

Pseudo Dirichlet sets and a new cardinal invariant

Shizuo Kamo (Sakai) (加茂 静夫)

Abstract

Z. Bukovská [5] proved that $\mathfrak{p} \leq \text{non}(\mathcal{PD})$, where $\mathcal{PD}$ denotes the set of all pseudo-Dirichlet sets. In this paper, we shall show that $\mathfrak{p}$ can be replaced by $\mathfrak{h}$ in this inequality. It is known that $\mathfrak{p} < \mathfrak{h}$ is consistent (see [1]). So, the equality $\mathfrak{p} = \text{non}(\mathcal{PD})$ can not be proved. This is a partial answer of problem 2 in [6]. Next, we shall introduce a certain cardinal invariant $\mathfrak{f}$ and show that $\text{add}(\mathcal{N}) \leq \mathfrak{f} \leq \text{non}(\mathcal{PD})$. Also, we shall construct two generic models such that one satisfies the inequality $\mathfrak{b} < \mathfrak{f}$ and another satisfies the inequality $\mathfrak{f} < \text{non}(\mathcal{PD})$.

1 Introduction

Throughout this paper, we shall use the standard terminologies for forcing of set theory and cardinal invariants on ω (see [3]). For each $a \in \mathbf{R}$, we denote by $\|a\|$ the distance of a and the set of integers $\mathbf{Z}$. Let A be a subset of the unit interval $[0, 1]$. A is called a *pseudo Dirichlet set*, if there exists an $X \in [\omega]^\omega$ such that

$$\forall a \in A \forall^\infty n \in X \left(\|na\| < \frac{1}{|X \cap n| + 1} \right).$$

We denote the set of all pseudo Dirichlet sets by $\mathcal{PD}$. Z. Bukovská [5] showed that $\mathfrak{p} \leq \text{non}(\mathcal{PD})$. Let $\mathfrak{h}$ be the least cardinal κ such that the boolean algebra $\mathcal{P}(\omega)/\text{fin}$ does not satisfy the κ -distributive law.

Theorem 1.1 $\mathfrak{h} \leq \text{non}(\mathcal{PD})$.

Proof For each $a \in [0, 1]$, let $\|a\|^*$ denote the unique real number r such that $0 \leq r < 1$ and $a = r \pmod{\mathbf{Z}}$. To show this theorem, let $A \subset [0, 1]$ and $|A| < \mathfrak{h}$.

For each $a \in A$, take a maximal almost disjoint set $W_a \subset [\omega]^\omega$ such that

$$\forall n \in X \forall m \in X \setminus n (||na||^* - ||ma||^* < \frac{1}{|X \cap n| + 1}), \text{ for all } X \in W_a.$$

Since $|A| < \mathfrak{h}$, there exists a maximal almost disjoint set W such that W is a refinement of all W_a 's. Take a $Y \in W$. Choose some $Y' = \{y_i \mid i < \omega\} \in [Y]^\omega$ such that

$$|Y \cap [y_i, y_{i+1})| \geq i \text{ and } y_{i+1} - y_i < y_{i+2} - y_{i+1}, \text{ for all } i < \omega.$$

Let $Z = \{y_{i+1} - y_i \mid i < \omega\}$. We complete the proof by showing that

$$\forall^\infty n \in Z (||na|| < \frac{1}{|Z \cap n| + 1}), \text{ for all } a \in A.$$

Let $a \in A$. Since W is a refinement of W_a , there exists an $X \in W_a$ such that $Y \subset^* X$. Take an $i < \omega$ such that $Y \setminus y_i \subset X$. Then, for any $j \in [i+1, \omega)$, it holds that

$$|(y_{j+1} - y_j)a| \leq ||y_{j+1}a||^* - ||y_ja||^* \leq \frac{1}{|X \cap y_j| + 1} < \frac{1}{j+1}. \quad \square$$

2 Combinatorial principle wIn_2

T. Bartoszynski [2] introduced the notion of slalom and, using this, investigated systematically the relations between combinatorics and cardinal invariants which are associated by the null ideal $\mathcal{N}$ and the meager ideal $\mathcal{M}$. The following statement In_2 and the theorem are some of them.

