

CFT on $\mathbb{P}^1$ and Monodromy Representations
of the Braid Groups

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§0. From the differential equations of N -point functions of vertex operators in the conformal field theory on $\mathbb{P}^1$, arise the monodromy representations of the braid group B_N . In the meeting of last year, I reported that these monodromy representations give "all" irreducible representations of the Hecke algebra $H_N(q)$ of type A_{N-1} (obtained by H. Wenzl [W]) associated with the affine Lie algebra of type $A_n^{(1)}$. In this meeting, I will report that associated with the affine Lie algebras of type $B_n^{(1)}$, $C_n^{(1)}$ and $D_n^{(1)}$, the monodromy representations of the group B_N give "all" irreducible representations of the Birman-Wenzl-Murakami algebra, the q -analogue of Brauer's centralizer algebras. Very important is Jimbo-Miwa-Okado's

calculations[JMO], and in the case of type $C_n^{(1)}$ the representations are equivalent to the ones obtained by J.Murakami[M].

§1. Let g be the simple Lie algebra of type X_n , and $\hat{g} = g \oplus \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$ be the affine Lie algebra of type $X_n^{(1)}$. Fix an integer $\ell \geq 1$ and introduce the number $\kappa = \ell + g$, where g is the dual Coxeter number of $\hat{g}$.

Denote by P_+ the set of dominant integral weights of g and by P_ℓ the set of elements $\lambda \in P_+$ satisfying $(\theta, \lambda) \leq \ell$, where θ is the maximum root. For a weight $\lambda \in P_\ell$, we denote by V_λ the irreducible representation of g of highest weight λ , by $\mathcal{H}_\lambda$ the integrable representation of $\hat{g}$ of highest weight $\ell\Lambda_0 + \lambda$ and by $|\lambda\rangle$ the (fixed) highest weight cyclic vector of V_λ and $\mathcal{H}_\lambda$.

The Virasoro algebra also acts on $\mathcal{H}_\lambda$ by the Sugawara forms $L(m)$, $m \in \mathbb{Z}$, and the space $\mathcal{H}_\lambda$ is graded by means of the eigenspace decomposition w.r.t. the operator $L(0)$:

$$\mathcal{H}_\lambda = \sum_{d \in \mathbb{Z}_{\geq 0}} \mathcal{H}_\lambda(d), \quad \mathcal{H}_\lambda(d) = \{v \in \mathcal{H}_\lambda; L(0)v = (\Delta_\lambda + d)v\},$$

where $\Delta_\lambda = \frac{(\lambda, \lambda + 2\rho)}{2\kappa}$ and ρ is the half sum of positive roots of g . Note that $\dim \mathcal{H}_\lambda(d) < \infty$ and $\mathcal{H}_\lambda(0) \cong V_\lambda$.

There are dual right $\mathfrak{g}$ and $\hat{\mathfrak{g}}$ -module V_λ^+ and $\mathcal{H}_\lambda^+$ of V_λ and $\mathcal{H}_\lambda$, and the nondegenerate invariant bilinear form $\langle \cdot | \cdot \rangle$ on $V_\lambda^+ \times V_\lambda$ and $\mathcal{H}_\lambda^+ \times \mathcal{H}_\lambda$ with the normalized condition $\langle \lambda | \lambda \rangle = 1$ where $\langle \lambda |$ is a fixed highest weight vector of $V_\lambda^+ = \mathcal{H}_\lambda^+(0)$ and $\mathcal{H}_\lambda^+$.

A triple $v = \begin{bmatrix} \lambda \\ \mu_2 \mu_1 \end{bmatrix}$ of weights in P_ℓ is called a vertex and is drawn as

$$v = \begin{array}{c} \lambda \\ \downarrow \\ \mu_2 \text{ --- } \leftarrow \text{ --- } \leftarrow \text{ --- } \mu_1 \end{array} .$$

A multi-valued, holomorphic function

$$\Phi(z) : V_\lambda \otimes \mathcal{H}_{\mu_1} \longrightarrow \hat{\mathcal{H}}_{\mu_2} = \prod_{d \in \mathbb{Z}_{\geq 0}} \mathcal{H}_{\mu_2}^{(d)}$$

on $\mathbb{P}^1 \setminus \{0, \infty\}$ is called a vertex operator of type $\begin{bmatrix} \lambda \\ \mu_2 \mu_1 \end{bmatrix}$

(sometimes called of weight λ), if it satisfies the following:

$$(\text{Gauge Condition}) \quad [X(m), \Phi(z)(u \otimes \cdot)] = z^m \Phi(z)(Xu \otimes \cdot)$$

$$(X \in \mathfrak{g}, m \in \mathbb{Z}, u \in V_\lambda);$$

$$(\text{Eq. of Motion}) \quad [L(m), \Phi(z)] = z^m \left\{ z \frac{d}{dz} + (m+1) \Delta_\lambda \right\} \Phi(z),$$

where $X(m) = X \otimes t^m$ and the number Δ_λ is called the *conformal dimension* of the vertex operator $\Phi(z)$.

Denote by $\mathcal{V}er(v)$ the space of all vertex operators of type v , and introduce the space

$$\mathcal{V}er(\lambda) = \sum_{\mu_1, \mu_2 \in P_\ell} \mathcal{V}er\left(\begin{matrix} \lambda \\ \mu_2, \mu_1 \end{matrix}\right)$$

of all vertex operators of weight λ .

