

# A Galois embedding from polymorphic types into existential types – Extended Abstract –

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## Abstract

We show that there exist translations between polymorphic  $\lambda$ -calculus and a subsystem of minimal logic with existential types, which form a Galois connection and moreover a Galois embedding. From a programming point of view, this result means that polymorphic functions can be represented by abstract data types.

## 1 Introduction

We show that polymorphic types can be interpreted by the use of second order existential types. For this, we prove that there exist translations between polymorphic  $\lambda$ -calculus  $\lambda_2$  and subsystem of minimal logic with existential types, which form a Galois connection and moreover a Galois embedding. From a programming point of view, this result means that polymorphic functions can be represented by abstract data types and vice versa.

Peter Selinger [Seli01] has introduced control categories and established an isomorphism between call-by-name and call-by-value  $\lambda\mu$ -calculi. The isomorphism reveals duality not only on logical connectives ( $\wedge, \vee$ ) like de Morgan but also on reduction strategies (call-by-name and call-by-value), input-output relations (demand- and data-driven) and inference rules (introduction and elimination).

Philip Wadler [Wad03] introduced the dual calculus in the style of Gentzen's sequent calculus LK, such that the duality explicitly appears on antecedent and succedent in the sequent of the propositional calculus.

Our main interest is a neat connection and proof duality between polymorphic types and existential types. It is logically quite natural like de Morgan's

duality, and computationally still interesting, since dual of polymorphic functions with universal type can be regarded as abstract data types with existential type [MP85]. Instead of classical systems like [Pari92], even intuitionistic systems can enjoy that polymorphic types can be interpreted by existential types and vice versa. This interpretation also contains proof duality, such that the universal introduction rule is interpreted by the use of the existential elimination rule, and the universal elimination by the existential introduction. Moreover, we established not only a Galois connection but also a Galois embedding from polymorphic  $\lambda$ -calculus (Girard-Reynolds) into a calculus with existential types.

## 2 Polymorphic $\lambda$ -calculus $\lambda 2$

We give the definition of polymorphic  $\lambda$ -calculus *à la* Church as second order intuitionistic logic, denoted by  $\lambda 2$ . This calculus is also known as the system F. The syntax of types is defined from type variables denoted by  $X$ , using  $\Rightarrow$  or  $\forall$  over type variables. The syntax of  $\lambda 2$ -terms is defined from individual variables denoted by  $x$ , using term-applications, type-applications or  $\lambda$ -abstractions over individual variables or type variables.

**Definition 1 (Types)**

$$A ::= X \mid A \Rightarrow A \mid \forall X. A$$

**Definition 2 (Pseudo-terms)**

$$\Lambda 2 \ni M ::= x \mid \lambda x : A. M \mid M M \mid \lambda X. M \mid M A$$

**Definition 3 (Reduction rules)**  $(\beta) (\lambda x : A. M) M_1 \rightarrow M[x := M_1]$

$$(\beta_t) (\lambda X. M) A \rightarrow M[X := A]$$

$$(\eta) \lambda x. M x \rightarrow M \text{ if } x \notin FV(M)$$

$$(\eta_t) \lambda X. M X \rightarrow M \text{ if } X \notin FV(M)$$

A set of free variables in  $M$  is denoted by  $FV(M)$ . The one step reduction relation is denoted by  $\rightarrow_{\lambda 2}$ . We write  $\rightarrow_{\lambda 2}^+$  or  $\rightarrow_{\lambda 2}^*$  to denote the transitive closure or the reflexive and transitive closure of  $\rightarrow_{\lambda 2}$ , respectively. We employ the notation  $=_{\lambda 2}$  for the symmetric, reflexive and transitive closure of the one step reduction  $\rightarrow_{\lambda 2}$  defined above. We write  $\equiv$  for a syntactical identity modulo renaming of bound variables. Let  $R$  be  $\beta$ ,  $\beta_t$ ,  $\eta$  or  $\eta_t$ . Then we often write  $\rightarrow_R$  to denote the corresponding subset of  $\rightarrow_{\lambda 2}$ .

The typing judgement of  $\lambda 2$  takes the form of  $\Gamma \vdash M : A$ , where  $\Gamma$  is a set of declarations in the form of  $x : A$  with distinct variables as subjects.

**Definition 4 (Type assignment rules)**

$$\frac{x : A \in \Gamma}{\Gamma \vdash x : A}$$

$$\begin{array}{c}
\frac{\Gamma, x:A_1 \vdash M : A_2}{\Gamma \vdash \lambda x:A_1. M : A_1 \Rightarrow A_2} (\Rightarrow I) \quad \frac{\Gamma \vdash M_1 : A_1 \Rightarrow A_2 \quad \Gamma \vdash M_2 : A_1}{\Gamma \vdash M_1 M_2 : A_2} (\Rightarrow E) \\
\frac{\Gamma \vdash M : A}{\Gamma \vdash \lambda X. M : \forall X. A} (\forall I)^* \quad \frac{\Gamma \vdash M : \forall X. A}{\Gamma \vdash M A_1 : A[X := A_1]} (\forall E)
\end{array}$$

where  $(\forall I)^*$  denotes the eigenvariable condition  $X \notin FV(\Gamma)$

### 3 Minimal logic with second order sum

Next, we introduce the counter calculus  $\lambda^\exists$  as minimal logic consisting of negations, conjunctions and second order sums. Such a calculus seems to be logically weak and has never been considered as far as we know. However,  $\lambda^\exists$  turns out strong enough to interpret  $\lambda 2$  and interesting to investigate polymorphism.

**Definition 5 (Types)**

$$A ::= \perp \mid X \mid \neg A \mid A \wedge A \mid \exists X. A$$

**Definition 6 (Pseudo-terms)**

$$\begin{aligned}
\Lambda^\exists \ni M &::= x \mid \lambda x:A. M \mid M M \\
&\mid \langle M, M \rangle \mid \text{let } \langle x, x \rangle = M \text{ in } M \\
&\mid \langle A, M \rangle_{\exists X. A} \mid \text{let } \langle X, x \rangle = M \text{ in } M
\end{aligned}$$

**Definition 7 (Reduction rules)**  $(\beta) (\lambda x:A. M) M_1 \rightarrow M[x := M_1]$

$(\eta) \lambda x:A. M x \rightarrow M$  if  $x \notin FV(M)$

$(\text{let}_\wedge) \text{let } \langle x_1, x_2 \rangle = \langle M_1, M_2 \rangle \text{ in } M \rightarrow M[x_1 := M_1, x_2 := M_2]$

$(\text{let}_{\wedge_\eta}) \text{let } \langle x_1, x_2 \rangle = M_1 \text{ in } M[z := \langle x_1, x_2 \rangle] \rightarrow M[z := M_1]$   
if  $x_1, x_2 \notin FV(M)$

$(\text{let}_\exists) \text{let } \langle X, x \rangle = \langle A_1, M_2 \rangle_{\exists X. A} \text{ in } M \rightarrow M[X := A_1, x := M_2]$

$(\text{let}_{\exists_\eta}) \text{let } \langle X, x \rangle = M_1 \text{ in } M[z := \langle X, x \rangle] \rightarrow M[z := M_1]$   
if  $X, x \notin FV(M)$

A simultaneous substitution for free variables  $x_1, x_2$  or  $X, x$  is denoted by  $[x_1 := M_1, x_2 := M_2]$  or  $[X := A, x := M]$ , respectively. We also write  $=_{\lambda^\exists}$  for the reflexive, symmetric and transitive closure of the one step reduction  $\rightarrow_{\lambda^\exists}$  defined above. We may sometimes omit type annotations from terms.