Definition 2.1 For $h \in {}^\omega\omega$ and $F \subset {}^\omega\omega$, define the statement $\text{In}_2(F, h)$ by

$$\text{In}_2(F, h) \equiv \exists \varphi \in \prod_{n < \omega} [\omega]^{\leq h(n)} \forall f \in F \forall^\infty n < \omega (f(n) \in \varphi(n)).$$

The statement $\text{In}_2(F, \text{id}_\omega)$ is denoted by $\text{In}_2(F)$, where id_ω is the identity function on ω .

Theorem 2.1 (Bartoszynski [2]) $\text{add}(\mathcal{N}) = \min \{ |F| \mid F \subset {}^\omega\omega \text{ and not } \text{In}_2(F) \}.$

In this section, we shall introduce the statement wIn_2 which is some variant of In_2 . And we shall study relations between wIn_2 and $\text{non}(\mathcal{PD})$.

Definition 2.2 For $H, h \in {}^\omega\omega$ and $F \subset \prod_{n < \omega} H(n)$, define the statement $\text{wIn}_2(F, h, H)$

by

$$\text{wIn}_2(F, h, H) \equiv \exists \varphi \in \prod_{n < \omega} [H(n)]^{\leq h(n)} \forall f \in F \forall^\infty n < \omega (f(n) \in \varphi(n)).$$

$\text{wIn}_2(F, H)$ denotes the statement $\text{wIn}_2(F, \text{id}_\omega, H)$. Let

$$\mathbf{f} = \min \{ |F| \mid \exists H \in {}^\omega\omega (F \subset \prod_{n < \omega} H(n) \text{ and not } \text{wIn}_2(F, H)) \}.$$

The following lemma can be easily proved by the result of Bartoszynski.

Lemma 2.2 $\text{add}(\mathcal{N}) = \min \{ \mathbf{b}, \mathbf{f} \}.$ □

The main result of this section is the following theorem.

Theorem 2.3 $\mathbf{f} \leq \text{non}(\mathcal{PD}).$

To show this theorem, we need some notations and lemmas.

A sequence $\langle I_n \mid n < \omega \rangle$ is called an *interval partition* of ω , if there exists an increasing function $f \in {}^\omega\omega$ such that $f(0) = 0$ and, for all $n < \omega$, $I_n = \{ k < \omega \mid f(n) \leq k < f(n+1) \}.$

The next lemma can be deduced from [6, Proposition 1]. But, for a convenience for the reader, we give a proof.

Lemma 2.4 Let $n < \omega$ and $0 < m, k < \omega$. Then, there exists some $p < \omega$ such that

$$\forall a_0, \dots, a_{m-1} \in [0, 1] \exists s < \omega (n \leq s < p \text{ and } \forall i < m (\|sa_i\| < \frac{1}{k})).$$

Proof By induction on $1 \leq m < \omega$.

Case 1. $m = 1$.

We claim that $p = nk + 1$ satisfies the condition. To show this, let $a \in [0, 1]$.

Define the mapping $\sigma : X = \{ nj \mid j = 1, \dots, k \} \rightarrow k$ by, for each $j < k$,

$$\sigma(nj) = \text{“ the unique } i \text{ such that } \frac{i}{k} \leq \|nja\|^* < \frac{i+1}{k} \text{”}.$$

If there exists some nj such that $\sigma(nj) = 0$ or $k - 1$, then $s = nj$ is a required one.

Otherwise, there exist $i < j \leq k$ such that $\sigma(ni) = \sigma(nj)$ and $s = n(j - i)$ is a required one.

Case 2. $m = m' + 1$.

By induction hypothesis, there exist $0 = p_0 < p_1 < \dots < p_k$ such that

$$\forall a_0, \dots, a_{m'-1} \in [0, 1] \exists s < \omega (p_j + n \leq s < p_{j+1} \text{ and } \forall i < m' (\|sa_i\| < \frac{1}{2k})),$$

for $j < k$.

