

Note on shape theory

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§1. Shape of compacta.

In [2], [3] K. Borsuk introduced the notion of shapes of metric compacta. Let X and Y be compacta lying in the Hilbert cube $Q (= \prod_{n=1}^{\infty} I_n, I_n \text{ a copy of the interval } I = [0,1], n \in \mathbb{N}, \text{ where } \mathbb{N} \text{ is the set of positive integers})$. A sequence $\underline{f} = \{f_n\}$ of maps ($=$ continuous maps) $f_n : Q \rightarrow Q, n \in \mathbb{N}$, is said to be a fundamental sequence of X to Y if for each neighborhood V of Y (in Q) there is a neighborhood U of X such that $f_n|_U \simeq f_{n+1}|_U$ in V for almost all n , that is, there is a homotopy $H : U \times I \rightarrow V$ such that $H(x,0) = f_n(x)$ and $H(x,1) = f_{n+1}(x)$ for $x \in U$. We write $\underline{f} : X \rightarrow Y$. Setting $i_n(x) = x$ for each $x \in Q$, for each compactum $X \subset Q$ $\underline{i}_X = \{i_n\} : X \rightarrow X$ is a fundamental sequence which is called the fundamental identity sequence of X .

Two fundamental sequences $\underline{f}, \underline{g} : X \rightarrow Y$ are said to be homotopic if for each neighborhood V of Y there is a neighborhood U of X such that $f_n|_U \simeq g_n|_U$ in V for almost all n . We denote it by $\underline{f} \simeq \underline{g}$. The collection of all fundamental sequences ho-

motopic to a given fundamental sequence $\underline{f}$ is said to be the fundamental class with the representative $\underline{f}$ and it is denoted by $[\underline{f}]$.

The composition $\underline{h} = \underline{g}\underline{f} : X \rightarrow Z$ of fundamental sequences $\underline{f} : X \rightarrow Y$ and $\underline{g} : Y \rightarrow Z$ is defined as the fundamental sequence consisting of maps $h_n = g_n f_n : Q \rightarrow Q$. If $\underline{f} \simeq \underline{f}' : X \rightarrow Y$ and $\underline{g} \simeq \underline{g}' : Y \rightarrow Z$, then $\underline{g}\underline{f} \simeq \underline{g}'\underline{f}' : X \rightarrow Z$.

Each map $f : X \rightarrow Y$ defines a fundamental sequence $\underline{f} : X \rightarrow Y$ as follows. Take any extension $h : Q \rightarrow Q$ of f and put $f_n = h$ for each n . Then $\underline{f} = \{f_n\}$ is a fundamental sequence of X to Y . We call $\underline{f}$ the fundamental sequence induced by f .

Proposition 1. Let X and Y be 0-dimensional compacta. Then every fundamental sequence $\underline{f} : X \rightarrow Y$ is induced by a map $f : X \rightarrow Y$ and f is uniquely determined by $\underline{f}$.

Proof. Let $\underline{f} = \{f_n\} : X \rightarrow Y$. From the definition of a fundamental sequence and the compactness of Q the sequence $\{f_n(x)\}$, $x \in X$, converges some point $f(x)$ of Y . Obviously the correspondence $x \rightarrow f(x)$, $x \in X$, defines a map $f : X \rightarrow Y$ and it induces $\underline{f}$.

Compacta X and Y in Q are said to be fundamentally equivalent if there exist two fundamental sequences $\underline{f} : X \rightarrow Y$ and $\underline{g} : Y \rightarrow X$ such that $\underline{g}\underline{f} \simeq \underline{i}_X$ and $\underline{f}\underline{g} \simeq \underline{i}_Y$. Then we write $X \simeq_F Y$. If we assume only that the relation $\underline{g}\underline{f} \simeq \underline{i}_X$ holds, then we say that Y fundamentally dominates X and we write $Y \geq_F X$. If X and Y are homeomorphic, then $X \simeq_F Y$, because if $\underline{f} : X \rightarrow Y$ and $\underline{g} : Y \rightarrow X$ are fundamental sequences induced by f and $g = f^{-1}$ then

$gf \simeq i_X$ and $fg \simeq i_Y$. Also, if X and Y are homotopically equivalent, then $X \simeq_F Y$.