**Definition 8 (Type assignment rules)**

$$\frac{x:A \in \Gamma}{\Gamma \vdash x:A}$$

$$\begin{array}{c}
\frac{\Gamma, x:A \vdash M : \perp}{\Gamma \vdash \lambda x:A. M : \neg A} (\neg I) \quad \frac{\Gamma \vdash M_1 : \neg A \quad \Gamma \vdash M_2 : A}{\Gamma \vdash M_1 M_2 : \perp} (\neg E) \\
\frac{\Gamma \vdash M_1 : A_1 \quad \Gamma \vdash M_2 : A_2}{\Gamma \vdash \langle M_1, M_2 \rangle : A_1 \wedge A_2} (\wedge I) \\
\frac{\Gamma \vdash M_1 : A_1 \wedge A_2 \quad \Gamma, x_1:A_1, x_2:A_2 \vdash M : A}{\Gamma \vdash \text{let } \langle x_1, x_2 \rangle = M_1 \text{ in } M : A} (\wedge E) \\
\frac{\Gamma \vdash M : A[X := A_1]}{\Gamma \vdash \langle A_1, M \rangle_{\exists X.A} : \exists X.A} (\exists I) \quad \frac{\Gamma \vdash M_1 : \exists X.A_1 \quad \Gamma, x:A_1 \vdash M : A}{\Gamma \vdash \text{let } \langle X, x \rangle = M_1 \text{ in } M : A} (\exists E)^*
\end{array}$$

where  $(\exists E)^*$  denotes the eigenvariable condition  $X \notin FV(\Gamma, A)$

## 4 CPS-translation and soundness

For a CPS-translation from  $\lambda 2$ -calculus into  $\lambda^\exists$ -calculus, we define an embedding of types (types for denotations of proof terms), denoted by  $A^k$ , and types for continuations, denoted by  $A^*$ , which also work for denotation of  $\lambda 2$ -types.

**Definition 9 (Embedding of types)**

$$A^k = \neg A^*$$

**Definition 10 (Types for continuations and denotation of types)**

- (1)  $X^* = X$
- (2)  $(A_1 \Rightarrow A_2)^* = A_1^k \wedge A_2^*$
- (3)  $(\forall X.A)^* = \exists X.A^*$

The definition above inherits the propositional case from Hofmann-Streicher [HS97] and Selinger [Seli01]. The operator  $*$  exactly takes logical duality when one reads  $A_1 \Rightarrow A_2$  as  $\neg A_1 \vee A_2$ . It is remarked that in terms of classical logic, we have  $(A_1 \Rightarrow A_2)^k \leftrightarrow (A_2^* \Rightarrow A_1^*)$ , which means that a function is interpreted as an inverse function over continuations.

**Lemma 1** *We have  $A^*[X := A_1^*] = (A[X := A_1])^*$  and  $(A[X := A_1])^k = A^k[X := A_1^*]$ .*

*Proof.* By induction on the structure of  $A$ . □

The definition of denotation of proof terms, denoted by  $\llbracket M \rrbracket$ , is given by induction on the typing derivation of  $M$ .

**Definition 11 (Denotation of  $\lambda 2$ -terms)**

- (i)  $\llbracket x \rrbracket = x$  if  $\Gamma \vdash x : A$
- (ii)  $\llbracket \lambda x:A_1. M \rrbracket = \lambda k:(A_1 \Rightarrow A_2)^*. (\text{let } \langle x, c \rangle = k \text{ in } \llbracket M \rrbracket c)$   
if  $\Gamma \vdash \lambda x:A_1. M : A_1 \Rightarrow A_2$

- (iii)  $\llbracket M_1 M_2 \rrbracket = \lambda a : A_2^*. \llbracket M_1 \rrbracket \langle \llbracket M_2 \rrbracket, a \rangle$   
*if*  $\Gamma \vdash M_1 M_2 : A_2$
- (iv)  $\llbracket \lambda X. M \rrbracket = \lambda k : (\forall X. A)^*. (\text{let } \langle X, c \rangle = k \text{ in } \llbracket M \rrbracket c)$   
*if*  $\Gamma \vdash \lambda X. M : \forall X. A$
- (v)  $\llbracket M A_1 \rrbracket = \lambda k : (A[X := A_1])^*. \llbracket M \rrbracket \langle A_1^*, k \rangle_{\exists X. A^*}$   
*if*  $\Gamma \vdash M A_1 : A[X := A_1]$

The definition above interprets each proof term with type  $A$  as a functional element with type  $A^k$  (space of denotations of type  $A$ ), which takes, as an argument, a continuation with type  $A^*$ . The cases of application say that continuations are in the form of a pair  $\langle \llbracket M \rrbracket, a \rangle$  or  $\langle A^*, a \rangle$  consisting of a denotation and a continuation in this order. The cases of  $\lambda$ -abstraction mean that after the interpretation,  $\lambda$ -abstraction is waiting for a first component of a continuation (i.e., a denotation of its argument), and the second component becomes a rest continuation to the result. It should be remarked that  $(\forall I)$  and  $(\forall E)$  are respectively interpreted by dual  $(\exists E)$  and  $(\exists I)$ , i.e., we call proof duality.

We may simply write  $\langle R_1, R_2, \dots, R_n, M \rangle$  for  $\langle R_1, \langle R_2, \dots, R_n, M \rangle \rangle$ , where we let  $\langle M \rangle \equiv M$ , and  $R_i$  is either  $M$  or  $A$ .

**Example 1** Let  $xM_1 \dots M_n$  be with type  $A_{m+1}$  and  $A$  be  $A_1 \Rightarrow \dots \Rightarrow A_{m+1}$ :

$$\begin{aligned}
& \llbracket \lambda x_1 : A_1 \dots \lambda x_m : A_m. xM_1 \dots M_n \rrbracket \\
& \rightarrow_{\beta}^+ \lambda k_1 : A^*. \text{let } \langle x_1, k_2 \rangle = k_1 \text{ in} \\
& \quad \text{let } \langle x_2, k_3 \rangle = k_2 \text{ in} \\
& \quad \dots \\
& \quad \text{let } \langle x_m, k_{m+1} \rangle = k_m \text{ in } x \langle \llbracket M_1 \rrbracket, \dots, \llbracket M_n \rrbracket, k_{m+1} \rangle \\
& \text{where } k_i : A_i^k \wedge (A_{i+1} \Rightarrow \dots \Rightarrow A_{m+1})^*
\end{aligned}$$

**Lemma 2** We have  $\llbracket M[x := N] \rrbracket = \llbracket M \rrbracket[x := \llbracket N \rrbracket]$  and  $\llbracket M[X := A] \rrbracket = \llbracket M \rrbracket[X := A^*]$ .

