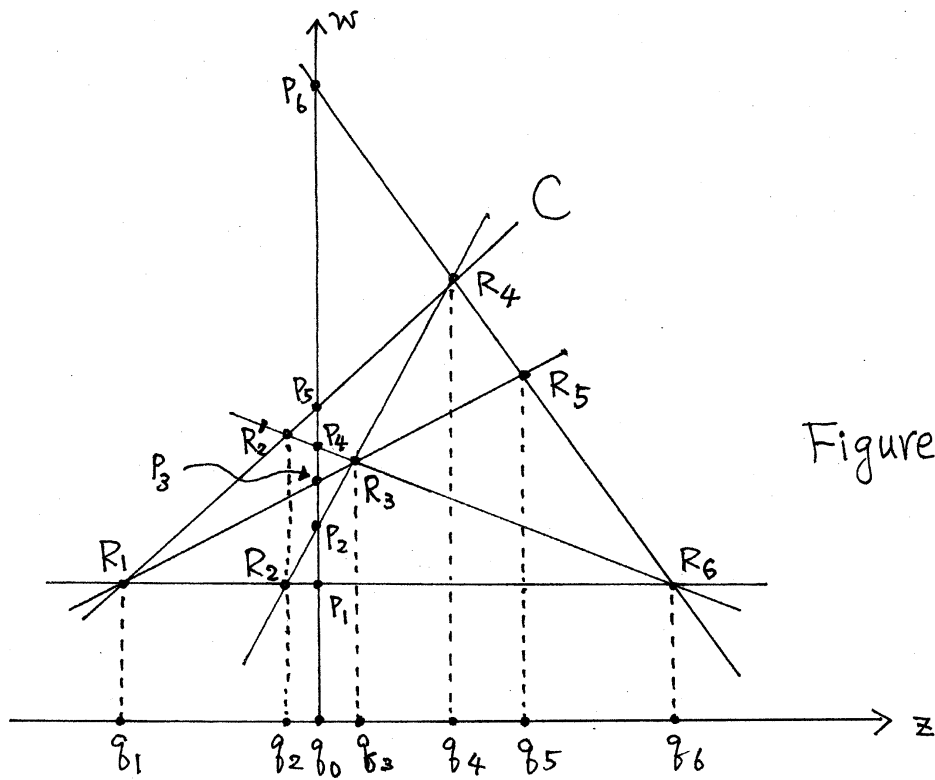


Figure 5

$$\gamma'_1 = (\sigma_2 \sigma_3 \sigma_2)(\gamma_1) = \gamma_1,$$

$$\gamma'_2 = (\sigma_2 \sigma_3 \sigma_2)(\gamma_2) = \gamma_2^{-1} \gamma_3^{-1} \gamma_4 \gamma_3 \gamma_2,$$

$$\gamma'_3 = (\sigma_2 \sigma_3 \sigma_2)(\gamma_3) = \gamma_2^{-1} \gamma_3 \gamma_2,$$

$$\gamma'_4 = (\sigma_2 \sigma_3 \sigma_2)(\gamma_4) = \gamma_2,$$

$$\gamma'_5 = (\sigma_2 \sigma_3 \sigma_2)(\gamma_5) = \gamma_5,$$

$$\gamma'_6 = (\sigma_2 \sigma_3 \sigma_2)(\gamma_6) = \gamma_6. \quad \dots (5)$$

On the other hand, by Theorem 1,  $\gamma_i = \theta(\delta_3) \gamma_i$  ( $1 \leq i \leq 6$ ). This implies that

$$\gamma_4 \gamma_3 \gamma_2 = \gamma_3 \gamma_2 \gamma_4 = \gamma_2 \gamma_4 \gamma_3.$$