We show that $p = p_k$ satisfies the condition. So, let $a_0, \dots, a_{m'} \in [0, 1]$. By the choice of p_j (for $j < k$), there exist $s_0, \dots, s_{k-1} < \omega$ such that

$$p_j + n \leq s_j < p_{j+1} \text{ and } \forall i < m' (\|s_j a_i\| < \frac{1}{2k}), \text{ for } j < k.$$

Then, it holds that

$$\|s_j a_{m'}\| < \frac{1}{k}, \text{ for some } j < k$$

or

$$\|s_j a_{m'} - s_{j'} a_{m'}\| < \frac{1}{k}, \text{ for some } j < j' < k.$$

In either cases, similar to case 1, we can take a required element s . □

Corollary 2.5 *There is an interval partition $\langle I_n \mid n < \omega \rangle$ which satisfies*

$$(*) \left\{ \begin{array}{l} \text{For any } n < \omega \text{ and } a_0, \dots, a_{n-1} \in [0, 1), \text{ there exists some } k \in I_n \text{ such that} \\ \|ka_i\| < 2^{-n}, \text{ for all } i < n. \end{array} \right.$$

□

Proof of Theorem 2.3 Take an interval partition $\langle I_n \mid n < \omega \rangle$ which satisfies

(*) in the previous corollary. Define $H \in {}^\omega \omega$ by

$$H(n) = 2^n \sum_{k \leq n} |I_k|, \text{ for all } n < \omega.$$

To show the theorem, let $A \subset [0, 1]$ and $|A| < \mathfrak{f}$. For each $a \in A$, define $f_a \in \prod_{n < \omega} H(n)$

by

$$\frac{f_a(n)}{H(n)} \leq a < \frac{f_a(n) + 1}{H(n)}, \text{ for all } n < \omega.$$

Since $|A| < \mathfrak{f}$, there exists a $\varphi \in \prod_{n < \omega} [H(n)]^n$ such that

$$\forall a \in A \forall^\infty n < \omega (f_a(n) \in \varphi(n)).$$

For each $n < \omega$, take $s_n \in I_n$ such that

$$\|s_n \frac{j}{H(n)}\| < 2^{-n}, \text{ for all } j \in \varphi(n).$$

We complete the proof by showing that

$$\forall a \in A \forall^\infty n < \omega (\|s_n a\| < 2^{-n+1}).$$

So, let $a \in A$. Take an $m < \omega$ such that

$$\forall n \geq m (f_a(n) \in \varphi(n)).$$

Then, for any $n \geq m$, since $\frac{f_a(n)}{H(n)} \leq a < \frac{f_a(n)+1}{H(n)}$, it holds that

$$s_n \frac{f_a(n)}{H(n)} \leq s_n a < s_n \frac{f_a(n)+1}{H(n)}.$$

So,

$$\|s_n a\| \leq \|s_n \frac{f_a(n)}{H(n)}\| + \frac{s_n}{H(n)} < 2^{-n+1}.$$

□

Note that what we really proved is $\min\{|F| \mid \text{not } \text{wIn}_2(F, H)\} \leq \text{non}(\mathcal{PD})$,

where H is a function defined in the proof of Theorem 2.3.

3 The cardinal invariant $\mathbf{f}$

In the previous section, we introduced the cardinal invariant $\mathbf{f}$ and showed the equality $\text{add}(\mathcal{N}) = \min\{\mathbf{b}, \mathbf{f}\}$. Both of $\text{add}(\mathcal{N})$ and $\mathbf{b}$ appear in the Cichoń's diagram. It seems to be an interesting problem to check the relations between $\mathbf{f}$ and other cardinals in the diagram. Since it is known that $\mathcal{PD} \subset \mathcal{N} \cap \mathcal{M}$, it holds that $\mathbf{f} \leq \min\{\text{non}(\mathcal{N}), \text{non}(\mathcal{M})\}$. So, $\mathbf{f}$ seems to be not so large. If the inequality $\mathbf{f} \leq \mathbf{b}$ always holds, then $\mathbf{f}$ is equal to $\text{add}(\mathcal{N})$ and $\mathbf{f}$ does not become a new cardinal invariant. In this section, we shall show that there exists a generic model which satisfies the inequality $\mathbf{b} < \mathbf{f}$.