**Proposition 1 (Soundness)**

- (i) If we have  $\Gamma \vdash_{\lambda 2} M : A$ , then  $\Gamma^k \vdash_{\lambda \exists} \llbracket M \rrbracket : A^k$ .
- (ii) For well-typed  $M_1, M_2 \in \Lambda 2$ , if we have  $M_1 \rightarrow_{\lambda 2} M_2$  then  $\llbracket M_1 \rrbracket \rightarrow_{\lambda \exists}^+ \llbracket M_2 \rrbracket$ .

*Proof.* If we have  $\Gamma \vdash M : A$ , then  $\Gamma^k \vdash \llbracket M \rrbracket : A^k$  by induction on the derivation together with Definition 11. We show two cases of (1)  $\lambda X. M$  and (2)  $MA$ .

(1) Suppose the following figure of  $\lambda 2$ , where  $X$  is never free in the context  $\Gamma$ .

$$\frac{M : A}{\lambda X. M : \forall X. A} (\forall I)^*$$

Then we have the proof figure of  $\lambda^\exists$ , where the eigenvariable condition of  $(\exists E)$  can be guaranteed by that of  $(\forall I)$ .

$$\frac{\frac{\frac{[k : \exists X.A^*]}{\text{let } \langle X, c \rangle = k \text{ in } \llbracket M \rrbracket c : \perp} \quad \frac{\llbracket M \rrbracket : A^k \quad [c : A^*]}{\llbracket M \rrbracket c : \perp} \quad (\exists E)^*}{\lambda k : (\exists X.A^*).( \text{let } \langle X, c \rangle = k \text{ in } \llbracket M \rrbracket c ) : (\forall X.A)^k} (\forall I)$$

(2) Suppose that

$$\frac{M : \forall X.A}{MA_1 : A[X := A_1]} (\forall E)$$

Then we have the following proof figure:

$$\frac{\frac{\llbracket M \rrbracket : (\forall X.A)^k \quad \frac{[a : (A[X := A_1])^* = A^*[X = A_1^*]]}{\langle A_1^*, a \rangle_{\exists X.A^*} : \exists X.A^*} \quad (\exists I)}{\llbracket M \rrbracket \langle A_1^*, a \rangle_{\exists X.A^*} : \perp}}{\lambda a : (A[X := A_1])^* . \llbracket M \rrbracket \langle A_1^*, a \rangle_{\exists X.A^*} : (A[X := A_1])^k} (\exists I)$$

The other cases for  $(\Rightarrow I)$  and  $(\Rightarrow E)$  are the same as above.

Next, we can prove that if we have  $M_1 \rightarrow_{\lambda 2} M_2$  then  $\llbracket M_1 \rrbracket \rightarrow_{\lambda \exists}^+ \llbracket M_2 \rrbracket$  by induction on the derivation of well-typed terms. We show the cases of (3)  $(\beta_t)$  where  $\lambda X.M : \forall X.A$  and (4)  $(\eta_t)$ .

$$\begin{aligned} (3) \quad & \llbracket (\lambda X.M)A_1 \rrbracket \\ &= \lambda a : (A[X := A_1])^* . (\lambda k : (\exists X.A^*).( \text{let } \langle X, c \rangle = k \text{ in } \llbracket M \rrbracket c ) \langle A_1^*, a \rangle) \\ &\rightarrow_{\beta} \lambda a : (A[X := A_1])^* . ( \text{let } \langle X, c \rangle = \langle A_1^*, a \rangle \text{ in } \llbracket M \rrbracket c ) \\ &\rightarrow_{\text{let } \exists} \lambda a : (A[X := A_1])^* . \llbracket M \rrbracket [X := A_1^*] a \\ &= \lambda a : (A[X := A_1])^* . \llbracket M[X := A_1] \rrbracket a \quad \text{from Lemma 1} \\ &\rightarrow_{\eta} \llbracket M[X := A_1] \rrbracket \end{aligned}$$

$$\begin{aligned} (4) \quad & \llbracket \lambda X.MX \rrbracket \\ &= \lambda k : (\forall X.A)^* . ( \text{let } \langle X, c \rangle = k \text{ in } (\lambda a : (A[X := X])^* . \llbracket M \rrbracket \langle X, a \rangle ) c ) \\ &\rightarrow_{\beta} \lambda k : (\forall X.A)^* . ( \text{let } \langle X, c \rangle = k \text{ in } \llbracket M \rrbracket \langle X, c \rangle ) \\ &\rightarrow_{\text{let } \exists, \eta} \lambda k : (\forall X.A)^* . \llbracket M \rrbracket k \\ &\rightarrow_{\eta} \llbracket M \rrbracket \end{aligned}$$

□

## 5 Inverse translation and Galois embedding

We introduce a generation rule of  $\mathcal{R}$  à la [SF93], which describes the image of the CPS-translation closed under the reduction rules. We write  $R \in \mathcal{R}, \mathcal{R}^*$  for both  $R \in \mathcal{R}$  and  $R \in \mathcal{R}^*$ , and  $R_1, \dots, R_n \in \mathcal{R}$  for  $R_i \in \mathcal{R}$  ( $1 \leq i \leq n$ ).

**Definition 12 (Inductive Generation of  $\mathcal{R}$ )**

$$x \in \mathcal{R}, \mathcal{R}^* \quad A^* \in \mathcal{R}^*$$

$$\frac{R \in \mathcal{R} \quad R_1, \dots, R_n \in \mathcal{R}^* \quad a \notin FV(RR_1 \dots R_n) \quad n \geq 0}{\lambda a.R\langle R_1, \dots, R_n, a \rangle \in \mathcal{R}, \mathcal{R}^*}$$

$$\frac{\lambda a.W, R_1 \in \mathcal{R} \quad R_2, \dots, R_n \in \mathcal{R}^* \quad b \notin FV(R_1 \dots R_n W) \quad n \geq 0}{\lambda b.(\text{let } \langle x, a \rangle = \langle R_1, \dots, R_n, b \rangle \text{ in } W) \in \mathcal{R}, \mathcal{R}^*}$$

$$\frac{\lambda a.W \in \mathcal{R} \quad A_1^*, R_2, \dots, R_n \in \mathcal{R}^* \quad b \notin FV(R_2 \dots R_n W) \quad n \geq 0}{\lambda b.(\text{let } \langle X, a \rangle = \langle A_1^*, R_2, \dots, R_n, b \rangle \text{ in } W) \in \mathcal{R}, \mathcal{R}^*}$$

From the inductive definition above,  $R \in \mathcal{R}$  is in the form of either  $x$  or  $\lambda a.W$  for some  $W$ . It is important that terms with the pattern of  $\lambda a.W \in \mathcal{R}$  have the form such that the continuation variable  $a$  appears exactly once in  $W$  (linear continuation), since our source calculus is intuitionistic.