Hence the relations (5) can be rewritten as

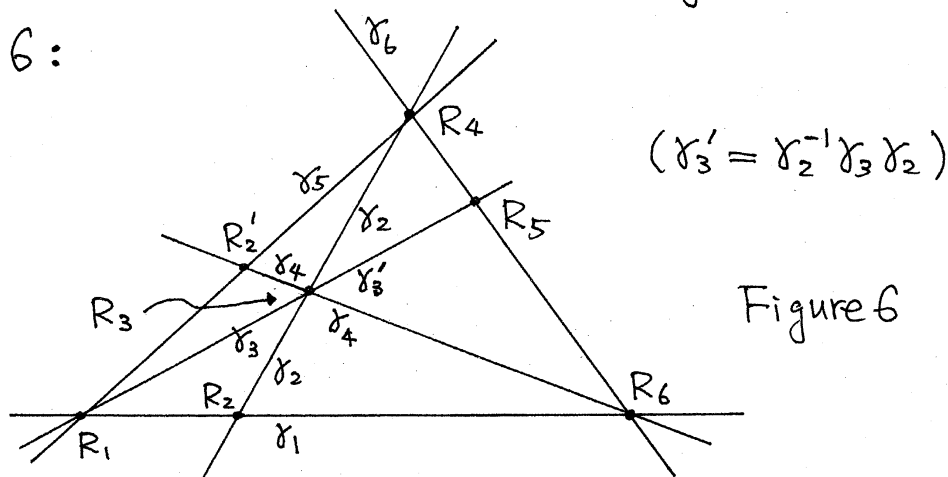
$$\gamma'_2 = \gamma_4,$$

$$\gamma'_3 = \gamma_2^{-1} \gamma_3 \gamma_2,$$

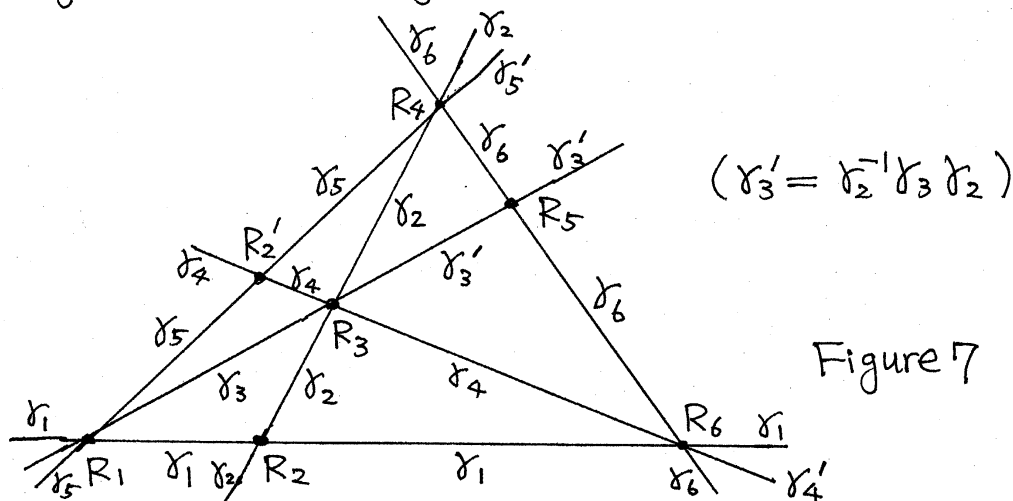
$$\gamma'_4 = \gamma_2,$$

$$\gamma'_i = \gamma_i \quad (i=1, 5, 6).$$

We write these data on the line segments as in Figure 6:



In a similar way, we write these data on every line segment as in Figure 7:



(What we write in Figure 7 are nothing but the so called contiguity relations.)

Observing these data in Figure 7, we can write down the fundamental groups:

$$\pi_1(\mathbb{C}^2 - C, p_0) = \langle \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6 \mid$$

$$\gamma_5 \gamma_3 \gamma_1 = \gamma_3 \gamma_1 \gamma_5 = \gamma_1 \gamma_5 \gamma_3,$$

$$\gamma_2 \gamma_1 = \gamma_1 \gamma_2, \quad \gamma_5 \gamma_4 = \gamma_4 \gamma_5,$$

$$\gamma_4 \gamma_3 \gamma_2 = \gamma_3 \gamma_2 \gamma_4 = \gamma_2 \gamma_4 \gamma_3,$$

$$\gamma_6 \gamma_5 \gamma_2 = \gamma_5 \gamma_2 \gamma_6 = \gamma_2 \gamma_6 \gamma_5,$$

$$\gamma_6 \cdot \gamma_2^{-1} \gamma_3 \gamma_2 = \gamma_2^{-1} \gamma_3 \gamma_2 \cdot \gamma_6,$$

$$\gamma_6 \gamma_4 \gamma_1 = \gamma_4 \gamma_1 \gamma_6 = \gamma_1 \gamma_6 \gamma_4 \rangle.$$