Definition 3.1 For each $H \in {}^\omega\omega$, define the forcing notion $Q(H)$ by

$$Q(H) = \{p \in \prod_{n < \omega} [H(n)]^{\leq n} \mid \exists k < \omega \forall n < \omega (|p(n)| \leq k) \},$$

$$q \leq p \text{ iff } \forall n < \omega (p(n) \subset q(n)).$$

Define $\tau_H : Q(H) \rightarrow \omega$ by

$$\tau_H(p) = \min\{ k < \omega \mid \forall n < \omega (|p(n)| \leq k) \}.$$

Using the density argument, the following lemma can be proved easily.

Lemma 3.1 *Let $H \in {}^\omega\omega$ and $\mathcal{G}$ be V -generic on $Q(H)$. In $V[\mathcal{G}]$, define $\varphi \in \prod_{n < \omega} \mathcal{P}(H(n))$*

by

$$\varphi(n) = \bigcup \{ p(n) \mid p \in \mathcal{G} \}.$$

Then, it holds that

- (1) $|\varphi(n)| \leq n$, for all $n < \omega$,
- (2) $\forall g \in (\prod_{n < \omega} H(n))^V \forall^\infty n < \omega (g(n) \in \varphi(n))$. □

Lemma 3.2 *$Q(H)$ satisfies the ω_1 -chain condition.*

Proof Let $W \subset Q(H)$ and $|W| = \omega_1$. Replace W by a certain subset of W , if necessary, we can assume that, for some $k < \omega$,

$$\tau_H(p) = k \text{ and } p \restriction 2k = p' \restriction 2k, \text{ for all } p, p' \in W.$$

Then, every elements of W are mutually compatible. □

Lemma 3.3 *Every unbounded family of functions in ${}^\omega\omega \cap V$ is still unbounded in $V^{Q(H)}$.*

Bartszinski and Judah [3, Theorem 6.4.13] proved that any finite support iteration by forcing notions which preserved the unboundedness in ${}^\omega\omega$ does not add a dominating function. So, starting a ground model which satisfies CH, by choosing appropriate H 's, we can construct an ω_2 -stage finite support iteration P such that V^P satisfies $\mathfrak{b} = \omega_1$ and $\mathfrak{f} = \omega_2$.

In order to prove Lemma 3.3, we need a result of Brendle and Judah [4]. Let P be a forcing notion which satisfies the ω_1 -chain condition and $\tau : P \rightarrow \omega$ be a homomorphism. Following Brendle and Judah [4], we say that (P, τ) is *nice*, if it

satisfies

$$\left\{ \begin{array}{l} \text{For any predence set } \{p_i \mid i < \omega\} \subset P, \text{ it holds that} \\ \forall m < \omega \exists n < \omega \forall q \in P \text{ (if } \tau(q) \leq m, \text{ then } \exists i < n \text{ (} q \upharpoonright p_i \text{)).} \end{array} \right.$$

Theorem 3.4 (Brendle and Judah [4]) *Let (P, τ) be a nice forcing notion. Then, every unbounded family of functions in ${}^\omega\omega \cap V$ is still unbounded in $V^{Q(H)}$. $\square$*

Proof of Lemma 3.3 It suffices to show that $(Q(H), \tau_H)$ is nice. So, let $\{p_i \mid i < \omega\}$ be a predence subset of $Q(H)$ and $m < \omega$. To get a contradiction, assume that, for each $n < \omega$, there exists a condition $q_n \in Q(H)$ such that

$$\tau_H(q_n) \leq m \text{ and } \forall i < n \text{ (} q_n \perp p_i \text{)}.$$