**Lemma 3 (Subject reduction property w.r.t.  $\mathcal{R}$ )** *The category  $\mathcal{R}$  is closed under the reduction rules of  $\lambda^\exists$ .*

*Proof.* Substitutions associated to the reduction rules are closed with respect to  $\mathcal{R}$ .  $\square$

Typing rules for  $R \in \mathcal{R}$  are defined in terms of those for  $\lambda^\exists$  as follows, denoted by  $\vdash_{\lambda^\exists_{\mathcal{R}}}$ . Here, we write  $R$  or  $\lambda a.W$  for denotations with type  $A^k$ , and  $C_a$  for continuations with type  $A^*$ , where  $C_a$  contains exactly one occurrence of the continuation variable  $a$  at the tail position:

$$C_a ::= a \mid \langle R, C_a \rangle \mid \langle A^*, C_a \rangle_{\exists X.A^*}$$

$$R ::= x \mid \lambda a.RC_a \mid \lambda a.\text{let } \langle x, a \rangle = C_a \text{ in } W \mid \lambda a.\text{let } \langle X, a \rangle = C_a \text{ in } W$$

where we write  $W$  for  $R \equiv \lambda a.W$ .

**Definition 13 (Typing rules for  $\mathcal{R}$ )**

$$\frac{x:A^k \in \Gamma^k}{\Gamma^k \vdash x:A^k} \quad \frac{}{\Gamma^k, a:A^* \vdash a:A^*}$$

$$\frac{\Gamma^k \vdash R:A^k \quad \Gamma^k, a:A_1^* \vdash C_a:B^*}{\Gamma^k, a:A_1^* \vdash \langle R, C_a \rangle:(A \Rightarrow B)^*} (\wedge I) \quad \frac{\Gamma^k, a:A_1^* \vdash C_a:A^*[X:=B^*]}{\Gamma^k, a:A_1^* \vdash \langle B^*, C_a \rangle_{\exists X.A^*}:(\forall X.A)^*} (\exists I)$$

$$\frac{\Gamma^k \vdash R:A^k \quad \Gamma^k, a:A_1^* \vdash C_a:A^*}{\Gamma^k \vdash \lambda a:A_1^*.RC_a:A_1^k} (\neg EI)$$

$$\frac{\Gamma^k, x:A^k \vdash \lambda b:B^*.W : B^k \quad \Gamma^k, a:A_1^* \vdash C_a : (A \Rightarrow B)^*}{\Gamma^k \vdash \lambda a:A_1^*.\text{let } \langle x, b \rangle = C_a \text{ in } W : A_1^k} (\wedge E)$$

$$\frac{\Gamma^k \vdash \lambda b:B^*.W : B^k \quad \Gamma^k, a:A_1^* \vdash C_a : (\forall X.B)^*}{\Gamma^k \vdash \lambda a:A_1^*.\text{let } \langle X, b \rangle = C_a \text{ in } W : A_1^k} (\exists E)^*$$

where  $(\exists E)^*$  denotes the eigenvariable condition  $X \notin FV(\Gamma)$ .

**Lemma 4 (Subject reduction property w.r.t. types)** *If we have  $R_1 : A^k$  and  $C_1 : B^*$ , respectively, together with  $R_1 \rightarrow_{\lambda\exists}^* R_2$  and  $C_1 \rightarrow_{\lambda\exists}^* C_2$ , then we also have  $R_2 : A^k$  and  $C_2 : B^*$ .*

*Proof.* The calculus  $\lambda^\exists$  has the subject reduction property.  $\square$

Following the patterns of  $\lambda a.W \in \mathcal{R}$ , we now give the definition of the inverse translation  $\sharp$  as  $(\lambda a.W)^\sharp = W^\sharp$ .

**Definition 14 (Inverse translation  $\sharp$  for  $\mathcal{R}$ )** (i)  $x^\sharp = x$ ;  $(A^*)^\sharp = A$

(ii)  $(R\langle R_1, \dots, R_n, a \rangle)^\sharp = R^\sharp R_1^\sharp \dots R_n^\sharp$

(iii)  $\bullet$   $(\text{let } \langle x, c \rangle = \langle R_1, \dots, R_n, a \rangle \text{ in } W)^\sharp = (\lambda x.W^\sharp)R_1^\sharp \dots R_n^\sharp$   
 $\bullet$   $(\text{let } \langle X, c \rangle = \langle R_1, \dots, R_n, a \rangle \text{ in } W)^\sharp = (\lambda X.W^\sharp)R_1^\sharp \dots R_n^\sharp$

**Proposition 2 (Completeness1)** (1) *If we have  $\Gamma^k \vdash_{\lambda_{\mathcal{R}}^\exists} R : A^k$ , then  $\Gamma \vdash_{\lambda 2} R^\sharp : A$ .*

(2) *If we have  $\Gamma^k, a:A^* \vdash_{\lambda_{\mathcal{R}}^\exists} C_a : B^*$ , then  $\Gamma, x:B \vdash_{\lambda 2} (\lambda a.xC_a)^\sharp : A$ .*

*Proof.* By simultaneous induction on the derivations.  $\square$

Let  $(\eta_a^-)$  be an  $\eta$ -expansion:  $R \rightarrow \lambda a:A^*.Ra$  where  $a \notin FV(R)$  and  $R \in \mathcal{R}$ . Then the set of well-typed  $\mathcal{R}$  becomes the image of the CPS-translation closed under the reduction rules, called *Univ*.

**Definition 15 (Universe of the CPS-translation)**

$$\text{Univ} \stackrel{\text{def}}{=} \{P \in \Lambda^\exists \mid \llbracket M \rrbracket \rightarrow_{\lambda_{\mathcal{R}}^\exists \eta_a^-}^* P \text{ for some well-typed } M \in \Lambda 2\}$$

**Lemma 5** *Univ is generated by  $\mathcal{R}$ , i.e.,  $\text{Univ} \subseteq \mathcal{R}$ .*

*Proof.* For well typed  $M \in \Lambda 2$ , we have  $\llbracket M \rrbracket \in \mathcal{R}$ , and moreover  $\mathcal{R}$  is closed under  $(\eta_a^-)$  and the reduction rules by Lemma 3.  $\square$

**Lemma 6** *For any  $P \in \text{Univ}$ , we have some  $\Gamma$  and  $A$  such that  $\Gamma^k \vdash_{\lambda\exists} P : A^k$ .*

*Proof.* From the definition of *Univ*, Proposition 1 and Lemma 4.  $\square$

**Proposition 3** (1) Let  $M \in \Lambda 2$  be a well-typed term. Then we have that  $\llbracket M \rrbracket^\sharp \equiv M$ .

(2) Let  $P \in \mathcal{R}$  be well-typed. Then we have that  $\llbracket P^\sharp \rrbracket \rightarrow_{\beta\eta_a^-}^* P$ .