It is known that the relation of the fundamental equivalence and the relation of the fundamental domination have an absolute character, that is, they do not depend on locations of compacta X and Y in Q . Since the relation of the fundamental equivalence is equivalence relation, the class of all compacta decomposes into mutually disjoint classes of compacta, called shapes. We denote by $Sh(X)$ the class containing X and we call it the shape of X . Also we write $Sh(X) \geq Sh(Y)$ if $X \geq_F Y$. If X and Y are ANR's (= compact ANR's for metric spaces), then it is known that $Sh(X) \geq Sh(Y)$ if and only if X dominates homotopically Y and $Sh(X) = Sh(Y)$ if and only if X and Y have same homotopy type. The shape of a space consisting of only one point is said to be trivial and denote by $Sh(1)$. If X is contractible, then it is obvious $Sh(X) = Sh(1)$.

Let X be a compactum contained in a compactum Y . A fundamental sequence $\underline{f} = \{f_n\} : Y \rightarrow X$ is said to be a fundamental retraction if $\underline{f}j_X \simeq i_X$, where $j_X : X \rightarrow Y$ is a fundamental sequence induced by the inclusion $j_X : X \subset Y$. If there is a fundamental retraction $\underline{f} : Y \rightarrow X$, then we call X a fundamental retract of Y . If there is a fundamental retraction $\underline{f} : Y \rightarrow X$ such that $\underline{f} \simeq i_Y$, then X is a fundamental deformation retract of Y . A compactum X is said to be a fundamental absolute retract (a fundamental absolute neighborhood retract), written as FAR (FANR), if X is a fundamental retract of an AR (an ANR).

The following theorem characterizes a compactum with trivial shape ([4],[8],[10]).

Theorem 1. (K.Borsuk, D.M.Hyman, S.Mardešić) For a compactum X the followings are equivalent.

- (1) X is of trivial shape.
- (2) X is an FAR.
- (3) For a certain imbedding $X \subset Q$ there is a sequence $\{X_n\}$ of neighborhoods of X such that each X_n is homeomorphic to Q, $X_{n+1} \subset \text{Interior } X_n$, $n \in \mathbb{N}$, and $\bigcap_n X_n = X$.

For a compactum X, Borsuk defined $Fd(X)$, the fundamental dimension of X, as the minimum of dimensions of all compacta Y with $Sh(Y) \geq Sh(X)$:

$$Fd(X) = \min_{Sh(Y) \geq Sh(X)} \dim Y$$

Obviously it holds that if $Sh(X) \leq Sh(Y)$ then $Fd(X) \leq Fd(Y)$ and if X and Y are compacta and $Y \neq \emptyset$ then $Fd(X) \leq Fd(X \times Y) \leq Fd(X) + Fd(Y)$.

Let X be a compactum. A closed subset Y of X is said to be a fundamental k-skeleton of X if $\dim Y \leq k$ and the homomorphisms $\check{H}_n(Y;G) \rightarrow \check{H}_n(X;G)$ and $\pi_n(Y, y_0) \rightarrow \pi_n(X, y_0)$ induced by the inclusion $(Y, y_0) \subset (X, y_0)$, y_0 a point of Y, are isomorphisms for $0 \leq n < k$ and an epimorphism for $n = k$, where $\check{H}_n(X;G)$ is the n-dimensional Čech homology group of X with coefficients in G and $\pi_n(X, y_0)$ is the n-dimensional fundamental group of (X, y_0) defined by Borsuk [3].

We do not know whether every compactum has a fundamental 0-skeleton or not. If X is a solenoid of Van Dantzig, then X has a fundamental 0-skeleton which is homeomorphic to a Cantor discontinuum. (See Corollary of Theorem 5).

§2. Approach to shapes by Mardešić and Segal.

By an ANR-sequence we imply an inverse sequence $\underline{X} = \{X_n, \pi_{nn+1}\}$ over the set of positive integers N , where X is an ANR and $\pi_{nn+1} : X_{n+1} \rightarrow X_n$ is a map, $n \in N$ ($\pi_{nm} = \pi_{n+1, n+2} \cdots \pi_{m-1, m}$ for $n < m$). Let $\underline{X} = \varprojlim \underline{X}$ and let $\pi_n : \underline{X} \rightarrow X_n$ be the projection. A map $\underline{f} : \underline{X} \rightarrow \underline{Y} = \{Y_n, \mu_{nn+1}\}$ consists of an increasing function $f : N \rightarrow N$ and of a collection of maps $f_n : X_{f(n)} \rightarrow Y_n$ such that

$$f_n \pi_{f(n)f(n')} \simeq \mu_{nn'} f_{n'}, \quad \text{for } n \leq n', n, n' \in N.$$