Introduce the subalgebra $\mathfrak{t}_\theta = \mathbb{C}\langle X_\theta, [X_\theta, X_{-\theta}], X_{-\theta} \rangle \cong \mathfrak{sl}(2; \mathbb{C})$ of $\mathfrak{g}$ and the subspace $\mathcal{V}(v)$ of $\text{Hom}_{\mathfrak{g}}(V_\lambda \otimes V_{\mu_1}, V_{\mu_2})$ defined by

$$\mathcal{V}(v) = \cap \text{Ker } \pi_{\mathfrak{t}_\theta}(j, j_1, j_2)$$

where the intersection is taken over the set $\{j, j_1, j_2 \in \frac{1}{2}\mathbb{Z}_{\geq 0}; j+j_1+j_2 > \ell\}$, and $\pi_{\mathfrak{t}_\theta}(j, j_1, j_2)(\varphi) \in \text{Hom}_{\mathfrak{t}_\theta}(W_j \otimes W_{j_1}, W_{j_2})$ is defined as

$$\pi_{\mathfrak{t}_\theta}(j, j_1, j_2)(\varphi) = \text{proj}_{W_{j_2}} \circ \varphi|_{W_j \otimes W_{j_1}} \quad (\varphi \in \text{Hom}_{\mathfrak{g}}(V_\lambda \otimes V_{\mu_1}, V_{\mu_2}))$$

where W_j, W_{j_1}, W_{j_2} are $\mathfrak{t}_\theta$ -simple submodules of $V_\lambda, V_{\mu_1}, V_{\mu_2}$ with spin j, j_1, j_2 respectively.

By Equation of Motion, Φ is expressed as a formal

Laurent series

$$\Phi(z) = \sum_{m \in \mathbb{Z}} \Phi(m) z^{-m - \hat{\Delta}(v)},$$

where $\hat{\Delta}(v) = \Delta_\lambda + \Delta_{\mu_1} - \Delta_{\mu_2}$ and $\Phi(m)$ is homogeneous of degree m , i.e.

$$\Phi(m): V_\lambda \otimes_{\mu_1} (d) \longrightarrow \mathcal{H}_{\mu_2} (d-m) \quad \text{for any } d.$$

The principal branch of $\Phi(z)$ is taken such as the value of $z^{-\hat{\Delta}(v)}$ is positive for $z \in \mathbb{R}_+ = \{z \in \mathbb{R}; z > 0\}$ and uniquely continued to the region $\mathbb{C}_+ = \{z \in \mathbb{C}; \text{Im} z > 0\}$, and we refer this for the value of $\Phi(z)$ on $\mathbb{C}_+$.

For any vertex operator $\Phi \in \mathcal{V}el(v)$, its initial term $\varphi = \Phi(0) \big|_{V_\lambda \otimes_{\mu_1} (0)} = \text{proj}_{V_{\mu_2}} \cdot z^{\hat{\Delta}(v)} \Phi(z) \big|_{V_\lambda \otimes_{\mu_1} V}$ belongs to $\mathcal{V}(v)$. Under this correspondence,

Theorem 1. *The space $\mathcal{V}el(v)$ of N -point functions of type v is isomorphic with the space $\mathcal{V}(v)$ of initial terms of type v .*

Call v ℓ CG (ℓ -constrained Clebsch-Gordan) vertex, if $\mathcal{V}(v) \neq 0$, and denote by (ℓCG) the set of all ℓ CG vertices.

For each $\varphi \in \mathcal{V}$, denote by Φ_φ the vertex operator with the initial term φ .

Notes. i) Even if we assume that $\lambda \in P_+$ and $\mu_i \in P_\ell$, $\mathcal{V}\left(\begin{smallmatrix} \lambda \\ \mu_2 \mu_1 \end{smallmatrix}\right) \neq 0$ implies that $\lambda \in P_\ell$.

ii) Operator product expansions of currents $X(z) =$

$\sum_{m \in \mathbb{Z}} X(m) z^{-m-1}$ ($X \in \mathfrak{g}$) and the energy-momentum tensor $T(z) =$

$\sum_{m \in \mathbb{Z}} L(m) z^{-m-2}$ with vertex operators allow the extension of

the vertex operators $\Phi(z)$ of type $\begin{smallmatrix} \lambda \\ \mu_2 \mu_1 \end{smallmatrix}$ to the operators

$\Phi(z): \mathcal{H}_\lambda \otimes \mathcal{H}_{\mu_1} \longrightarrow \hat{\mathcal{H}}_{\mu_2}$ by means of contour integrals.

(Nuclear Democracy)

iii) By the same arguments as in §3, the analytic

continuation of a vertex operator Φ of type $\begin{smallmatrix} \lambda \\ \mu_2 \mu_1 \end{smallmatrix}$ along the path γ_0 gives a vertex operator of type $\begin{smallmatrix} \lambda \\ \mu_1 \mu_2 \end{smallmatrix}$, where

$$\gamma_0(t) = z e^{\pi\sqrt{-1}t}, \quad t \in [0, 1], \quad z \in \mathbb{R}_+.$$