(3) If  $P \in \mathcal{R}$  is a normal form of  $\lambda^\exists$ , then  $P^\sharp$  is a normal form of  $\lambda 2$ .

*Proof.*

(1) By induction on the structure of well-typed  $M \in \Lambda 2$ .

(2) By case analysis on  $P \in \text{Univ}$ , following the definition of  $\sharp$ .

(3) Following the cases of (i), (ii) and (iii) of the definition of  $\sharp$ .

- Case of (ii):

Let  $P$  be  $\lambda a:A^*.R\langle R_1, \dots, R_n, a \rangle$ . Since  $P$  is a normal form of  $\lambda^\exists$ , we have  $R \equiv x$  and  $R_i$  is also in normal with  $n \geq 1$ , to say,  $R_i^{nf}$ . Then we have normal  $P^\sharp = x(R_1^{nf})^\sharp \dots (R_n^{nf})^\sharp$ .

- Case of (iv) with  $n = 0$ :

Let  $P$  be  $\lambda a:A^*.(let \langle x, c \rangle = a \text{ in } W)$ . Since  $P$  is in normal, so is  $W$  without  $(let_{\wedge_\eta})$  nor  $(let_{\exists_\eta})$  redexes, to say,  $W^{nf}$ . Then we have normal  $P^\sharp = \lambda x.(W^{nf})^\sharp$ .

□

**Proposition 4** We have  $\text{Univ} = \mathcal{R}$  with respect to well-typed terms.

*Proof.* We have  $\text{Univ} \subseteq \mathcal{R}$  from Lemma 5. Let  $P \in \mathcal{R}$  be well-typed. Then  $P^\sharp \in \Lambda 2$  is well-typed from Proposition 2. Proposition 3 implies that  $\llbracket P^\sharp \rrbracket \rightarrow_{\beta\eta_a^-}^* P$ , and hence  $P \in \text{Univ}$ . Therefore we have  $\mathcal{R} \subseteq \text{Univ}$ . □

**Lemma 7**  $(W[a := \langle \langle R_1, \dots, R_m, b \rangle \rangle])^\sharp = W^\sharp R_1^\sharp, \dots, R_m^\sharp$  provided  $a \in FV(W)$ .

*Proof.* Following the case analysis on the definition  $\sharp$ . We show one case of  $W = (let \langle x, c \rangle = C_a \text{ in } W')$ , where  $C_a = \langle S_1, \dots, S_n, a \rangle$ . Let  $\theta$  be  $[a := \langle R_1, \dots, R_n, b \rangle]$ . We have  $W\theta = (let \langle x, c \rangle = \langle S_1, \dots, S_n, R_1, \dots, R_m, b \rangle \text{ in } W')$ , and then we have  $(W\theta)^\sharp = (\lambda x.(W')^\sharp) S_1^\sharp \dots S_n^\sharp R_1^\sharp \dots R_m^\sharp = (W)^\sharp R_1^\sharp \dots R_m^\sharp$ . □

**Proposition 5 (Completeness2)** Let  $P, Q \in \text{Univ}$ .

- (1) If  $P \rightarrow_\beta Q$  then  $P^\sharp \equiv Q^\sharp$ .
- (2) If  $P \rightarrow_\eta Q$  then  $P^\sharp \equiv Q^\sharp$ .
- (3) If  $P \rightarrow_{let_\wedge} Q$  then  $P^\sharp \rightarrow_\beta Q^\sharp$ .
- (4) If  $P \rightarrow_{let_\exists} Q$  then  $P^\sharp \rightarrow_{\beta_t} Q^\sharp$ .
- (5) If  $P \rightarrow_{let_{\wedge_\eta}} Q$  then  $P^\sharp \rightarrow_\eta Q^\sharp$ .

(6) If  $P \rightarrow_{\text{let}\exists_n} Q$  then  $P^\sharp \rightarrow_{\eta_t} Q^\sharp$ .

*Proof.* By induction on the structure of  $P$ , following the case analysis on the definition  $\sharp$ . The cases (1,2) are straightforward. We show some of the cases; (4):

Let  $P$  be  $\lambda a.\text{let } \langle X, c \rangle = \langle A^*, R_1, \dots, R_n, a \rangle \text{ in } W$ .

$$\begin{aligned} P^\sharp &= (\lambda X.W^\sharp)AR_1^\sharp \dots R_n^\sharp \\ &\rightarrow_{\beta_t} W^\sharp[X := A]R_1^\sharp \dots R_n^\sharp \\ &= (W[X := A])^\sharp R_1^\sharp \dots R_n^\sharp = (W[X := A][c := \langle R_1, \dots, R_n, a \rangle])^\sharp \end{aligned}$$

(6): Let  $P$  be  $\lambda a.\text{let } \langle X, c \rangle = a \text{ in } R\langle R_1, \dots, R_n, X, c \rangle$  where  $X, c \notin FV(RR_1 \dots R_n)$ .

$$P^\sharp = \lambda X.R^\sharp R_1^\sharp \dots R_n^\sharp X \rightarrow_{\eta_t} R^\sharp R_1^\sharp \dots R_n^\sharp$$

Let  $Q$  be  $\lambda a.\text{let } \langle X, c \rangle = a \text{ in let } \langle Y, b \rangle = \langle R_1, \dots, R_n, X, c \rangle \text{ in } W$  where  $X, c \notin FV(R_1 \dots R_n W)$ .

$$Q^\sharp = \lambda X.(\lambda Y.W^\sharp)R_1^\sharp \dots R_n^\sharp X \rightarrow_{\eta_t} (\lambda Y.W^\sharp)R_1^\sharp \dots R_n^\sharp$$

□

**Theorem 1** (i)  $\Gamma \vdash_{\lambda 2} M : A$  if and only if  $\Gamma^k \vdash_{\lambda \exists} \llbracket M \rrbracket : A^k$ .

(ii)  $P \in \text{Univ}$  if and only if  $\Gamma \vdash_{\lambda 2} P^\sharp : A$  for some  $\Gamma, A$ .

(iii) Let  $M_1, M_2$  be well-typed  $\lambda 2$ -terms.

$M_1 =_{\lambda 2} M_2$  if and only if  $\llbracket M_1 \rrbracket =_{\lambda \exists} \llbracket M_2 \rrbracket$ .

In particular,  $M_1 \rightarrow_{\lambda 2} M_2$  if and only if  $\llbracket M_1 \rrbracket \rightarrow_{\beta} \rightarrow_{\text{let}} \rightarrow_{\eta} \llbracket M_2 \rrbracket$ .

(iv) Let  $P_1, P_2 \in \text{Univ}$ .  $P_1 =_{\lambda \exists} P_2$  if and only if  $P_1^\sharp =_{\lambda 2} P_2^\sharp$ .

*Proof.* (i,ii) From Propositions 1 and 2. (iii, iv) From Propositions 1 and 5. □

**Theorem 2** The inverse translation  $\sharp : \text{Univ} \rightarrow \Lambda 2$  is bijective, in the following sense:

(1) If we have  $P_1^\sharp =_{\lambda 2} P_2^\sharp$  then  $P_1 =_{\lambda \exists} P_2$  for  $P_1, P_2 \in \text{Univ}$ .