Two maps $\underline{f}, \underline{g} : \underline{X} \rightarrow \underline{Y}$ are said to be homotopic, $\underline{f} \simeq \underline{g}$, if for each $n \in N$ there is an $n' \in N$, $n' \geq f(n), g(n)$, such that

$$f_n \pi_{f(n)n'} \simeq g_n \pi_{g(n)n'}.$$

The composite $\underline{gf} : \underline{X} \rightarrow \underline{Z}$ of $\underline{f} : \underline{X} \rightarrow \underline{Y}$ and $\underline{g} : \underline{Y} \rightarrow \underline{Z} = \{Z_n, \nu_{nn+1}\}$ is a map of sequences $\underline{h} : \underline{X} \rightarrow \underline{Z}$, where $h = fg : N \rightarrow N$ and $h_n = g_n f_{g(n)} : X_{fg(n)} \rightarrow Z_n$. The identity map of sequences $\underline{i}_X : \underline{X} \rightarrow \underline{X}$ is given by the identity $1_N : N \rightarrow N$ and the map $i_{X_n} : X_n \rightarrow X_n$, $n \in N$. Two compacta X and Y are said to be of the same shape in the sense of ANR-systems, written as $\overline{\text{Sh}}(X) = \overline{\text{Sh}}(Y)$, provided there exist ANR-systems $\underline{X}$ and $\underline{Y}$ with $X = \varprojlim \underline{X}$ and $Y = \varprojlim \underline{Y}$ and maps $\underline{f} : \underline{X} \rightarrow \underline{Y}$ and $\underline{g} : \underline{Y} \rightarrow \underline{X}$ such that $\underline{gf} \simeq \underline{i}_X$ and $\underline{fg} \simeq \underline{i}_Y$.

Mardešić and Segal [15, 16] gave the following useful cha-

racterization of shapes.

Theorem 2. (S.Mardešić and J.Segal) Let X and Y be compacta. Then $\text{Sh}(X) = \text{Sh}(Y)$ if and only if $\overline{\text{Sh}}(X) = \overline{\text{Sh}}(Y)$.

§ 2. Shape of decomposition spaces.

According to Borsuk [4,p.266], a compactum X is said to be approximatively k -connected if for a certain imbedding $X \subset Q$ and for every neighborhood V of X in Q there is a neighborhood U of X such that every map of a k -sphere S^k into U is null homotopic in V . It is known that the approximative k -connectedness is the shape invariant.

Theorem 3. (Kodama) Let f be a map of a compactum X onto a compactum Y with $\dim Y \leq n$ such that for each $y \in Y$ $f^{-1}(y)$ is approximatively k -connected, $k = 0, 1, \dots, n$. Then $\text{Sh}(X) \geq \text{Sh}(Y)$. Moreover, if $\dim X \leq n$ then $\text{Sh}(X) = \text{Sh}(Y)$.

In the proof of Theorem 3 ([11]) an argument in the proof of Theorem of [9] is used essentially.

The following corollary is a generalization of Borsuk [4, Theorem (6.1)].

Corollary 1. An n -dimensional compactum X is of trivial shape if and only if X is approximatively k -connected for $k = 0, 1, \dots, n$.

For the proof it is sufficient to apply Theorem 3 to the case where Y is a space consisting of one point.

Corollary 2. (R.B.Sher) If X and Y are finite dimensional and f is a map of X onto Y such that $f^{-1}(y)$ is of trivial shape

This is an immediate consequence of Theorems 1 and 3.

For a compactum X , denote by $\square(X)$ the set of all components of X . We consider $\square(X)$ as the decomposition space of X . Then it is a compactum. As an application of Theorem 3, we obtain the following theorem by Borsuk [3, Theorem (8.1)].

Corollary 3. (Borsuk) Let X, Y be compacta in Q . Then for every fundamental sequence $f : X \rightarrow Y$ there is a unique (continuous) map $\Lambda_f : \square X \rightarrow \square Y$ such that for each component X_0 of X $f : X_0 \rightarrow \Lambda_f(X_0)$ is a fundamental sequence. Moreover Λ_f depends only on the fundamental class f and this dependence is covariant, that is, if $g : Y \rightarrow Z$ is a fundamental sequence then $\Lambda_{gf} = \Lambda_g \Lambda_f$.