By Theorem 2,  $\pi_1(\mathbb{P}^2 - C, p_0)$  is generated by  $\gamma_1, \dots, \gamma_6$  with the same relations as in  $\pi_1(\mathbb{C}^2 - C, p_0)$ , plus one more relation

$$\gamma_6 \gamma_5 \gamma_4 \gamma_3 \gamma_2 \gamma_1 = 1.$$

If we take coordinate system  $(z, w)$  such that one of the 6 lines is  $L_\infty$  (the line at infinity), then the expression of  $\pi_1(\mathbb{P}^2 - C)$  becomes simpler:

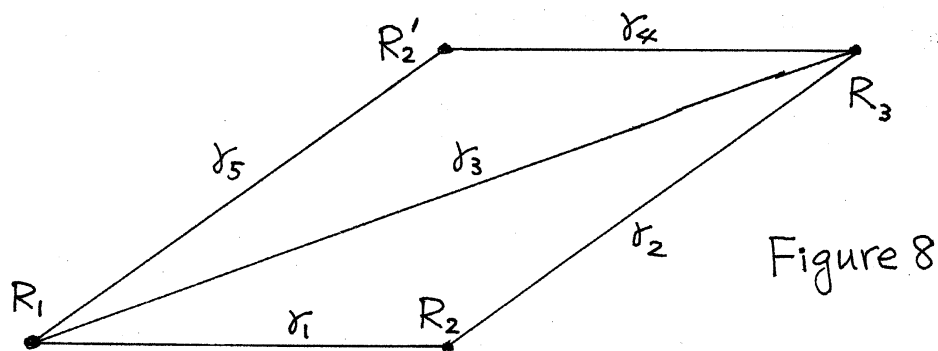


Figure 8

$$\pi_1(\mathbb{P}^2 - C, p_0) = \langle \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5 \mid$$

$$\gamma_5 \gamma_3 \gamma_1 = \gamma_3 \gamma_1 \gamma_5 = \gamma_1 \gamma_5 \gamma_3,$$

$$\gamma_5 \gamma_4 = \gamma_4 \gamma_5, \quad \gamma_2 \gamma_1 = \gamma_1 \gamma_2,$$

$$\gamma_4 \gamma_3 \gamma_2 = \gamma_3 \gamma_2 \gamma_4 = \gamma_2 \gamma_4 \gamma_3 \rangle.$$

Remark The expression of  $\pi_1(\mathbb{P}^2 - C, p_0)$  is changed if we change the position of the reference point  $p_0$ .

Example 2 Let  $C$  be the strongly real curve consisting of an irreducible conic and 3 lines which are tangent to the conic as in Figure 9:

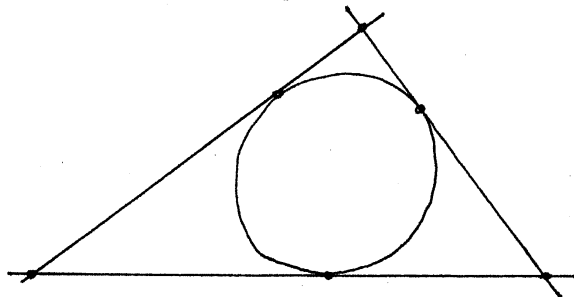


Figure 9

We take  $g_0$  so that  $g_3 < g_0 < g_4$  and assume that  $g_0$  is the origin as in Figure 10:

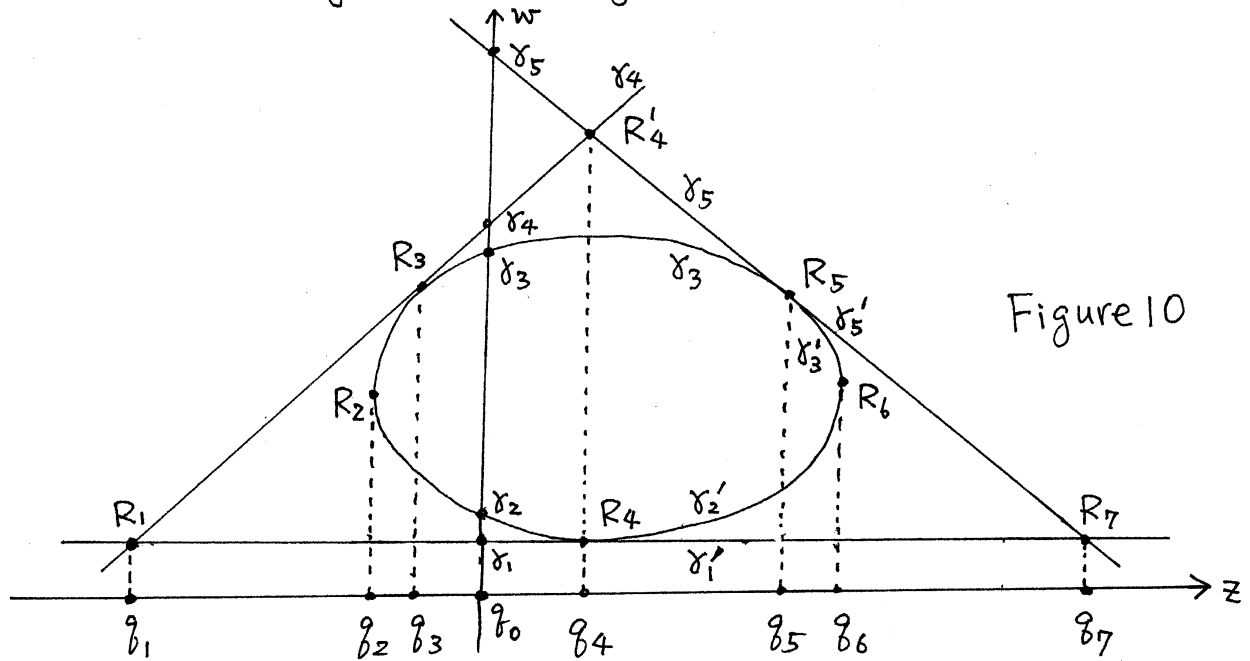


Figure 10

(We write  $\delta_i$  directly instead of  $P_i$  in Figure 10).

We observe the braid monodromy  $\theta(\delta_4)$ ,  $\theta(\delta_5)$ ,  $\theta(\delta_6)$  and  $\theta(\delta_7)$ .

The half monodromy of  $\theta(\delta_4) = \sigma_1^4 \sigma_4^2$  is  $\sigma_1^2 \sigma_4$ .

Hence

$$(\delta_2 \delta_1)^2 = (\delta_1 \delta_2)^2, \quad \delta_5 \delta_4 = \delta_4 \delta_5$$

and

$$\delta'_1 = \delta_2 \delta_1 \delta_2^{-1},$$

$$\delta'_2 = \delta_1^{-1} \delta_2 \delta_1,$$

$$\delta'_4 = \delta_5,$$

$$\delta'_5 = \delta_4.$$

(We write  $\gamma_1'$  and  $\gamma_2'$  on the curve segments in Figure 10.)

In a similar way, we observe  $\theta(\delta_5)$  and get the following relations:

$$(\gamma_5 \gamma_3)^2 = (\gamma_3 \gamma_5)^2,$$

$$\gamma_3' = \gamma_5 \gamma_3 \gamma_5^{-1},$$

$$\gamma_5' = \gamma_3^{-1} \gamma_5 \gamma_3.$$

(We write  $\gamma_3'$  and  $\gamma_5'$  on the curve segments in Figure 10.)

We also observe  $\theta(\delta_6)$  and get the following relation:

$$\gamma_3' = \gamma_2', \text{ i.e. } \gamma_5 \gamma_3 \gamma_5^{-1} = \gamma_1^{-1} \gamma_2 \gamma_1.$$

But we must be careful about  $\theta(\delta_7)$ , for  $w_2(z)$  and  $w_3(z)$  become imaginary numbers.