(2) For any well-typed  $M \in \Lambda 2$ , we have some  $P \in \text{Univ}$  such that  $P^\sharp \equiv M$ .

*Proof.* Let  $M$  be a well-typed term of  $\Lambda 2$ . Then we can take  $P$  as  $\llbracket M \rrbracket$ , so that we have  $P^\sharp \equiv M$ . □

**Definition 16 (Galois connection)** Let  $\rightarrow_S^*$  and  $\rightarrow_T^*$  be pre-orders on  $S$  and  $T$  respectively, and  $f : S \rightarrow T$  and  $g : T \rightarrow S$  be maps. Two maps  $f$  and  $g$  form a Galois connection between  $S$  to  $T$  whenever  $f(M) \rightarrow_T^* P$  if and only if  $M \rightarrow_S^* g(P)$ , see also [SW97].

It is known that the definition above is equivalent to the following clauses:

- (i)  $M \rightarrow_S^* g(f(M))$
- (ii)  $f(g(P)) \rightarrow_T^* P$
- (iii)  $M_1 \rightarrow_S^* M_2$  implies  $f(M_1) \rightarrow_T^* f(M_2)$
- (iv)  $P_1 \rightarrow_T^* P_2$  implies  $g(P_1) \rightarrow_S^* g(P_2)$

**Definition 17 (Galois embedding)** Two maps  $f$  and  $g$  form a Galois embedding into  $T$  if they form a Galois connection and  $g(f(M)) \equiv M$ .

**Theorem 3** *The translations  $\llbracket \cdot \rrbracket$  and  $\sharp$  form a Galois connection between  $\lambda_2$  and  $\text{Univ}$ , and moreover, they establish a Galois embedding into  $\text{Univ}$ .*

*Proof.* From Propositions 1, 3, and 5. □

It is remarked that a Galois embedding is the dual notion of a reflection:  $f$  and  $g$  form a reflection in  $S$  if they form a Galois connection and  $f(g(P)) \equiv P$ . In fact, let  $M \rightarrow^- N$  (expansion) be  $N \rightarrow M$  (reduction). Then  $\rightarrow^*$  is a pre-order, and  $\langle \sharp, \llbracket \cdot \rrbracket, \rightarrow_{\lambda_2}^*, \rightarrow_{\lambda_2}^- \rangle$  forms a reflection.

Let  $\sharp \text{Univ}$  be  $\{P^\sharp \mid P \in \text{Univ}\}$ . Let  $\llbracket \sharp \text{Univ} \rrbracket$  be  $\{\llbracket M \rrbracket \mid M \in \sharp \text{Univ}\}$ .

**Corollary 1 (Kernel of  $\lambda_2$ )** *For any  $P \in \llbracket \sharp \text{Univ} \rrbracket$ , we have  $P \equiv \llbracket P^\sharp \rrbracket$ .*

*Proof.* Let  $\Lambda_2$  be a set of well-typed  $\lambda_2$ -terms. From the theorem above, we have  $\sharp \text{Univ} = \Lambda_2$  and  $\llbracket \sharp \text{Univ} \rrbracket = \llbracket \Lambda_2 \rrbracket$ . Hence, any  $P \in \llbracket \Lambda_2 \rrbracket$  is in the form  $P \equiv \llbracket M \rrbracket$  for some  $M \in \Lambda_2$ , such that  $\llbracket P^\sharp \rrbracket \equiv \llbracket \llbracket M \rrbracket^\sharp \rrbracket \equiv \llbracket M \rrbracket \equiv P$ . □

**Corollary 2 (Normalization of  $\lambda_2$ )** *The weak normalization of  $\lambda_2$  is inherited from that of  $\lambda^\exists$  ( $\lambda_{\mathcal{R}}^\exists$ ). Moreover, the strong normalization of  $\lambda_2$  is implied by that of  $\lambda^\exists$  ( $\lambda_{\mathcal{R}}^\exists$ ).*

*Proof.* The weak normalization of  $\lambda_2$  is implied by Theorem 3 ( $\llbracket \cdot \rrbracket$  and  $\sharp$  form a Galois connection) together with Proposition 3. The strong normalization of  $\lambda_2$  is implied by Proposition 1 (soundness). □

**Corollary 3 (Church-Rosser of  $\lambda_2$ )** *The Church-Rosser property of  $\lambda_2$  is inherited from that of  $\mathcal{R}$ .*

*Proof.* The Church-Rosser property of  $\lambda_2$  is implied by Theorem 3. □

We remark that the system  $\lambda^\exists$  can be regarded logically as a subsystem of  $F$ , in the sense that the connectives  $\wedge$  and  $\exists$  together with the reduction rules can be coded by universal types of  $F$  [GTL89]. Our result, in turn, means that universal types can be interpreted by the use of existential types. Moreover, proof duality appears in the proof such that  $(\forall I) \leftrightarrow (\exists E)$  and  $(\forall E) \leftrightarrow (\exists I)$ .

## 6 Proof duality between polymorphic functions and abstract data types

We discuss the proof duality in detail. If we have  $\Gamma \vdash_{\lambda 2} A$  in  $\lambda 2$ , then classical logic has  $A^* \vdash \Gamma^*$ , turning assumptions into conclusions and vice versa. In terms of intuitionistic logic, we can expect that  $\neg\Gamma^*, A^* \vdash \perp$ . In fact, we have  $\neg\Gamma^*, a:A^* \vdash_{\lambda\exists} \underline{\underline{M}} : \perp$  if  $\Gamma \vdash_{\lambda 2} M : A$ , under the following definition.

**Definition 18 (Modified CPS-translation)**

- (i)  $\underline{\underline{x}} = xa$
- (ii)  $\underline{\underline{\lambda x:A_1.M}} = \text{let } \langle x, a \rangle = a \text{ in } \underline{\underline{M}}$
- (iii)  $\underline{\underline{M_1 M_2}} = \underline{\underline{M_1}}[a := \langle \lambda a:A_1^*. \underline{\underline{M_2}}, a \rangle]$  for  $M_2 : A_1$
- (iv)  $\underline{\underline{\lambda X.M}} = \text{let } \langle X, a \rangle = a \text{ in } \underline{\underline{M}}$
- (v)  $\underline{\underline{M A_1}} = \underline{\underline{M}}[a := \langle A_1^*, a \rangle]$

**Lemma 8** *Let  $M \in \Lambda 2$  be a well-typed term.*

- (1) *We have  $\llbracket M \rrbracket a \rightarrow_{\beta\eta_a}^* \underline{\underline{M}}$  and  $\llbracket M \rrbracket \rightarrow_{\beta\eta_a}^* \lambda a. \underline{\underline{M}}$ .*
- (2)  $\llbracket M \rrbracket^\sharp = (\underline{\underline{M}})^\sharp \equiv M$
- (3) *If  $M$  is a normal form of  $\lambda 2$ , then  $\underline{\underline{M}}$  is a normal form of  $\lambda^\exists$  without  $(\eta_a)$ . The form of normal  $\underline{\underline{M}}$  without  $(\eta_a)$  is described by  $NF$  as follows:*

$$\begin{aligned}
 NF &::= xa \\
 &\quad | \text{let } \langle \chi, a \rangle = k \text{ in let } \langle \chi, a \rangle = k \text{ in } \dots \\
 &\quad \text{let } \langle \chi, a \rangle = k \text{ in } x \langle Nf, \dots, Nf, a \rangle
 \end{aligned}$$

where  $Nf ::= A^* \mid \lambda a. NF$ , and we write  $\chi$  for either  $x$  or  $X$ .