Proof. Let $\pi_X : X \rightarrow \square X$ and $\pi_Y : Y \rightarrow \square Y$ be the decomposition maps. Since $\pi_X^{-1}(x)$ is a continuum for each $x \in \square X$, it is approximatively 0-connected. Since $\dim \square X = 0$, by Theorem 3 there is a fundamental sequence $h : \square X \rightarrow X$ such that $\pi_X h \simeq i_{\square X}$. Consider $\pi_Y f h : \square X \rightarrow \square Y$. By Proposition 1, $\pi_Y f h$ is induced by a map $\Lambda_f : \square X \rightarrow \square Y$. It is obvious that Λ_f satisfies Corollary 3.

The following generalizes Sher [21, Theorem 12] and it is given by a similar method as in a proof of Theorem 3 (cf. [11]).

Corollary 4. Let (X, x_0) and (Y, y_0) be pointed compacta. Let f be a map of (X, x_0) onto (Y, y_0) . If $f^{-1}(y)$ is approximatively k -connected for each $y \in Y$ and $k = 0, 1, \dots, n$, then the induced homomorphism $f_* : \pi_k(X, x_0) \rightarrow \pi_k(Y, y_0)$ is an isomorphism for $k = 0, 1, \dots, n$, where π_k is the k -dimensional fundamen-

tal group of Borsuk [3].

Corollary 5. Let f be a map of a compactum X onto an n -dimensional compactum Y such that $f^{-1}(y)$ is approximately k -connected for each $y \in Y$ and $k = 0, 1, \dots, n$. Then $Fd(X) \geq Fd(Y)$.

4. Δ -spaces and fundamental dimension.

A compactum X is said to be a Δ -space if there is an inverse sequence $\{K_n, \pi_{n+1}\}$ of finite simplicial complexes such that $X = \varprojlim \{K_n\}$ and each bonding map $\pi_{n+1}: K_{n+1} \rightarrow K_n$ is simplicial.

Theorem 4. (Kodama) (1) Every 0-dimensional compactum and every finite polytope are Δ -spaces.

(2) There is a 1-dimensional AR with property (Δ) which is not a Δ -space.

(3) Every Δ -space is dimensionally full-valued for paracompact spaces (cf. [14]).

In the shape category every compactum has a Δ -space as its representative as shown by the following.

Theorem 5. (Kodama) For each compactum X there is a Δ -space X' such that $Sh(X) = Sh(X')$ and $Fd(X) = Fd(X')$.

Corollary. For every compactum X there is a compactum X' such that X' contains X as a fundamental deformation retract and X' has a fundamental k -skeleton for each $k = 0, 1, 2, \dots$.

We only give a proof of Corollary. For a given compactum X , find a Δ -space Y by Theorem 5 such that $Sh(X) = Sh(Y)$. By Moszyńska [20] there is a compactum X' such that both X and Y

are fundamental deformation retracts of X' . Let $\{K_n, \pi_{n+1}\}$ be an inverse sequence of finite simplicial complexes such that $Y = \varprojlim \{K_n\}$ and each π_{n+1} is simplicial. Denote by K_n^i the i -skeleton of K_n , $i = 0, 1, \dots$. Then $\{K_n^i, \pi_{n+1}\}$ forms an inverse sequence. Put $Y_i = \varprojlim \{K_n^i\}$, $i = 0, 1, \dots$. Then it is obvious that Y_i is a fundamental i -skeleton of X' .

As shown in the above, every Δ -space has a fundamental k -skeleton for each $k = 0, 1, 2, \dots$. On the other hand, consider the 2-dimensional continuum $Q(\sigma)$ constructed in [10, p.390]. It is easy to know that $Q(\sigma)$ has no fundamental 1-skeleton. Also, we can see that every ANR has a fundamental i -skeleton for $i = 0, 1$. The following example is a trivial modification of the example constructed by Borsuk [1].

Example. There is an infinite dimensional ANR X which does not have a fundamental k -skeleton for each $k = 2, 3, \dots$.

To find such an ANR X , let S^2 be a 2-sphere and let A be an arc in S^2 . Take a map f from A onto the Hilbert cube Q and let X be the adjunction space obtained by S^2 , Q and f . Then X is an infinite dimensional ANR. If X_k is a fundamental k -skeleton of X for $k \geq 2$, then X_k has to contain a subset $S^2 - A$ of X . Since $S^2 - A$ is dense in X , we have $X_k = X$. Thus there is no fundamental k -skeleton of X , $k = 2, 3, \dots$.

Proposition 2. If X is a compactum in an n -dimensional euclidean space R^n , then $Fd(X) \leq n - 1$.