When  $z$  moves to the right on the real axis with  $\delta_5 < z < \delta_6 - \varepsilon$ , then the points in (3) move as in Figure 11:

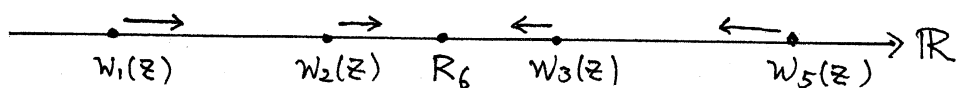
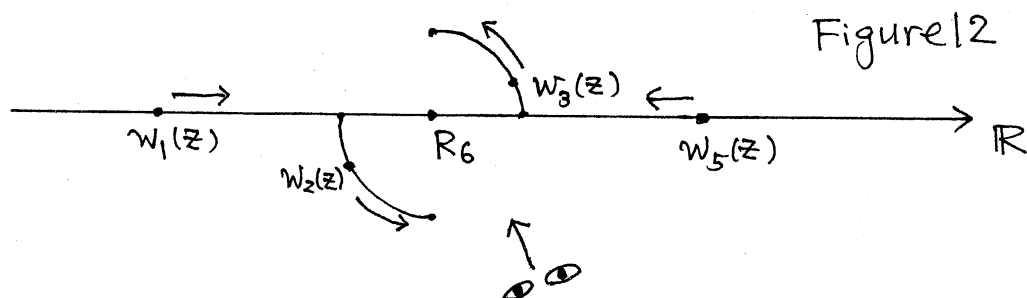
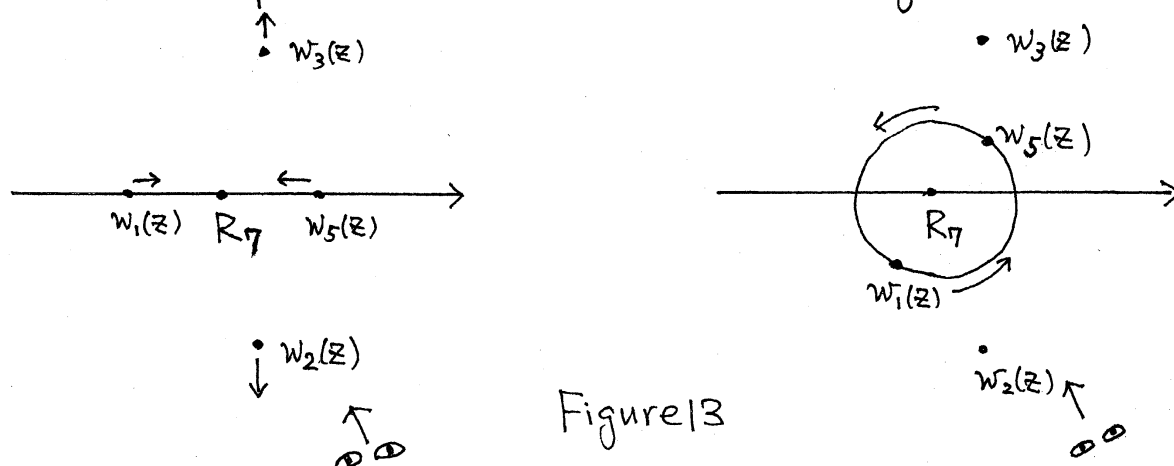


Figure 11

But when  $z$  moves on the lower hemicircle around  $\gamma_6$ , then the points in (3) move as in Figure 12 in which our eyes are at the reference point and observe the movements of points in order to determine the braid:



When  $z$  further moves along  $\theta(\delta_7)$  with  $z > \gamma_6 + \varepsilon$ , then the points in (3) move as in Figure 13:



Hence from the point  $\gamma_6 - \varepsilon$ ,  $\theta(\delta_7)$  equals to

$$\sigma = \sigma_1^{-1} \sigma_3^{-1} \sigma_2^2 \sigma_3 \sigma_1.$$

The relations  $\sigma(\gamma_j') = \gamma_j'$  ( $j = 1, 2, 3, 5$ ) and

the relation  $\gamma_3' = \gamma_2'$  imply that

$$(\gamma_2'^{-1} \gamma_5' \gamma_2') \gamma_1' = \gamma_1' (\gamma_2'^{-1} \gamma_5' \gamma_2').$$