This gives an isomorphism C_{γ_0} of $\mathcal{V}_{\begin{smallmatrix} \lambda \\ \mu_2 \mu_1 \end{smallmatrix}}$ to $\mathcal{V}_{\begin{smallmatrix} \lambda \\ \mu_1 \mu_2 \end{smallmatrix}}$

and the corresponding isomorphism

$$C_{\gamma_0}: \mathcal{V}_{\begin{smallmatrix} \lambda \\ \mu_2 \mu_1 \end{smallmatrix}} \longrightarrow \mathcal{V}_{\begin{smallmatrix} \lambda \\ \mu_1 \mu_2 \end{smallmatrix}}$$

is given by

$$C_{\gamma_0} = e^{\pi\sqrt{-1} \hat{\Delta}(v)} T ,$$

where T is the transposition:

$$T : \text{Hom}(V_{\lambda} \otimes V_{\mu_1}, V_{\mu_2}) \longrightarrow \text{Hom}(V_{\mu_1} \otimes V_{\lambda}, V_{\mu_2})$$

$$(T\phi)(u \otimes v) = \phi(v \otimes u).$$

A vertex operator $\Phi(z)$ of type v is also considered as an operator from $\mathcal{H}_{\mu_1}$ to $\hat{\mathcal{H}}_{\mu_2}$ parametrized by V_{λ} , i.e.

$$\Phi(u; z)(v) = \Phi(z)(u \otimes v) \quad (u \in V_{\lambda}, v \in \mathcal{H}_{\mu_1}).$$

§2. It is convenient to introduce the spaces $\mathcal{H} = \sum_{\lambda \in P_{\ell}} \mathcal{H}_{\lambda}$ and $\hat{\mathcal{H}} = \sum_{\lambda \in P_{\ell}} \hat{\mathcal{H}}_{\lambda}$ and consider vertex operators as linear operators of $\mathcal{H}$ to $\hat{\mathcal{H}}$. The vacuum $|0\rangle$ of $\mathcal{H}_0$ is called a *Virasoro vacuum*, since $L(m)|0\rangle = 0$ for $m \geq -1$. Note that $V_0 = \mathbb{C}|0\rangle$.

For an N -ple $\Lambda = (\lambda_N, \dots, \lambda_1)$ of weights in P_{ℓ} , denote

$$V_{\Lambda} = V_{\lambda_N} \otimes \dots \otimes V_{\lambda_1} \quad \text{and} \quad V_g^V(\Lambda) = \text{Hom}_g(V_{\Lambda}, \mathbb{C}).$$

For any vertex operators $\Phi^i(z_i)$ of weight λ_i ($1 \leq i \leq N$),

$$\langle 0 | \Phi^N(z_N) \cdots \Phi^1(z_1) | 0 \rangle$$

is the coefficient of $|0\rangle$ the iterated application

$\Phi^N(z_N) \cdots \Phi^1(z_1) | 0 \rangle$ to the vector $|0\rangle$, and this is a $V_g^V(\Lambda)$ -valued formal Laurent series in $z_N, \dots, z_1$ called the *N-point function of weight Λ* and is denoted by $\langle \Phi^N(z_N) \cdots \Phi^1(z_1) \rangle$. Denote by $\mathcal{P}_{el}(\Lambda)$ the space of all N-point functions of weight Λ .

The space $V_g^V(\Lambda)$ is decomposed as

$$V_g^V(\Lambda) = \sum_{\mu} V_g^V(\Lambda)_{\mu}, \quad \mu = (\mu_{N-1}, \dots, \mu_1) \in (P_+)^{N-1};$$

$$V_g^V(\Lambda)_{\mu} \xleftarrow[\cong]{C_{\Lambda}} \text{Hom}_g(V_{\lambda_N} \otimes V_{\mu_{N-1}}, V_0) \otimes \cdots \otimes \text{Hom}_g(V_{\lambda_i} \otimes V_{\mu_{i-1}}, V_{\mu_i})$$

$$\otimes \cdots \otimes \text{Hom}_g(V_{\lambda_1} \otimes V_0, V_{\mu_1}),$$

where the identification C_{Λ} is given by

$$C_{\Lambda}(\varphi_N \otimes \cdots \otimes \varphi_1)(u_N \otimes \cdots \otimes u_1)$$

$$= \langle 0 | \varphi_N(u_N \otimes \varphi_{N-1}(\cdots \otimes \varphi_2(u_2 \otimes \varphi_1(u_1 \otimes |0\rangle)) \cdots) \rangle$$

$$= \langle 0 | \varphi_N(u_N) \cdots \varphi_1(u_1) (|0\rangle) \rangle,$$

for $\varphi_i \in \text{Hom}_g(V_{\lambda_i} \otimes V_{\mu_{i-1}}, V_{\mu_i}) \cong \text{Hom}_g(V_{\lambda_i}, \text{Hom}(V_{\mu_{i-1}}, V_{\mu_i}))$

($1 \leq i \leq N$; $\mu_N = \mu_0 = 0$), and $u_N \otimes \cdots \otimes u_1 \in V_{\Lambda}$.

Introduce the subspace $\mathcal{P}(\Lambda)$ of $V_g^V(\Lambda)$ defined, through C_Λ , by

$$\mathcal{P}(\Lambda) = \sum_{\mu} \mathcal{P}(\Lambda)_{\mu}, \quad \mu = (\mu_{N-1}, \dots, \mu_1) \in (P_{\mathfrak{g}})^{N-1};$$

where

$$\mathcal{P}(\Lambda)_{\mu} = \mathcal{P}(v_N(\mu)) \otimes \dots \otimes \mathcal{P}(v_i(\mu)) \otimes \dots \otimes \mathcal{P}(v_1(\mu)) \subset V_g^V(\Lambda)$$