Since $\{q_n \upharpoonright k \mid n < \omega\}$ is a finite set for every $k < \omega$, we can choose $X_k \in [\omega]^\omega$ by induction on $k < \omega$ such that

$$X_{k+1} \subset X_k \text{ and } \forall n, n' \in X_k \text{ (} q_n \upharpoonright (k+1) = q_{n'} \upharpoonright (k+1) \text{)}.$$

Define $r \in Q(H)$ by

$$r(k) = q_n(k), \text{ for some/all } n \in X_k.$$

Note that $\tau_H(r) \leq m$. Since $\{p_i \mid i < \omega\}$ is predence, there exists $i < \omega$ such that r is compatible with p_i . Let $k = \tau_H(p_i) + m$. Take $n \in X_k$ such that $i < n$. Since $i < n$, it holds that p_i and q_n are incompatible. Since $\tau_H(p_i) + \tau_H(q_n) \leq k$, it holds that $\exists j < k \text{ (} |p_i(j) \cup q_n(j)| > j \text{)}$. By this and the fact that $r \upharpoonright k = q_n \upharpoonright k$, r is incompatible with p_i . This is a contradiction. $\square$

4 Consistency of $\mathfrak{f} < \text{non}(\mathcal{PD})$

Concerning about the cardinal invariant associated by In_2 , T. Bartoszynski [2] pointed out implicitly that, if two functions $h_0, h_1 \in {}^\omega\omega$ satisfies that

$$\lim_{n < \omega} h_i(n) = \infty, \text{ for } i = 0, 1,$$

then $\min\{|F| \mid \text{not } \text{In}_2(F, h_0)\} = \min\{|F| \mid \text{not } \text{In}_2(F, h_1)\}$.

In this section, we shall show that, for any $H \in {}^\omega\omega$, $\mathfrak{f}$ may not be equal to $\min\{|F| \mid \text{not } \text{wIn}_2(F, H)\}$. Using this, we shall prove the consistency of $\mathfrak{f} <$

$\text{non}(\mathcal{PD})$. Henceforce, $H \in {}^\omega\omega$ is an arbitrary, but fixed function on ω . For each $k < \omega$, let

$$T_k (= T_k^H) = \{ q \in Q(H) \mid \tau_H(q) \leq k \}.$$

Define $H_0, H_1 : \omega \times \omega \rightarrow \omega$ by

$$H_0(k, m) = \min \left\{ l < \omega \mid \begin{array}{l} \forall \delta : l \rightarrow [\omega_2]^{\leq k} \exists S \in [l]^m \exists v \in [\omega_2]^{\leq k} \\ \forall i, j \in S \text{ (if } i \neq j, \text{ then } \delta(i) \cap \delta(j) = v) \end{array} \right\},$$

$$H_1(k, m) = \min \left\{ l < \omega \mid \begin{array}{l} \forall \delta : l \rightarrow T_k \exists S \in [l]^m \exists q \in Q(H) \\ \forall i \in S \text{ (} q \leq \delta(i) \text{)} \end{array} \right\}.$$

Note that H_0 is a recursive function. And, H_1 is an H -recursive function, since it holds that

$$\exists q' \in Q(H) \forall q \in S \text{ (} q' \leq q \text{)} \quad \text{iff} \quad \forall i < mk \text{ (} |\bigcup_{q \in S} q(i)| \leq i \text{)}, \text{ for any}$$

$$S \in [T_k]^m.$$

Define $H_2, H^* : \omega \rightarrow \omega$ by

$$H_2(k) = \underbrace{H_1(k, H_1(k, H_1(\cdots, H_1(k, k+1)\cdots)))}_{k \text{ times}},$$

$$H^*(k) = H_0(k, H_2(k)).$$

Define an ω_2 -stage finite support iteration P_α (for $\alpha \leq \omega_2$) associated with $\dot{Q}_\alpha$ (for $\alpha < \omega_2$) by

$$\Vdash_\alpha \dot{Q}_\alpha = Q(H), \quad \text{for all } \alpha < \omega_2.$$