*Proof.* By induction on the structure of  $M$ , and Proposition 3. □

The notion of path is defined as in Prawitz [Pra65], together with inference rules.

**Definition 19 (Path)** A sequence consisting of formulae and inference rules  $A_1(R_1)A_2(R_2)\dots A_{n-1}(R_{n-1})A_n$  is defined as a path in the deduction  $\Pi$  of  $\lambda 2$  or  $\lambda^\exists$ , as follows:

- (i)  $A_1$  is a top-formula in  $\Pi$ ;

- (ii)  $A_i$  ( $i < n$ ) is not the minor premiss of an application of  $(\Rightarrow E)$  or  $(\neg E)$ , and either
- 1)  $A_i$  is not a major premiss of  $(\wedge E)$  or  $(\exists E)$ , and  $A_{i+1}$  is the formula occurrence immediately below  $A_i$  by an application of  $(R_i)$ , or
  - 2)  $A_i$  is the major premiss of an application of  $(\wedge E)$  or  $(\exists E)$ , and  $A_{i+1}$  is the assumption discharged in  $\Pi$  by  $(\wedge E)$  or  $(\exists E)$ , to say,  $(R_i)$ ;
- (iii)  $A_n$  is either a minor premiss of  $(\Rightarrow E)$  or  $(\neg E)$ , or the end-formula of  $\Pi$ .

We write  $(I)$  for either  $(\Rightarrow I)$  or  $(\forall I)$ , and  $(E)$  for either  $(\Rightarrow E)$  or  $(\forall E)$ . We also define inference rule correspondence as follows:  $(\Rightarrow I)^* = (\wedge E)$ ,  $(\Rightarrow E)^* = (\wedge I)$ ,  $(\forall I)^* = (\exists E)$ ,  $(\forall E)^* = (\exists I)$ .

**Theorem 4 (Proof duality)** *Let  $\Pi$  be a normal deduction of  $\Gamma \vdash_{\lambda 2} M : A$ , and let  $\pi$  be a path  $A_1(E_1)A_2(E_2) \dots A_i(E_i)A_{i+1}(I_{i+1}) \dots A_{n-1}(I_{n-1})A_n$  in the normal deduction. Then, in the deduction of  $\neg\Gamma^*, a : A^* \vdash_{\lambda 3} \underline{M} : \perp$ , there exists a path  $\pi^*$ , as follows:*

$$\pi^* = A_n^*(I_{n-1})^* A_{n-1}^* \dots (I_{i+1})^* A_{i+1}^* (E_i)^* A_i^* \dots (E_2)^* A_2^* (E_1)^* A_1^*.$$

*Proof.* By induction on the normal derivation of  $\Gamma \vdash_{\lambda 2} M : A$ . We show here some of the cases:

(0) Case of  $n = 1$ :

For  $x : A$ , we have the following deduction:

$$\frac{x : \neg A^* \quad a : A^*}{xa : \perp} (\neg E)$$

which means that the corresponding path ends with the minor premiss of  $(\neg E)$ , just before  $\perp$ .

(1)  $A_n$  ( $n = i + 1$ ) is derived by an elimination rule.

Case of  $(\Rightarrow E)$ :

From a normal deduction  $\Pi$ ,  $(B \Rightarrow A_n)$  cannot be derived by an introduction rule:

$$\frac{\frac{\Pi_1}{M_1 : B \Rightarrow A_n} \quad \frac{\Pi_2}{M_2 : B}}{M_1 M_2 : A_n} (\Rightarrow E)$$

Then we have a path  $\pi_1^d$  from  $(B \Rightarrow A_n)^*$ , corresponding to the path  $\pi_1$  to  $(B \Rightarrow A_n)$ :

$$\begin{array}{ccc} a : (B \Rightarrow A_n)^* & & a : B^* \\ \Sigma_1 & & \Sigma_2 \\ \underline{M_1} : \perp & & \underline{M_2} : \perp \end{array}$$

The figure below says that we have a path  $\pi^d = (A_n)^*(\Rightarrow E)^*(B \Rightarrow A_n)^*\pi_1^d$ , corresponding to the path  $\pi = \pi_1(B \Rightarrow A_n)(\Rightarrow E)(A_n)$ :

$$\frac{\frac{\frac{[a : B^*]}{\Sigma_2} \quad \frac{\underline{\underline{M_2}} : \perp}{\lambda a. \underline{\underline{M_2}} : \neg B^*} (\neg I) \quad a : A_n^*}{\langle \lambda a. \underline{\underline{M_2}}, a \rangle : \neg B^* \wedge A_n^*} (\wedge I)}{\underline{\underline{M_1}}[a := \langle \lambda a. \underline{\underline{M_2}}, a \rangle] : \perp} \Sigma_1$$

Case of  $(\exists E)$ :

From a normal deduction of  $\Pi$ ,  $\forall X.A_n$  cannot be derived from an introduction rule:

$$\frac{\frac{\Pi_1}{M : \forall X.A_n}}{MB : A_n[X := B]} (\forall E)$$

Then we have a path  $\pi_1^d$  from  $(\forall X.A_n)^*$ , corresponding to the path  $\pi_1$  to  $\forall X.A_n$ , as follows:

$$\frac{a : (\forall X.A_n)^*}{\underline{\underline{M}} : \perp} \Sigma_1$$

The following figure shows that we have a path  $\pi^d = (A_n[X := B])^*(\forall E)^*(\forall X.A_n)^*\pi_1^d$ , corresponding to the path  $\pi = \pi_1(\forall X.A_n)(\forall E)(A_n[X := B])$ :

$$\frac{\frac{a : (A_n[X := B])^*}{\langle B^*, a \rangle : \exists X.A_n^*} (\exists I)}{\underline{\underline{M}}[a := \langle B^*, a \rangle] : \perp} \Sigma_1$$

(2)  $A_n$  ( $n = i + 2$ ) is derived by an introduction rule.

