

PARAFERMION VERTEX OPERATOR ALGEBRAS AND W-ALGEBRAS

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ABSTRACT. We prove the conjectural isomorphism between the level k $\widehat{\mathfrak{sl}}_2$ -parafermion vertex operator algebra and the $(k+1, k+2)$ minimal series W_k -algebra for all $k \in \mathbb{N}$. As a consequence, we obtain the conjectural isomorphism between the $(k+1, k+2)$ minimal series W_k -algebra and the coset vertex operator algebra $SU(k)_1 \otimes SU(k)_1 / SU(k)_2$.

1. INTRODUCTION

A parafermion vertex operator algebra $K(\mathfrak{g}, k)$ is by definition the commutant of the Heisenberg vertex operator subalgebra $M_{\mathfrak{h}}(k, 0)$ in the simple vertex operator algebra $L_{\widehat{\mathfrak{g}}}(k, 0)$ of the level k integrable highest weight module for an affine Kac-Moody algebra $\widehat{\mathfrak{g}}$, where $\mathfrak{g}$ is a finite dimensional simple Lie algebra and $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Some basic properties of $K(\mathfrak{g}, k)$ were studied in [21, 22]. Their arguments heavily depend on the properties of the parafermion vertex operator algebra $K(\mathfrak{sl}_2, k)$, which were obtained in [16, 17]. Thus, for the study of the general parafermion vertex operator algebras, it is essential to understand the case $\mathfrak{g} = \mathfrak{sl}_2$.

The parafermion vertex operator algebra $K(\mathfrak{sl}_2, k)$ is also known as a W -algebra. It was conjectured over 20 years ago in the physics literature [10] that the parafermion vertex operator algebra $K(\mathfrak{sl}_2, k)$ is isomorphic to the $(k+1, k+2)$ -minimal series W -algebra [29, 4] associated with $\mathfrak{sl}_k$. The purpose of this paper is to prove this conjecture. As a consequence, the rationality of $K(\mathfrak{sl}_2, k)$ is established.

In [16, 17] it was shown that $K(\mathfrak{sl}_2, k)$ is isomorphic to the simple quotient of the W -algebra $W(2, 3, 4, 5)$ of [10, 33] for $k \geq 5$. This relationship between $K(\mathfrak{sl}_2, k)$ and the simple quotient of $W(2, 3, 4, 5)$ is clear because a set of generators and the operator product expansions among the generators of these two vertex operator algebras are known [10, 16, 17], and they coincide with each other. On the other hand, the W -algebra $\mathcal{W}^\ell(\mathfrak{sl}_k)$ was constructed in a different manner and the explicit operator product expansions of generators of $\mathcal{W}^\ell(\mathfrak{sl}_k)$ are not known in general. Therefore, we should take another approach to establish a correspondence between the vertex operator algebras $K(\mathfrak{sl}_2, k)$ and $\mathcal{W}^\ell(\mathfrak{sl}_k)$.

The key idea is the use of a decomposition $L_{\widehat{\mathfrak{sl}}_2}(k, 0) = \bigoplus_{j=0}^{k-1} V_{\mathbb{Z}\gamma-j\gamma/k} \otimes M^j$, where $V_{\mathbb{Z}\gamma-j\gamma/k}$ is a simple module for a vertex operator algebra $V_{\mathbb{Z}\gamma}$ associated with a rank one lattice $\mathbb{Z}\gamma$, $\langle \gamma, \gamma \rangle = 2k$, and M^j is a simple module for $M^0 = K(\mathfrak{sl}_2, k)$. That is, we consider not only the vertex operator algebra $K(\mathfrak{sl}_2, k)$ but also some of its simple modules and take tensor product with $V_{\mathbb{Z}\gamma-j\gamma/k}$. We shall characterize $L_{\widehat{\mathfrak{sl}}_2}(k, 0)$ as a unique vertex operator algebra that admits such a decomposition, see Section 3 for the precise statement. The fusion rules among the simple $V_{\mathbb{Z}\gamma}$ -modules play an important role in the argument here. Furthermore, we shall apply the result to show that under a certain assumption, a

vertex operator algebra having similar properties as the $(k+1, k+2)$ -minimal series W -algebra $\mathcal{W}_{k+1, k+2}(\mathfrak{sl}_k)$ associated with $\mathfrak{sl}_k$ is in fact isomorphic to the parafermion vertex operator algebra $K(\mathfrak{sl}_2, k)$.