Similar considerations work for  $\theta(\delta_1)$ ,  $\theta(\delta_2)$  and  $\theta(\delta_3)$ .  
Thus we have

$$\begin{aligned} \pi_1(\mathbb{P}^2 - C, p_0) = & \langle \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5 \mid \\ & (\gamma_2 \gamma_1)^2 = (\gamma_1 \gamma_2)^2, \quad \gamma_5 \gamma_4 = \gamma_4 \gamma_5, \quad \gamma_3' = \gamma_2', \\ & (\gamma_2'^{-1} \gamma_5' \gamma_2') \gamma_1' = \gamma_1' (\gamma_2'^{-1} \gamma_5' \gamma_2'), \quad (\gamma_5 \gamma_3)^2 = (\gamma_3 \gamma_5)^2, \\ & (\gamma_4 \gamma_3)^2 = (\gamma_3 \gamma_4)^2, \quad \gamma_3'' = \gamma_2, \\ & \gamma_1 (\gamma_2^{-1} \gamma_4'' \gamma_2) = (\gamma_2^{-1} \gamma_4'' \gamma_2) \gamma_1, \\ & \gamma_5 \gamma_4 \gamma_3 \gamma_2 \gamma_1 = 1 \rangle, \end{aligned}$$

where

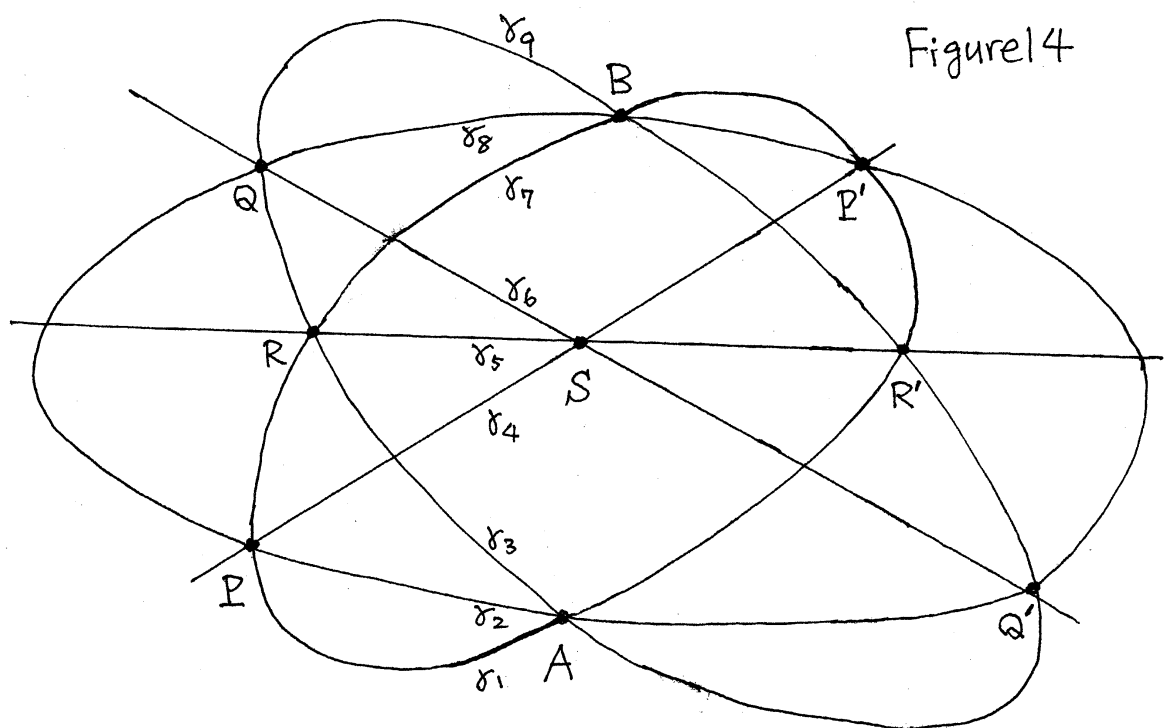
$$\begin{aligned} \gamma_1' &= \gamma_2 \gamma_1 \gamma_2^{-1}, & \gamma_2' &= \gamma_1^{-1} \gamma_2 \gamma_1, \\ \gamma_3' &= \gamma_5 \gamma_3 \gamma_5^{-1}, & \gamma_5' &= \gamma_3^{-1} \gamma_5 \gamma_3, \\ \gamma_3'' &= \gamma_4 \gamma_3 \gamma_4^{-1}, & \gamma_4'' &= \gamma_3^{-1} \gamma_4 \gamma_3. \end{aligned}$$