Case of  $(\Rightarrow I)$ :

$$\frac{\frac{[x : B]}{\Pi_1} (E)}{\lambda x. M : B \Rightarrow A_{n-1}} (\Rightarrow I)$$

Then we have the path  $\pi_1^d$  from  $A_{n-1}^*$ , corresponding to the path  $\pi_1$  to  $A_{n-1}$ :

$$\frac{x : \neg B^* \quad a : A_{n-1}^*}{\underline{\underline{M}} : \perp} \Sigma_1$$

Now we have the following figure, so that there exists a path  $\pi^d = (B \Rightarrow A_{n-1})^*(\Rightarrow I)^*(A_{n-1}^*)\pi_1^d$ , corresponding to the path  $\pi = \pi_1(A_{n-1})(\Rightarrow I)(B \Rightarrow A_{n-1})$ :

$$\frac{[x : \neg B^*] \quad [a : A_{n-1}^*] \quad \frac{\frac{a' : \neg B^* \wedge A_{n-1}^*}{\text{let } \langle x, a \rangle = a' \text{ in } \underline{\underline{M}} : \perp} \quad \frac{\Sigma_1 \quad \underline{\underline{M}} : \perp}{\underline{\underline{M}} : \perp}}{(\wedge E)}$$

Case of  $(\forall I)$ :

$$\frac{\frac{\frac{\Pi_1 \quad A_{n-2}}{M : A_{n-1}} (E)}{\lambda X. M : \forall X. A_{n-1}} (\forall I)}$$

Then we have the path  $\pi_1^d$  from  $A_{n-1}^*$ , corresponding to the path  $\pi_1$  to  $A_{n-1}$ , as follows:

$$\frac{a : A_{n-1}^* \quad \Sigma_1 \quad \underline{\underline{M}} : \perp}{\underline{\underline{M}} : \perp}$$

Now we have the path  $\pi^d = (\forall X. A_{n-1})^* (\forall I)^* (A_{n-1})^* \pi_d^1$ , corresponding to the path  $\pi(A_{n-1})(\forall I)(\forall X. A_{n-1})$  to  $\forall X. A_{n-1}$ :

$$\frac{[a : A_{n-1}^*] \quad \frac{\frac{a' : \exists X. A_{n-1}^* \quad \underline{\underline{M}} : \perp}{\text{let } \langle X, a \rangle = a' \text{ in } \underline{\underline{M}} : \perp} \quad \Sigma_1 \quad \underline{\underline{M}} : \perp}{(\exists E)}$$

(3)  $A_n$  ( $n > i + 2$ ) is derived by an introduction rule.

$$\frac{\frac{\frac{\Pi_1 \quad A_{n-2}}{A_{n-1}} (I_{n-2})}{M : A_n} (I_{n-1})$$

Case  $I_{n-1}$  of  $(\Rightarrow I)$ :

$$\frac{[x : A] \quad \frac{\frac{\Pi_1 \quad A_{n-2}}{M : A_{n-1}} (I_{n-2})}{\lambda x. M : A \Rightarrow A_{n-1}} (\Rightarrow I)}$$

From the induction hypothesis, there exists the following deduction

$$\frac{a : A_{n-1}^* \quad \frac{x : \neg A^* \quad \vdots \quad \perp}{\underline{\underline{M}} : \perp}}{(I_{n-1})^*}$$

where we have a path  $\pi_1^d$  from  $A_{n-1}^*$ , corresponding to the path  $\pi_1$  to  $A_{n-1}$ . Then in the following deduction:

$$\frac{a' : \neg A^* \wedge A_{n-1}^* \quad \frac{[a : A_{n-1}^*] \quad \frac{\vdots}{\perp} (I_{n-1})^*}{\underline{M} : \perp} (\Rightarrow I)^*}{\text{let } \langle x, a \rangle = a' \text{ in } \underline{M} : \perp} (\Rightarrow I)^*$$

we obtain the path  $\pi^d = (A \Rightarrow A_{n-1})^* (\Rightarrow I)^* (A_{n-1})^* \pi_1^d$ , corresponding to the path  $\pi = \pi_1(A_{n-1})(\Rightarrow I)(A \Rightarrow A_{n-1})$  to  $(A \Rightarrow A_{n-1})$ .

Case  $I_{n-1}$  of  $(\forall I)$ :

$$\frac{\frac{\Pi_1}{A_{n-2}} (I_{n-2})}{M : A_{n-1}} (\forall I)$$

Then in the following deduction:

$$\frac{a : A_{n-1}^* \quad \frac{\vdots}{\perp} (I_{n-2})^*}{\underline{M} : \perp}$$

we have a path  $\pi_1^d$  from  $A_{n-1}^*$ , corresponding to the path  $\pi_1$  to  $A_{n-1}$ . Now the following deduction gives the path  $\pi^d = (\forall X.A_{n-1})^* (\forall I)^* (A_{n-1})^* \pi_1^d$ , corresponding to the path  $\pi = \pi_1(A_{n-1})(\forall I)(\forall X.A_{n-1})$ :

$$\frac{a' : \exists X.A_{n-1}^* \quad \frac{[a : A_{n-1}^*] \quad \frac{\vdots}{\perp} (I_{n-2})^*}{\underline{M} : \perp} (\forall I)^*}{\text{let } \langle X, a \rangle = a' \text{ in } \underline{M} : \perp}$$

□

It is remarked that from the theorem above, inference rules in a path of normal deductions of  $\lambda 2$  are reversely applied in the corresponding path of  $\lambda^\exists$ , under the correspondence between  $(\forall I)$  and  $(\exists E)$ ;  $(\forall E)$  and  $(\exists I)$ ; etc. Moreover, together with lemma 8, normal forms have the following shape:

$$\begin{aligned} NF &::= xa \\ &\mid \text{let } \langle \chi, a_1 \rangle = a \text{ in let } \langle \chi, a_2 \rangle = a_1 \text{ in } \dots \\ &\quad \text{let } \langle \chi, a_n \rangle = a_{n-1} \text{ in } x(Nf, \dots, Nf, a_n) \end{aligned}$$

where  $Nf ::= A^* \mid \lambda a.NF$ .

## 7 Concluding remarks

It is remarked that  $\lambda^\exists$  can be regarded as a subsystem of  $\lambda 2$ , in the sense that  $\wedge$  and  $\exists$  with reduction rules can be impredicatively coded in  $\lambda 2$ . We have established a Galois embedding from polymorphic  $\lambda 2$  into  $\lambda^\exists$ , in which proof duality appears such that polymorphic functions with  $\forall$ -type can be interpreted by abstract data types with  $\exists$ -type [MP85] and vice versa. Moreover, inference rules in a path of normal deductions of  $\lambda 2$  are reversely applied in the corresponding dual paths of  $\lambda^\exists$ , under the correspondence between  $(\forall I)$  and  $(\exists E)$ ;  $(\forall E)$  and  $(\exists I)$ ; etc. The involved CPS-translation is similar to that of [Plot75], [HS97], [Seli01] or [Fuji03]. However, relating to extensionality, the case of conjunction-elimination is essentially distinct from them, and this point is important for the completeness. Although none of two through [Plot75], [HS97] and ours in this paper are  $\beta\eta$ -equal, we remark that they are isomorphic to each other in the simply typed case, from the work on answer type polymorphism by Thielecke [Thie04]. Our definition of the CPS-translation can work even for polymorphic  $\lambda\mu$ -calculus (second order classical logic) [Pari92].

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