We note that by the level rank duality $K(\mathfrak{sl}_2, k)$ is isomorphic to the coset vertex algebra $\text{Com}_{L_{\widehat{\mathfrak{sl}_k}(1,0)} \otimes L_{\widehat{\mathfrak{sl}_k}(1,0)}}(L_{\widehat{\mathfrak{sl}_k}}(2, 0))$ ([37]). Therefore our result gives another conjectural isomorphism ([36, 11])

$$\mathcal{W}_{k+1, k+2}(\mathfrak{sl}_k) \cong \text{Com}_{L_{\widehat{\mathfrak{sl}_k}(1,0)} \otimes L_{\widehat{\mathfrak{sl}_k}(1,0)}}(L_{\widehat{\mathfrak{sl}_k}}(2, 0))$$

for all $k \geq 2$.

This paper is the final version of our unpublished preprint “A characterization of parafermion vertex operator algebras”.

The organization of the paper is as follows. In Section 2, we prepare some materials which will be necessary in later sections. We review intertwining operators and simple current extensions. We also recall some properties of the lattice vertex operator algebra $V_{\mathbb{Z}\gamma}$ and its simple modules as well as the construction of the parafermion vertex operator algebra $K(\mathfrak{sl}_2, k)$. In Section 3, we study a vertex operator algebra of the form $V = \bigoplus_{j=0}^{k-1} V_{\mathbb{Z}\gamma - j\gamma/k} \otimes M^j$. Under a certain hypothesis, we show that V is isomorphic to $L_{\widehat{\mathfrak{sl}_2}}(k, 0)$. In Section 4, we obtain a characterization of the parafermion vertex operator algebra $K(\mathfrak{sl}_2, k)$. In Section 5, we recall some results on W -algebras established in [1, 3, 4, 6]. Finally, in Section 6, we apply the result of Section 4 to show that $K(\mathfrak{sl}_2, k)$ is isomorphic to $\mathcal{W}_{k+1, k+2}(\mathfrak{sl}_k)$. A correspondence of the simple modules for $K(\mathfrak{sl}_2, k)$ with those for $\mathcal{W}_{k+1, k+2}(\mathfrak{sl}_k)$ is discussed as well. In Appendix A, we prove Proposition 5.2 that was stated in [24] without a proof.

2. PRELIMINARIES

We use standard notation for vertex operator algebras and their modules [30, 31, 38]. Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra and (M, Y_M) its module. Then

$$Y_M(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1}$$

is the vertex operator associated with $v \in V$. The linear operator v_n of M is called a component operator. The eigenspace with eigenvalue n for the operator $L(0) = \omega_1$ is denoted by $M_{(n)}$. An element $w \in M_{(n)}$ is said to be of weight n or ω -weight n and we write $\text{wt } w = n$. In this paper we always assume that $\text{wt } w \in \mathbb{Q}$. The top level of M means the nonzero weight subspace $M_{(n)}$ of smallest possible n . We denote the top level of M by $M(0)$. The weight of the top level $M(0)$ is called the top weight of M . The generating function

$$\text{ch } M = \sum_{n \in \mathbb{Q}} (\dim M_{(n)}) q^n$$

of $\dim M_{(n)}$ is called the character of M . A vertex operator algebra V is said to be of CFT-type if $V = \bigoplus_{n \geq 0} V_{(n)}$ and $V_{(0)} = \mathbb{C}\mathbf{1}$.

Let $M' = \bigoplus_{n \in \mathbb{Q}} M_{(n)}^*$ be the restricted dual space of a V -module M , where $M_{(n)}^*$ is the ordinary dual space of $M_{(n)}$. The adjoint vertex operator $Y_{M'}(v, z) \in (\text{End } M')[[z, z^{-1}]]$ is defined by

$$\langle Y_{M'}(v, z)w', w \rangle_M = \langle w', Y_M(e^{zL(1)}(-z^{-2})^{L(0)}v, z^{-1})w \rangle_M$$

for $v \in V$, $w \in M$ and $w' \in M'$, where $\langle \cdot, \cdot \rangle_M$ is the natural pairing of M' and M [30, (5.2.4)]. Then $(M', Y_{M'})$ is a V -module [30, Theorem 5.2.1] called the contragredient or dual module of M . If M and M' are isomorphic as V -modules, then M is said to be self-dual. The vertex operator algebra V is said to be self-dual if V is isomorphic to its dual V' as a V -module.

For an automorphism g of V , we define a V -module $(M \circ g, Y_{M \circ g})$ by setting $M \circ g = M$ as vector spaces and $Y_{M \circ g}(v, z) = Y_M(gv, z)$ for $v \in V$. Then $M \mapsto M \circ g$ induces a permutation on the set of simple V -modules. The V -module M is said to be g -stable if $(M \circ g, Y_{M \circ g})$ is isomorphic to (M, Y_M) .