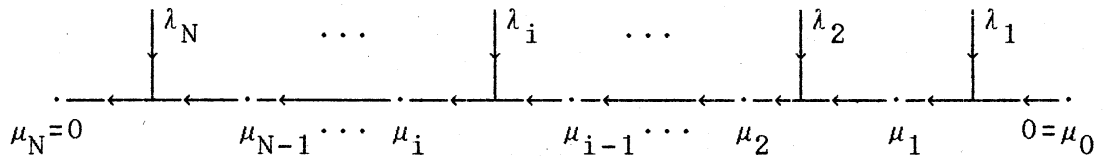
and

$$v_N(\mu) = \begin{pmatrix} \lambda_N \\ 0 \end{pmatrix}, \dots, v_i(\mu) = \begin{pmatrix} \lambda_i \\ \mu_i \end{pmatrix}, \dots, v_1(\mu) = \begin{pmatrix} \lambda_1 \\ \mu_1 \end{pmatrix}.$$

Then the space $\mathcal{P}(\Lambda)$ is isomorphic to $\mathcal{P}_{el}(\Lambda)$ of N-point functions of weight Λ as follows: to each $\varphi =$

$C(\varphi_N \otimes \dots \otimes \varphi_1) \in \mathcal{P}(\Lambda)$, assign the N-point function

$$\Phi_{\varphi_N \otimes \dots \otimes \varphi_1}(z) = \langle \Phi_{\varphi_N}(z_N) \dots \Phi_{\varphi_1}(z_1) \rangle \in \mathcal{P}_{el}(\Lambda).$$



Now introduce a system $KZ(\Lambda)$ of differential equations on $\text{Hom}_g(V_\Lambda, \mathbb{C})$ -valued functions $\Phi(z)$ on $X_N = \{z = (z_N, \dots, z_1) \in \mathbb{C}^N; z_i \neq z_j \text{ (i} \neq \text{j)}\}$

$$KZ(\Lambda) \quad \left(\kappa \frac{\partial}{\partial z_i} - \sum_{\substack{k=1 \\ k \neq i}}^N \frac{\Omega_{ik}}{z_i - z_k} \right) \Phi(z) = 0 \quad (1 \leq i \leq N)$$

due to Knizhnik-Zamolodchikov[KZ], where

$$\Omega_{ik} = \sum_{a=1}^{\dim g} \rho_i(X^a) \rho_k(X_a),$$

ρ_i denotes the g -action on the i -th component of $\text{Hom}(V_\Lambda, \mathbb{C})$ and $\{X^a\}$ and $\{X_a\}$ are dual bases of g .

Further introduce an additional ℓ -constraint condition, i.e. a system $\ell C(\Lambda)$ of algebraic equations

$$\ell C(\Lambda) \quad \sum_{|\mathbf{m}_i|=L_i} \binom{L_i}{\mathbf{m}_i} \prod_{k \neq i} (z_k - z_i)^{-m_k} \Phi(z) (X_\theta^{m_N} u_N, \dots, |\lambda_i\rangle, \dots, X_\theta^{m_1} u_1) = 0, \quad (1 \leq i \leq N)$$

for any $u_k \in V_{\lambda_k}$ ($k \neq i$), where $\mathbf{m}_i = (m_N, \dots, \hat{m}_i, \dots, m_1) \in (\mathbb{Z}_{\geq 0})^{N-1}$,

$|\mathbf{m}_i| = \sum_{k \neq i} m_k$, $L_i = \ell - (\lambda_i, \theta) + 1$ and $\binom{L_i}{\mathbf{m}_i}$ is the multinomial coefficient.

Remark. The system $KZ(\Lambda)$ of differential equations is completely integrable because of the infinitesimal pure braid relations among the operators Ω_{ik} (see [A]). The system $\ell C(\Lambda)$ is compatible with the system $KZ(\Lambda)$.

Any N -point function of weight Λ satisfies the systems

of $KZ(\Lambda)$ and $\mathcal{LC}(\Lambda)$. Hence

Theorem 2.

i) For any N -ple Λ of weights in P_ℓ , any N -point function $\langle \Phi^N(z_N) \cdots \Phi^1(z_1) \rangle$ of weight Λ is absolutely convergent in the region $\mathcal{R}_N$, and is analytically continued to a multivalued holomorphic function on X_N , where $\mathcal{R}_N$ is defined by

$$\mathcal{R}_N = \{z = (z_N, \dots, z_1) \in \mathbb{C}_+^N; |z_N| > \cdots > |z_1|\} \subset X_N.$$

ii) The solution space of the joint system $KZ(\Lambda)$ and $\mathcal{LC}(\Lambda)$ is isomorphic with $\mathcal{V}el(\Lambda)$, hence with $\mathcal{V}(\Lambda)$.

Note. If $v = \begin{bmatrix} \lambda \\ \mu \end{bmatrix} \in (CG)$, then $\mu = \lambda$, $\hat{\Delta}(v) = 0$, and $\mathcal{V}(v) \cong \text{Hom}_g(V_\lambda, V_\lambda) = \mathbb{C}id$.

If $v = \begin{bmatrix} \lambda \\ 0 \end{bmatrix} \in (CG)$, then $\mu = \lambda^+$, $\hat{\Delta}(v) = 2\Delta_\lambda$, and $\mathcal{V}(v) \cong \text{Hom}_g(V_\lambda \otimes V_{\lambda^+}, \mathbb{C}) = \mathbb{C}v$, where the anti-weight λ^+ of λ is defined as $-\lambda^+ (= w_0 \lambda)$ is the lowest weight of V_λ and v is normalised as $v(|\lambda\rangle \otimes w_0 |\lambda^+\rangle) = 1$, where w_0 is the longest element of the Weyl group of g .