Let $P(H) = P_{\omega_2}$. It holds that

$$V^{P(H)} \models \forall F \subset \prod_{n < \omega} H(n) \text{ (if } |F| \leq \omega_1, \text{ then } \text{wIn}_2(F, H) \text{)}.$$

The purpose of this section is to show

Theorem 4.1 $V^{P(H)} \models \text{not } \text{wIn}_2((\prod_{n < \omega} H^*(n))^V, H^*).$

Corollary 4.2 *Suppose that $V \models \text{CH}$. Let $H \in {}^\omega\omega$ be the function which is defined in the proof of Theorem 2.3. Then, it holds that*

$$V^{P(H)} \models \mathfrak{f} = \omega_1 \text{ and } \text{non}(\mathcal{PD}) = \omega_2.$$

□

To show Theorem 4.1, we need some definitions and lemmas. Let

$$D = \{ p \in P(H) \mid \forall \alpha \in \text{supp}(p) (p \upharpoonright \alpha \text{ decides } \tau_H(p(\alpha))) \}.$$

The following lemma can be proved easily.

Lemma 4.3 *D is dense in P(H).* □

Define $\rho : D \rightarrow \omega$ by

$$\rho(p) = \min \left\{ k < \omega \mid \begin{array}{l} |\text{supp}(p)| \leq k \\ \text{and} \\ \forall \alpha \in \text{supp}(p) (p \upharpoonright \alpha \Vdash_{\alpha} \tau_H(p(\alpha)) \leq k) \end{array} \right\}.$$

For each $k < \omega$, let

$$D_k = \{ p \in D \mid \rho(p) \leq k \}.$$

Lemma 4.4 *Let $k < \omega$ and $\delta : H^*(k) \rightarrow D_k$. Then, there exist $p^+ \in P(H)$ and $P(H)$ -name $\dot{S}$ which satisfy (1), (2).*

- (1) $\Vdash \dot{S} \subset H^*(k)$ and $|\dot{S}| \geq k + 1$.
- (2) $\forall i < H^*(k) \forall p' \leq p^+ (\text{ if } p' \Vdash i \in \dot{S}, \text{ then } p' \leq \delta(i))$.

Proof Let $k < \omega$. Define l_m (for $m \leq k$) by

$$\begin{cases} l_0 &= k + 1 \\ l_{m+1} &= H_1(k, l_m) \end{cases}.$$

Note that $H^*(k) = H_0(k, l_k)$. Assume that $\delta : H^*(k) \rightarrow D_k$. Since $\langle \text{supp}(\delta(i)) \mid i < H^*(k) \rangle : H^*(k) \rightarrow [\omega_2]^{\leq k}$, by the choice of H_0 , there exist $S_0 \in [H^*(k)]^{l_k}$ and $v \in [\omega_2]^{\leq k}$ such that

$$\forall i, j \in S_0 (\text{ if } i \neq j, \text{ then } \text{supp}(\delta(i)) \cap \text{supp}(\delta(j)) = v).$$

Define $p \in P(H)$ by

$$\text{supp}(p) = \bigcup \{ \text{supp}(\delta(i)) \mid i \in S_0 \} \setminus v,$$

$$p(\alpha) = \delta(i)(\alpha), \text{ if } \alpha \in \text{supp}(\delta(i)) \text{ and } i \in S_0.$$

Let $n = |v|$ and $v = \{ \alpha_1, \dots, \alpha_n \}_<$. Note that $n \leq k$. By induction on $1 \leq m \leq n$,

choose P_{α_m} -names $\dot{S}_m, \dot{q}_m$ such that

$$(3) \quad \Vdash_{\alpha_m} \dot{S}_m \in [\dot{S}_{m-1}]^{l_{k-m}} \text{ and } \dot{q}_m \in \dot{Q}_{\alpha_m},$$

$$(4) \quad p \upharpoonright \alpha_m \cup \langle \dot{q}_j \mid 1 \leq j < m \rangle \Vdash_{\alpha_m} \dot{q}_m \leq \delta(i)(\alpha_m), \text{ for all } i \in \dot{S}_m.$$