Proof. Take a sequence $\{K_k\}$ of triangulable neighborhoods

of X in R^n such that $K_{k+1} \subset K_k$, $k = 1, 2, \dots$, and $\bigcap_k K_k = X$. Since K_k is an n -dimensional polyhedron in R^n , there is a subpolyhedron L_k of K_k such that L_k is a strong deformation retract of K_k and $\dim L_k \leq n-1$. By induction, we can find a simplicial subdivision $\tilde{L}_k$ of L_k and a simplicial map $\pi_{k+1}: \tilde{L}_{k+1} \rightarrow \tilde{L}_k$ such that $j_k \pi_{k+1} \simeq i_{k+1} j_{k+1} | \tilde{L}_{k+1}$ in K_k for $k = 1, 2, \dots$, where $j_k: L_k \rightarrow K_k$, $j_{k+1}: L_{k+1} \rightarrow K_{k+1}$ and $i_{k+1}: K_{k+1} \rightarrow K_k$ are the inclusions. Consider the inverse sequence $\{\tilde{L}_k, \pi_{k+1}\}$ and $X' = \varprojlim \tilde{L}_k$. It is known by Theorem 2 that $\text{Sh}(X) = \text{Sh}(X')$. Since $\dim X' \leq n-1$, we know $\text{Fd}(X) \leq n-1$.

Let $\mathcal{C} = \{X_\alpha | \alpha \in \Lambda\}$ be a collection of compacta. A member X_0 of $\mathcal{C}$ is said to be majorant for the shapes of members of $\mathcal{C}$ if $\text{Sh}(X_0) \geq \text{Sh}(X_\alpha)$ for each $X_\alpha \in \mathcal{C}$. For example, let $\mathcal{C}$ be the collection of all 0-dimensional compacta Y such that $\text{Sh}(Y) \leq \text{Sh}(X)$ for a given compactum X . Then the decomposition space $\square X$ of X consisting of all components of X is majorant for the shapes of members of $\mathcal{C}$. This follows from Corollary 3 of Theorem 3.

Proposition 3. (Watanabe) For the collection $\mathcal{R}$ of all compacta in R^1 a Cantor discontinuum is majorant for the shapes of members of $\mathcal{R}$.

This is a consequence of Proposition 2.

Theorem 6. (1) (S. Spieź) There is a compactum in R^2 which is majorant for the shapes of all compacta in R^2 .

(2) (K. Borsuk and W. Holsztyński) For the collection of

all solenoids $\mathcal{D}$ no compactum X_0 satisfies the condition $\text{Sh}(X) \leq \text{Sh}(X_0)$ for every $X \in \mathcal{D}$. Hence, if $\mathcal{C}$ is the collection of all compacta in R^3 , then there is no compactum which is majorant for the shapes of members of $\mathcal{C}$.

Problem 1. Let X be a compactum and let $\mathcal{C}_X$ be the collection of all compacta Y such that $\text{Sh}(X) > \text{Sh}(Y)$. Does there exist a compactum which is majorant for the shapes of members of $\mathcal{C}_X$?

The following problem is raised by Borsuk [3].

Problem 2. (Borsuk) Let X and Y be compacta. If $\text{Fd}(Y) > 0$, then does it hold $\text{Fd}(X \times Y) \geq \text{Fd}(X) + 1$?

It is likely true that the following holds. However it does not know yet.

Problem 3. For every compactum X , does it hold that $\text{Fd}(X \times S^1) = \text{Fd}(X) + 1$? Here S^1 is a 1-sphere.

§5. Movable compacta.

According to Borsuk [3,5], a compactum X in Q is said to be movable if for each neighborhood U of X there is a neighborhood V of X such that for every neighborhood W of X there is a homotopy $H : V \times I \rightarrow U$ satisfying the condition:

$$H(x, 0) = x \quad \text{and} \quad H(x, 1) \in W \quad \text{for each } x \in V.$$

A compactum X is said to be k-movable if for every neighborhood U of X there is a neighborhood V of X such that for every compactum $A \subset V$ with $\dim A \leq k$ and for every neighborhood W of X there is a homotopy $H : A \times I \rightarrow U$ satisfying the condition:

$$H(x,0) = x \quad \text{and} \quad H(x,1) \in W \quad \text{for } x \in A.$$

Mardešić and Segal [17] gave a characterization of movable compacta in terms of ANR sequences.