3-point functions are essentially nothing but vertex

operators. The assignment to $\varphi \in \mathcal{P}\left(\begin{smallmatrix} \lambda \\ \mu_2 \mu_1 \end{smallmatrix}\right)$ the element

$$v \otimes \varphi \otimes \text{id} \in \mathcal{P}(\mu_2^*, \lambda, \mu_1),$$

$$v \otimes \varphi \otimes \text{id}(|u\rangle \otimes |v\rangle \otimes |w\rangle)$$

$$= v(|u\rangle \otimes \varphi(|v\rangle \otimes |w\rangle)) \quad \left[|u\rangle \otimes |v\rangle \otimes |w\rangle \in V_{\mu_2^*} \otimes V_{\lambda} \otimes V_{\mu_1} \right],$$

gives the isomorphism between them. Hence the space

$\mathcal{P}er\left(\begin{smallmatrix} \lambda \\ \mu_2 \mu_1 \end{smallmatrix}\right)$ of vertex operators is isomorphic with the

space $\mathcal{P}er(\mu_2^*, \lambda, \mu_1)$ of 3-point functions. More precisely,

the classical sector $\text{proj}_{V_{\mu_2}} \circ \Phi_{\varphi}(z) \big|_{V_{\lambda} \otimes V_{\mu_1}}$ of the vertex

operator $\Phi_{\varphi}(z)$ is given by

$$\lim_{z_t \nearrow \infty} \lim_{z_s \searrow 0} z^{2\Delta_{\mu_2}} \langle \Phi_v(z_t) \Phi_{\varphi}(z) \Phi_{\text{id}}(z_s) \rangle.$$

§3. Denote by $\mathcal{P}er(v_2) \circ \mathcal{P}er(v_1)$ the space of compositions $\Phi^2(z_2) \Phi^1(z_1)$ of vertex operators Φ^i of type v_i . Then

$$\sum_{\mu \in P_{\ell}} \mathcal{P}er\left(\begin{smallmatrix} \lambda_2 \\ \mu_t \mu \end{smallmatrix}\right) \circ \mathcal{P}er\left(\begin{smallmatrix} \lambda_1 \\ \mu \mu_s \end{smallmatrix}\right) \cong \mathcal{P}er(\Lambda) \cong \mathcal{P}(\Lambda),$$

where $\Lambda = (\mu_t^*, \lambda_2, \lambda_1, \mu_s)$.

The composition $\Phi^2(z_2) \Phi^1(z_1)$ is determined by the classical sector $\text{proj}_{V_{\mu_t}} \circ \Phi^2(z_2) \Phi^1(z_1) \big|_{V_{\mu_s}} \in$

$\text{Hom}_g(V_{\lambda_2} \otimes V_{\lambda_1} \otimes V_{\mu_s}, V_{\mu_t})$ and it is given by

$$\lim_{z_t \nearrow \infty} \lim_{z_s \searrow 0} z_t^{2\Delta_{\mu_t}} \langle \Phi_\nu(z_t) \Phi^2(z_2) \Phi^1(z_1) \Phi_{\text{id}}(z_s) \rangle.$$

Hence by Theorem 2, the composition $\Phi^2(z_2) \Phi^1(z_1)$ is absolutely convergent in the range $\mathcal{R}_2 = \{(z_2, z_1) \in \mathbb{C}_+^2; |z_2| > |z_1| > 0\}$, so by the analytic continuation it defines the holomorphic (multivalued) function valued in $\text{Hom}(V_{\lambda_2} \otimes V_{\lambda_1}, \text{Hom}(\mathcal{H}_{\mu_s}, \hat{\mathcal{H}}_{\mu_t}))$ on the complex manifold $M_2 = \{(z_2, z_1) \in (\mathbb{C} \setminus \{0\})^2; z_1 \neq z_2\}$.

Denote by $\Phi^2(u_2; z_1) \Phi^1(u_1, z_2) = C_\gamma (\Phi^2(u_2; z_2) \Phi^1(u_1, z_1))$ its analytic continuation along the path γ :

$$\gamma(t) = \left[\frac{z_2 + z_1}{2} + e^{\pi\sqrt{-1}t} \frac{z_2 - z_1}{2}, \frac{z_2 + z_1}{2} - e^{\pi\sqrt{-1}t} \frac{z_2 - z_1}{2} \right], \quad t \in [0, 1]$$

for $(w, z) \in \mathcal{R}$, then the corresponding analytic continuation

$$T \langle \Phi_\nu(z_t) \Phi^2(z_1) \Phi^1(z_2) \Phi_{\text{id}}(z_s) \rangle$$

satisfies the systems $\text{KZ}(\text{TA})$ and $\text{LC}(\text{TA})$ as a $\text{Hom}(V_{\lambda_1} \otimes V_{\lambda_2}, \text{Hom}(\mathcal{H}_{\mu_s}, \hat{\mathcal{H}}_{\mu_t}))$ -valued function, where T is the trans-

position operator: $\text{Hom}(V_{\lambda_2} \otimes V_{\lambda_1}, A) \longrightarrow \text{Hom}(V_{\lambda_1} \otimes V_{\lambda_2}, A)$,

$$(T\varphi)(u_1 \otimes u_2) = \varphi(u_2 \otimes u_1) \quad (u_2 \otimes u_1 \in V_{\lambda_1} \otimes V_{\lambda_2}),$$

and $\text{TA} = (\mu_t^*, \lambda_1, \lambda_2, \mu_s)$. Hence the analytic continuation

along γ gives an isomorphism between the spaces of compositions of vertex operators:

Theorem 3. (Commutation Relations)

For $\Lambda = (\mu_t^*, \lambda_2, \lambda_1, \mu_s)$, $C(\Lambda) = C_\gamma(\Lambda)$ is an isomorphism :

$$\begin{array}{ccc}
 C_\gamma(\Lambda): \mathcal{V}(\Lambda) & \xrightarrow{\quad\quad\quad} & \mathcal{V}(T\Lambda) \\
 \parallel & & \parallel \\
 \sum_{\mu \in P_\ell} \mathcal{V}\left(\begin{smallmatrix} \lambda_2 \\ \mu_t \quad \mu \end{smallmatrix}\right) \otimes \mathcal{V}\left(\begin{smallmatrix} \lambda_1 \\ \mu \quad \mu_s \end{smallmatrix}\right) & \xrightarrow{\quad\quad\quad} & \sum_{\mu \in P_\ell} \mathcal{V}\left(\begin{smallmatrix} \lambda_1 \\ \mu_t \quad \mu \end{smallmatrix}\right) \otimes \mathcal{V}\left(\begin{smallmatrix} \lambda_2 \\ \mu \quad \mu_s \end{smallmatrix}\right) .
 \end{array}$$

Remark. The isomorphisms $C_\gamma(\Lambda)$, $\Lambda \in P_\ell^4$ enjoy the braid relations: For any $N \geq 1$, $\mu_t, \mu_s \in P_\ell$, introduce the space

$$\mathcal{V}(N; \mu_t, \mu_s) = \sum_{\lambda_1, \dots, \lambda_N \in P_\ell} \mathcal{V}(\mu_t^*, \lambda_N, \dots, \lambda_1, \mu_s) .$$

Define the operators C_i ($1 \leq i \leq N-1$) on $\mathcal{V}(N; \mu_t, \mu_s)$ such that

$$C_i \mathcal{V}(\mu_t^*, \lambda_N, \dots, \lambda_1, \mu_s) \subset \mathcal{V}(\mu_t^*, \lambda_N, \dots, \lambda_i, \lambda_{i+1}, \dots, \lambda_1, \mu_s)$$

and

$$\begin{aligned}
 & C_i(\varphi_N \otimes \dots \otimes \varphi_1) \\
 &= \varphi_N \otimes \dots \otimes \varphi_{i+2} \otimes C(\mu_{i+1}, \lambda_{i+1}, \lambda_i, \mu_{i-1})(\varphi_{i+1} \otimes \varphi_i) \otimes \varphi_{i-1} \otimes \dots \otimes \varphi_1
 \end{aligned}$$

$$\begin{aligned} \text{for } \varphi_N \otimes \dots \otimes \varphi_1 &\in \mathcal{P}(\mu_t^*, \lambda_N, \dots, \lambda_1, \mu_s)(\mu_{N-1}, \dots, \mu_1) \\ &= \mathcal{P}\left(\begin{smallmatrix} \lambda_N \\ \mu_t & \mu_{N-1} \end{smallmatrix}\right) \otimes \dots \otimes \mathcal{P}\left(\begin{smallmatrix} \lambda_i \\ \mu_i & \mu_{i-1} \end{smallmatrix}\right) \otimes \dots \otimes \mathcal{P}\left(\begin{smallmatrix} \lambda_1 \\ \mu_1 & \mu_s \end{smallmatrix}\right). \end{aligned}$$

Then

$$C_i C_{i+1} C_i = C_{i+1} C_i C_{i+1}$$

as isomorphisms of $\mathcal{P}(N; \mu_t, \mu_s)$ to itself.

§4. The composition $\Phi^2(u_2; w) \Phi^1(u_1; z)$ is singular at $w=z$ and its behaviour near $w=z$ is described as follows.

For $\Lambda = (\mu_t^*, \lambda_2, \lambda_1, \mu_s)$, the space $V_g^V(\Lambda)$ has another decomposition

$$V_g^V(\Lambda) \xleftarrow[F]{\cong} \sum_{\nu \in P_+} \text{Hom}_g(V_{\lambda_2} \otimes V_{\lambda_1}, V_\nu) \otimes \text{Hom}_g(V_\nu \otimes V_{\mu_s}, V_{\mu_t}),$$

where the identification F is given by

$$F(\varphi_2 \otimes \varphi_1)(u_2 \otimes u_1 \otimes u_s) = \varphi_1(\varphi_2(u_2 \otimes u_1) \otimes u_s) \quad (u_i \in V_{\lambda_i}, u_s \in V_\mu),$$

For $\varphi_2 \in \mathcal{P}\left(\begin{smallmatrix} \nu \\ \mu_t & \mu_s \end{smallmatrix}\right)$ and $\varphi_1 \in \mathcal{P}\left(\begin{smallmatrix} \lambda_2 \\ \nu & \lambda_1 \end{smallmatrix}\right)$, a "vertex operator"

$\Phi_{\varphi_2 \otimes \varphi_1}^f(z)$ of $\mathcal{H}_{\mu_s}$ to $\hat{\mathcal{H}}_{\mu_t}$ parametrized by $V_{\lambda_2} \otimes V_{\lambda_1}$ defined by

$$\Phi_{\varphi_2 \otimes \varphi_1}^f(u_2 \otimes u_1; z) = \Phi_{\varphi_1}(\varphi_2(u_2 \otimes u_1); z) \quad (u_i \in V_{\lambda_i}).$$