We must show that these can be chosen. Assume that $m \leq n$ and $\dot{S}_j, \dot{q}_j$ were chosen, for $j < m$. Since H_1 is absolute and $H_1(k, l_{k-m}) = l_{k-m+1}$, it holds that

$$p \restriction \alpha_m \cup \langle \dot{q}_j \mid 1 \leq j < m \rangle \Vdash_{\alpha_m} \exists q \in Q(H) \exists S \in [\dot{S}_{m-1}]^{l_{k-m}} \forall i \in S (q \leq \delta(i)(\alpha_m)).$$

Using this, it can be possible to choose $\dot{S}_m$, and $\dot{q}_m$.

Let $p^+ = p \cup \langle \dot{q}_m \mid 1 \leq m \leq n \rangle$, $\dot{S} = \dot{S}_n$. It is clear that this p^+ and $\dot{S}$ satisfy (1) in the lemma. In order to show that these satisfy (2), assume that

$$i < H^*(k) \text{ and } p' \leq p^+ \text{ and } p' \Vdash i \in \dot{S}.$$

Since $\Vdash_P \dot{S} = \dot{S}_n \subset \dot{S}_{n-1} \subset \dots \subset S_0$, $i \in S_0$. For each $m = 1, \dots, n$, since $\dot{S}_m$ is a P_{α_m} -name, it holds that $p' \restriction \alpha_m \Vdash_{\alpha_m} i \in \dot{S}_m$. By this, since $p' \leq p^+$, we have that

$$p' \restriction \alpha_m \Vdash_{\alpha_m} \dot{q}_m \leq \delta(i)(\alpha_m), \text{ for all } m = 1, \dots, n.$$

So, $p' \leq \delta(i)$. □

Lemma 4.5 *Let $k < \omega$. Assume that a $P(H)$ -name $\dot{a}$ satisfies*

$$\Vdash \dot{a} \in [H^*(k)]^{\leq k}.$$

Then, there exists some $j < H^(k)$ such that*

$$\forall p \in D_k (\text{ not } p \Vdash j \in \dot{a}).$$

Proof Suppose not. Take $\delta : H^*(k) \rightarrow D_k$ such that

$$\delta(j) \Vdash j \in \dot{a}, \text{ for all } j < H^*(k).$$

By the previous lemma, there exist $p^+ \in P(H)$ and $P(H)$ -name $\dot{S}$ such that

$$\Vdash \dot{S} \subset H^*(k) \text{ and } |\dot{S}| \geq k + 1,$$

$$\forall i < H^*(k) \forall p' \leq p^+ (\text{ if } p' \Vdash i \in \dot{S}, \text{ then } p' \leq \delta(i)).$$

Then, it holds that $p^+ \Vdash \dot{S} \subset \dot{a}$. This contradicts that $p^+ \Vdash |\dot{S}| \geq k + 1$ and $|\dot{a}| \leq k$. □

Proof of Theorem 4.1 Assume that $\Vdash_{P(H)} \dot{\varphi} \in \prod_{k < \omega} [H^*(k)]^k$. Using the previous

lemma, for each $k < \omega$, take a $j_k < H^*(k)$ such that

$$\forall p \in D_k (\text{ not } p \Vdash j_k \in \dot{\varphi}(k)).$$

We claim that $\Vdash \exists^\infty k < \omega (j_k \notin \dot{\varphi}(k))$. Suppose not. Then, there exist $p \in D$ and $n < \omega$ such that

$$p \Vdash \forall k > n (j_k \in \dot{\varphi}(k)).$$

Take $k > n$ such that $p \in D_k$. Then, it holds that $p \Vdash j_k \in \dot{\varphi}(k)$. But, this contradicts the choice of j_k . $\square$

Added in proof:

After the completion of this paper, Dr. Kada [7] have proved that $\mathfrak{d} < \text{non}(\mathcal{PD})$ is consistent with ZFC.

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Department of Mathematics

University of Osaka Prefecture

Gakuen-chou, Sakai, Japan

email:kamo@center.osakafu-u.ac.jp