Theorem 7. (Mardešić and Segal) A compactum X is movable if and only if there is an ANR sequence $\{X_n, \pi_{nn'}\}$ satisfying the following condition: $X = \varprojlim \{X_n\}$ and for each $n \in \mathbb{N}$ there is an n' , $n' \geq n$, such that every $n'' \geq n$ there is a map $\mu_{n''n'}: X_{n'} \rightarrow X_{n''}$ satisfying the homotopy relation $\mu_{n''n'} \pi_{nn''} \simeq \pi_{nn'}$.

For movable compacta, the followings are known.

Theorem 8. (Borsuk) (1) Let X and Y be compacta with $\text{Sh}(X) \geq \text{Sh}(Y)$. If X is movable (k-movable), then Y is movable (k-movable).

(2) If X is movable (k-movable), then the suspension ΣX of X is movable (k-movable).

(3) Every compactum in $\mathbb{R}^2$ is movable.

(4) If X_i is a movable compactum for $i = 1, 2, \dots$, then $\prod_i X_i$ is movable.

(5) Every FANR is movable.

Theorem 9. (Kodama and Watanabe) An n-dimensional and n-movable compactum is movable.

Theorem 10. (1) (Mardešić) An n-dimensional LC^{n-1} compactum is movable.

(2) (Borsuk) An LC^{n-1} compactum is n-movable.

Let X be a Δ -space. As we know from the proof of Corollary of Theorem 5, for each $k = 0, 1, \dots$, there is a fundamental k-skeleton X_k of X. It is easy to see X_k is i-movable

for $i = 0, 1, \dots, k-1$, if X is movable.

Problem 4. Let X be a movable Δ -space. For each $k = 1, 2, \dots$, does there exist a fundamental k -skeleton X_k of X which is movable?

K. Borsuk [5] raised the following problems:

(1) Is it true that if X is m -movable and Y is n -movable then $X \times Y$ is $(m+n)$ -movable?

(2) Does there exist, for each $n = 1, 2, \dots$, a continuum which is n -movable, but is not $(n+1)$ -movable?

(3) Does there exist a non-movable compactum which is n -movable for every $n = 1, 2, \dots$?

These were solved by Kodama and Watanabe [12].

Theorem 11. (Kodama and Watanabe) (1) For compacta X and Y , $X \times Y$ is k -movable if and only if both X and Y are k -movable.

(2) If X is k -movable, then ΣX is $(k+1)$ -movable.

(3) There is a continuum X such that X is k -movable for every $k = 1, 2, \dots$, but not movable.

To show (3) of Theorem 11, we remark that there is a non-movable continuum X_0 such that ΣX_0 is homeomorphic to X_0 [7]. Since an n -fold suspension of a compactum X is $(n-1)$ -movable by (2) of Theorem 11, the continuum X_0 mentioned above is k -movable for every $k = 1, 2, \dots$. Borsuk proved every solenoid is not 1-movable. It is known that a suspension of a solenoid is 1-movable but not 2-movable.

It is known that every 2-dimensional ANR is dimensionally

full-valued (cf. [14]).

Problem 5. Is every 2-dimensional movable compactum dimensionally full-valued ?

Let X be a compactum with metric d . Let K be a finite simplicial complex and let $V(K)$ be the set of vertices of K . For a map $f : V(K) \rightarrow X$, we mean by mesh f the maximum of diameters of $f(s \cap V(K))$ for every simplex s of K . Let $\epsilon > 0$. For maps $f, g : V(K) \rightarrow X$ with $\max(\text{mesh } f, \text{mesh } g) < \epsilon$, by $f \sim_{\epsilon} g$ we imply that there is a sequence of maps $h_i : V(K) \rightarrow X$, $i = 0, 1, \dots, n$, such that $f = h_0$, $g = h_n$, $\text{mesh } h_i < \epsilon$, $i = 0, 1, \dots, n$, and $\max\{d(h_i(v), h_{i+1}(v)) : v \in V(K)\} < \epsilon$, $i = 0, 1, \dots, n-1$.

Proposition 4. A compactum X is movable if and only if for every $\epsilon > 0$ there is a $\delta > 0$ satisfying the following conditions: For every finite simplicial complex K , every map $f : V(K) \rightarrow X$ with $\text{mesh } f < \delta$ and every $\nu > 0$ there is a subdivision K' of K and a map $g : V(K') \rightarrow X$ such that $\text{mesh } g < \nu$ and $f \sim_{\epsilon} g$, where $\pi : V(K') \rightarrow V(K)$ is a map defined by any projection of K' to K .

This proposition gives a simple proof of Theorem 10. In a similar form to Proposition 4 we can obtain a necessary and sufficient condition for a compactum X in order that X be an FANR.

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