Theorem 4. (Short range expansion or Fusion rule)

i) Near $w=z$ ($(w,z) \in \mathcal{R}_2$),

$$\begin{aligned} \Phi^2(u_2; w) \Phi^1(u_1; z) &= \sum_{\nu \in P_\ell} (w-z)^{-\hat{\Delta}(w_1)} \left[\Phi_{\psi_\nu}^f(u_2 \otimes u_1; z) + O(w-z) \right] \\ &\sim (w-z)^{-(\Delta_{\lambda_1} + \Delta_{\lambda_2})} \sum_{\nu \in P_\ell} (w-z)^{\Delta_\nu} \Phi_{\psi_\nu}^f(u_2 \otimes u_1; z), \end{aligned}$$

where $\psi_\nu \in \mathcal{P}\left(\begin{smallmatrix} \lambda_2 \\ \nu \end{smallmatrix}\right) \otimes \mathcal{P}\left(\begin{smallmatrix} \nu \\ \mu_t \mu_s \end{smallmatrix}\right)$, and $O(w-z)$ is holomorphic near $w=z$ and vanishes at $w=z$:

$$\sim \sum_{\nu \in P_\ell} (w-z)^{-\hat{\Delta}(w_1(\nu))}$$

The value of $(w-z)^{-\hat{\Delta}(w_1)}$ is chosen as it is positive for $(w,z) \in \mathcal{R}_2 \cap \mathbb{R}^2$.

ii) For $\Lambda = (\mu_t^*, \lambda_2, \lambda_1, \mu_s)$, the fusion gives an isomorphism

$$F(\Lambda): \mathcal{P}(\Lambda) \cong \sum_{\mu \in P_\ell} \mathcal{P}\left(\begin{smallmatrix} \lambda_2 \\ \mu_t \mu \end{smallmatrix}\right) \otimes \mathcal{P}\left(\begin{smallmatrix} \lambda_1 \\ \mu \mu_s \end{smallmatrix}\right) \longrightarrow \sum_{\nu \in P_\ell} \mathcal{P}\left(\begin{smallmatrix} \lambda_2 \\ \nu \end{smallmatrix}\right) \otimes \mathcal{P}\left(\begin{smallmatrix} \nu \\ \mu_t \mu_s \end{smallmatrix}\right)$$

defined by

$$F(\Lambda)(\varphi^2 \otimes \varphi^1) = \sum_{\nu \in P_\ell} \psi_\nu (\varphi^2 \otimes \varphi^1 \in \mathcal{P}\left(\begin{smallmatrix} \lambda_2 \\ \mu_t \mu \end{smallmatrix}\right) \otimes \mathcal{P}\left(\begin{smallmatrix} \lambda_1 \\ \mu \mu_s \end{smallmatrix}\right)),$$

where ψ_ν are the ones obtained in i) for $\Phi^i = \Phi_\varphi^i$.

Theorem 5.

For $\Lambda = (\mu_t^*, \lambda_2, \lambda_1, \mu_s)$, the following diagram commutes:

$$\begin{array}{ccc}
 \mathcal{P}(\Lambda) & \xrightarrow{F(\Lambda)} & \sum_{\nu \in P_\ell} \mathcal{P}\left(\begin{smallmatrix} \nu \\ \mu_t \mu_s \end{smallmatrix}\right) \otimes \mathcal{P}\left(\begin{smallmatrix} \lambda_2 \\ \nu \lambda_1 \end{smallmatrix}\right) \\
 \downarrow C_\gamma & & \downarrow \text{id} \otimes C_{\gamma_0} \\
 \mathcal{P}(T\Lambda) & \xrightarrow{F(T\Lambda)} & \sum_{\nu \in P_\ell} \mathcal{P}\left(\begin{smallmatrix} \nu \\ \mu_t \mu_s \end{smallmatrix}\right) \otimes \mathcal{P}\left(\begin{smallmatrix} \lambda_1 \\ \nu \lambda_2 \end{smallmatrix}\right)
 \end{array}$$

Remark. The equation $KZ(\Lambda)$ in the limit $z_4 \nearrow \infty$, $z_1 \searrow 0$ is reduced to a differential equation (reduced KZ-system) $RKZ(\Lambda)$ on $V_g(\Lambda)$ - functions of one variable $\xi = z_3/z_2$. The equation $RKZ(\Lambda)$ has only regular singularities at $\xi = 0, 1, \infty$. The isomorphisms $C_\gamma(\Lambda)$ and $F(\Lambda)$ are essentially nothing but the connection matrices from the space of its solutions regularized at $\xi = 0$ to the spaces of solutions regularized at $\xi = \infty$ and $\xi = 1$ respectively.

§5. Naturally arises a problem to determine the

isomorphisms $C_\gamma(\Lambda)$ and $F(\Lambda)$, but it is very difficult to carry out for all cases. We succeeded (last year) in the case where $\hat{g}$ is an affine Lie algebra of type $A_n^{(1)}$ and $\Lambda = (\mu_t, \square, \square, \mu_s)$, where $\square$ means a Young diagram consisting of one node and represent the vector representation of $g = \mathfrak{sl}(n+1, \mathbb{C})$.

Now let $\hat{g}$ be an affine Lie algebra of type $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$, and P_+^0 be the set of weights $\lambda \in P_+$ such that the simple g -module V_λ can appear in some tensor products of the vector representations $V_\square$ of $g = \mathfrak{sl}(n+1; \mathbb{C})$, $\mathfrak{o}(2n; \mathbb{C})$, $\mathfrak{sp}(2n; \mathbb{C})$, $\mathfrak{o}(2n+1; \mathbb{C})$ respectively.