We can rewrite it in a simpler form:

$$\begin{aligned} \pi_1(\mathbb{P}^2 - C, p_0) = & \langle \gamma_1, \gamma_2, \gamma_4 \mid (\gamma_2 \gamma_1)^2 = (\gamma_1 \gamma_2)^2, \\ & (\gamma_4 \gamma_2)^2 = (\gamma_2 \gamma_4)^2, \quad \gamma_4 \gamma_1 = \gamma_1 \gamma_4 \rangle. \end{aligned}$$

$$(\gamma_3 = \gamma_4^{-1} \gamma_2 \gamma_4, \quad \gamma_5 = (\gamma_2 \gamma_4 \gamma_2 \gamma_1)^{-1}).$$

Example 3 Consider the 3 irreducible conics meeting at two points  $A$  and  $B$  and 3 lines  $PP'$ ,  $QQ'$  and  $RR'$  meeting at a point  $S$  as in Figure 14:



This configuration express a special case of Euler's theorem for cubic curves. Let  $C$  be the strongly real curve of this configuration. Then we can calculate the fundamental group as in Examples 2. The result is

$$\pi_1(\mathbb{P}^2 - C, p_0) = \langle \delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6 \mid$$

$$\delta_i \delta_j = \delta_j \delta_i \quad (i, j = 1, 2, 3),$$

$$\delta_i \delta_j = \delta_j \delta_i \quad (i = 1, 2, 3, \quad j = 4, 5, 6),$$

$$\delta_6 \delta_5 \delta_4 = \delta_5 \delta_4 \delta_6 = \delta_4 \delta_6 \delta_5,$$

$$\delta_6 \delta_5 \delta_4 (\delta_3 \delta_2 \delta_1)^2 = 1 \rangle.$$

In this way, many examples of the fundamental groups for strongly real curves  $C$  can be calculated. Interesting configurations like Pappus' theorem, Desargues' theorem, Pascal's theorem, Brianchon's theorem, etc., are expressed by strongly real curves. Thus the fundamental groups of the complements of the configurations can be calculated.

### 6. Braid groups as fundamental groups of strongly real curves.

If  $C$  is a strongly real curve of degree 3 and of genus 0 with a simple cusp as in Figure 15, then

$$\pi_1(\mathbb{C}^2 - C, p_0) = \langle \gamma_1, \gamma_2 \mid \gamma_2 \gamma_1 \gamma_2 = \gamma_1 \gamma_2 \gamma_1 \rangle \\ \simeq B_3 \quad (3\text{-rd braid group}).$$

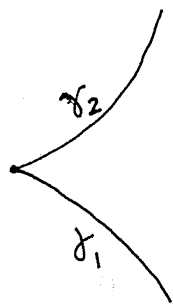


Figure 15

$$C: w^2 - z^3 = 0$$

is such a curve.

Next, if  $C$  is a strongly real curve of degree 4 and of genus 0 with 2 simple cusps and 4 nodes as in Figure 16, then

$$\begin{aligned} \pi_1(\mathbb{C}^2 - C, p_0) &= \langle \gamma_1, \gamma_2, \gamma_3 \mid \gamma_3 \gamma_2 \gamma_3 = \gamma_2 \gamma_3 \gamma_2, \\ &\quad \gamma_3 \gamma_1 \gamma_3 = \gamma_1 \gamma_3 \gamma_1, \gamma_2 \gamma_1 = \gamma_1 \gamma_2 \rangle \\ &\simeq B_4 \text{ (4-th braid group)}. \end{aligned}$$