For each $\tau \in P_\ell$, introduce the space

$$\mathcal{P}_N(\tau) = \sum_{\mu} \mathcal{P}_N(\tau)_\mu, \\ \mathcal{P}_N(\tau)_\mu = \mathcal{P}\left(\begin{smallmatrix} \square \\ \tau \end{smallmatrix} \mu_{N-1}\right) \otimes \cdots \otimes \mathcal{P}\left(\begin{smallmatrix} \square \\ \mu_i \end{smallmatrix} \mu_{i-1}\right) \otimes \cdots \otimes \mathcal{P}\left(\begin{smallmatrix} \square \\ \mu_1 \end{smallmatrix} 0\right),$$

where the summation is taken over the set $P_\ell^{N-1} \ni \mu = (\mu_1, \dots, \mu_{N-1})$. Then $\mathcal{P}_N(\tau)$ is the subspace of $\mathcal{P}(N; \tau, 0)$ which is invariant under the operators $C_i (1 \leq i \leq N-1)$.

The braid group B_N with N -strings of $\mathbb{C}$ has a system $\{b_i; 1 \leq i \leq N-1\}$ of generators with the fundamental relations:

$$(BR) \quad \begin{cases} b_i b_j = b_j b_i & (|i-j| \geq 2) \\ b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1} & (1 \leq i \leq N-2). \end{cases}$$

These generators b_i are represented by the curves on $\mathbb{C}$ defined by

$$b_i(t) = \left(N, N-1, \dots, i + \frac{1}{2}(1+e^{\pi\sqrt{-1}t}), i + \frac{1}{2}(1-e^{\pi\sqrt{-1}t}), \dots, 2, 1 \right) \\ t \in [0, 1],$$

We now define a monodromy representation π_N^τ of B_N on the space $\mathcal{V}_N(\tau)$ as $\pi_N^\tau(b_i) = C_i$ ($1 \leq i \leq N-1$). Then we get the main theorems.

Theorem 6.

If $\mathfrak{g}$ is of type A_n , then the monodromy representation $r^{1/(n+1)} \pi_N^\lambda$ in $\mathcal{V}_N(\tau)$ factors through the Iwahori-Hecke algebra $H_N(r)$, where $r = \exp(\frac{\pi\sqrt{-1}}{\ell+n+1})$.

Note. The algebra $H_N(r)$ is defined by generators $\{\tau_i, \tau_i^{-1} (1 \leq i \leq N-1)\}$ with the defining relations:
 $\tau_i \tau_i^{-1} = \tau_i^{-1} \tau_i = 1$, $\tau_i - \tau_i^{-1} = (r - r^{-1})$ ($1 \leq i \leq N-1$),
 $\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}$ and $\tau_i \tau_j = \tau_j \tau_i$ ($|i-j| \geq 2$).

Theorem 7. If $\mathfrak{g}$ is the simple Lie algebra of type B_n , C_n or D_n . Then the monodromy representation π_N^λ in $\mathcal{V}_N(\tau)$ factors through the Birman-Wenzl-Murakami algebra $C_N(\mathfrak{g}; r)$

where $r = \exp(\frac{\pi\sqrt{-1}}{\ell+g})$, $g(B_n) = 2n-1$, $g(C_n) = n+1$, $g(D_n) = 2n-2$; $C_N(B_n; r) = C_N(r^{-n-1/2}, r)$, $C_N(C_n; r) = C_N(r^n, r)$ and $C_N(D_n; r) = C_N(r^{-n}, r)$.

Note. The algebra $C_N(a, r)$ is defined by generators $\{\tau_i, \tau_i^{-1}, \varepsilon_i (1 \leq i \leq N-1)\}$ with the defining relations:

$$\tau_i \tau_i^{-1} = \tau_i^{-1} \tau_i = 1, \quad \tau_i \varepsilon_i = \varepsilon_i \tau_i = -(a^2 r)^{-1} \varepsilon_i,$$

$$\tau_i - \tau_i^{-1} = (r - r^{-1})(1 - \varepsilon_i) \quad (1 \leq i \leq N-1),$$

$$\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}, \quad \varepsilon_i \varepsilon_{i+1} \varepsilon_i = \varepsilon_i, \quad \varepsilon_{i+1} \varepsilon_i \varepsilon_{i+1} = \varepsilon_{i+1},$$

$$\tau_i^{\pm 1} \varepsilon_{i+1} \varepsilon_i = \tau_{i+1}^{\mp 1} \varepsilon_i, \quad \tau_{i+1}^{\pm 1} \varepsilon_i \varepsilon_{i+1} = \tau_i^{\mp 1} \varepsilon_{i+1},$$

$$\varepsilon_i \varepsilon_{i+1} \tau_i^{\pm 1} = \varepsilon_i \tau_{i+1}^{\mp 1}, \quad \varepsilon_{i+1} \varepsilon_i \tau_{i+1}^{\pm 1} = \varepsilon_{i+1} \tau_i^{\mp 1} \quad (1 \leq i \leq N-2),$$

$$\tau_i \tau_j = \tau_j \tau_i, \quad \varepsilon_i \tau_j = \tau_j \varepsilon_i, \quad \varepsilon_i \varepsilon_j = \varepsilon_j \varepsilon_i \quad (|i-j| \geq 2),$$

The proof is carried out by the explicit calculation of a differential equation of 4-point function in a very special case and the algebraic arguments for the algebras $H_N(r)$ and $C_N(a, r)$.

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