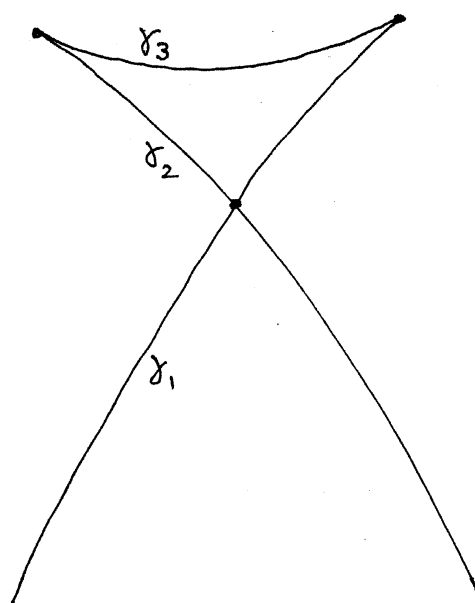


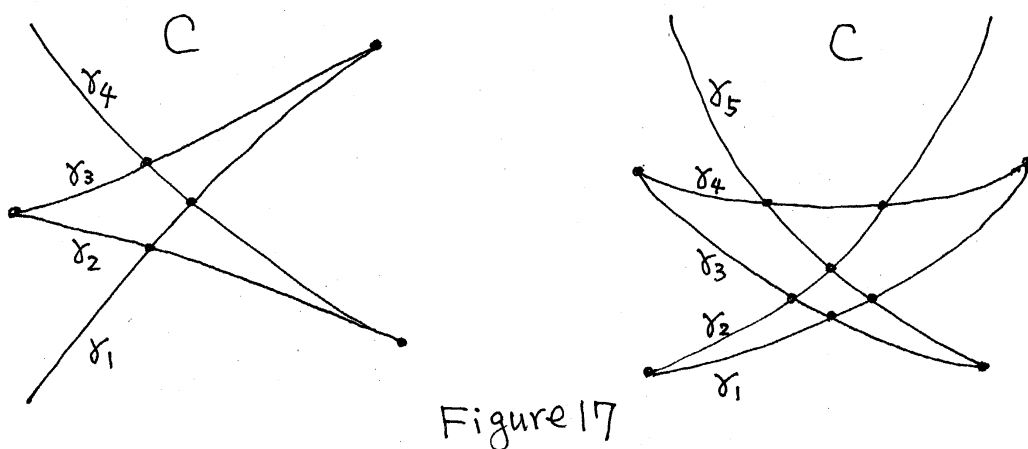
Figure 16

$$C: z^4 + 4z^2 - 6z^2w - 3w^2 + 4w^3 = 0$$

is such a curve. (This equation is due to M. Oka.)

In a similar way, if  $C$  are strongly real curves

of degree 5 and 6, respectively, and of genus 0, with only simple cusps and nodes as in Figure 17, then  $\pi_1(\mathbb{C}^2 - C, P_0)$  are isomorphic to  $B_5$  and  $B_6$ , respectively. (But we do not know the equations for such curves.)



We can continue this process and draw a strongly real curve whose complement in  $\mathbb{C}^2$  has the fundamental group isomorphic to the Artin braid group of  $d$  strings.

## 7. Finite branched covering of $\mathbb{P}^2$

One of the aim of Zariski [6] to study fundamental groups was to apply it to the study of algebraic functions of 2 variables, that is, of finite

branched coverings of  $\mathbb{P}^2$ .

Here a finite branched covering

$$f: X \longrightarrow \mathbb{P}^2$$

of  $\mathbb{P}^2$  is by definition, a finite proper holomorphic mapping of an irreducible normal projective surface  $X$  onto  $\mathbb{P}^2$ .  $f$  is determined by its permutation monodromy representation

$$\Phi_f: \pi_1(\mathbb{P}^2 - C, p_0) \longrightarrow S_d,$$

where  $d$  is the degree of  $f$ ,  $S_d$  is the  $d$ -th symmetric group and  $C$  is the branch curve of  $f$ . The image of  $\Phi_f$  is a transitive subgroup of  $S_d$ .

Conversely, given a homomorphism

$$\Phi: \pi_1(\mathbb{P}^2 - C, p_0) \longrightarrow S_d$$

whose image is transitive, there exist a unique (up to isomorphisms) finite branched covering

$$f: X \longrightarrow \mathbb{P}^2$$

such that (i) the branch curve is contained in

$C$  and (ii)  $\Phi_f = \Phi$ .

This follows from a Theorem of Grauert-Remmert [3].

Since we know about fundamental groups  $\pi_1(\mathbb{P}^2 - C, p_0)$  for many strongly real curves  $C$ , we can construct many examples of such homomorphisms  $\Phi$ .

We give here only one example. Let  $C$  be the configuration of 6 lines as in Example 1. We use the same notations as in Example 1. Put

$$\Phi(\gamma_1) = \Phi(\gamma_2) = (145236),$$

$$\Phi(\gamma_4) = \Phi(\gamma_5) = (135246),$$

$$\Phi(\gamma_3) = (146235),$$

$$\Phi(\gamma_6) = (136245).$$

Then it is easy to check that

$$\Phi: \pi_1(\mathbb{P}^2 - C, p_0) \longrightarrow S_6$$

is a homomorphism, whose image is a transitive subgroup of  $S_6$  of order 24.

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