2.1. Fusion rules. We review intertwining operators for later use. Let V be a vertex operator algebra and (U^i, Y_{U^i}) , $i = 1, 2, 3$ simple V -modules. Let $\mathcal{Y}(\cdot, z)$ be an intertwining operator of type $\binom{U^3}{U^1 U^2}$ [30, Section 5.4]. We define linear operators u_m^1 and $u^1(n)$ from U^2 to U^3 by

$$\begin{aligned} \mathcal{Y}(u^1, z)u^2 &= \sum_{m \in \mathbb{Q}} u_m^1 u^2 z^{-m-1} \\ &= \sum_{n \in \mathbb{Z}} u^1(n) u^2 z^{-n-h_1-h_2+h_3} \end{aligned}$$

for $u^i \in U^i$, $i = 1, 2$, $m \in \mathbb{Q}$ and $n \in \mathbb{Z}$, where h_i is the top weight of U^i . If u^1 is homogeneous, then the weight of the operator u_m^1 is $\text{wt } u_m^1 = \text{wt } u^1 - m - 1$ [30, (5.4.14)].

In the case $U^1 = V$ and $u^1 = \mathbf{1}$, it is well-known that $\mathcal{Y}(\mathbf{1}, z) : U^2 \rightarrow U^3$ is a homomorphism of V -modules (cf. page 285 of [39]).

Let $I\left(\binom{U^3}{U^1 U^2}\right) = I_V\left(\binom{U^3}{U^1 U^2}\right)$ be the space of intertwining operators of type $\binom{U^3}{U^1 U^2}$.

Lemma 2.1. *Let V be a self-dual vertex operator algebra and U^1, U^2 simple V -modules with integral weight. Assume that $I\left(\binom{V}{U^1 U^2}\right) \neq 0$. Then the following assertions hold.*

- (1) U^1 is isomorphic to the dual module $(U^2)'$ of U^2 .
- (2) Let $0 \neq \mathcal{Y}(\cdot, z) \in I\left(\binom{V}{U^1 U^2}\right)$. Then for any $0 \neq u^1 \in U^1(0)$, there exists $u^2 \in U^2(0)$ such that the weight 0 coefficient $u_{2h-1}^1 u^2$ of $\mathcal{Y}(u^1, z)u^2$ is nonzero: $0 \neq u_{2h-1}^1 u^2 \in V(0)$, where h is the top weight of U^1 .

Proof. Since the weights of U^1 and U^2 are integral and V is self-dual, there are one-to-one correspondences among the four spaces

$$I\left(\binom{V}{U^1 U^2}\right), \quad I\left(\binom{(U^2)'}{U^1 V'}\right), \quad I\left(\binom{(U^2)'}{V' U^1}\right), \quad I\left(\binom{(U^2)'}{V U^1}\right)$$

of intertwining operators by [30, Propositions 5.4.7 and 5.5.2]. More precisely, let $\psi : V \rightarrow V'$ be an isomorphism of V -modules and $0 \neq \mathcal{Y}(\cdot, z) \in I\left(\binom{V}{U^1 U^2}\right)$. We consider three intertwining operators

$$\mathcal{Y}^1(\cdot, z) \in I\left(\binom{(U^2)'}{U^1 V'}\right), \quad \mathcal{Y}^2(\cdot, z) \in I\left(\binom{(U^2)'}{V' U^1}\right), \quad \mathcal{Y}^3(\cdot, z) \in I\left(\binom{(U^2)'}{V U^1}\right)$$

defined by

$$\langle \mathcal{Y}^1(u^1, z)\psi(a), u^2 \rangle_{U^2} = \langle \psi(a), \mathcal{Y}(e^{zL(1)}(-z^{-2})^{L(0)}u^1, z^{-1})u^2 \rangle_V, \quad (2.1)$$

$$\mathcal{Y}^2(\psi(a), z)u^1 = e^{zL(-1)}\mathcal{Y}^1(u^1, -z)\psi(a), \quad (2.2)$$

$$\mathcal{Y}^3(a, z)u^1 = \mathcal{Y}^2(\psi(a), z)u^1, \quad (2.3)$$

respectively for $a \in V$, $u^1 \in U^1$ and $u^2 \in U^2$ [30, (5.5.4), (5.4.33)].

Let $a = \mathbf{1}$ and $0 \neq u^1 \in U^1(0)$. Then $\mathcal{Y}^3(\mathbf{1}, z) \neq 0$ by [18, Proposition 11.9]. Since U^1 and $(U^2)'$ are simple by our assumption and [30, Proposition 5.3.2], $\mathcal{Y}^3(\mathbf{1}, z) : U^1 \rightarrow (U^2)'$ is in fact an isomorphism of V -modules. Thus the assertion (1) holds.

We can choose $u^2 \in U^2(0)$ so that

$$\langle \mathcal{Y}^3(\mathbf{1}, z)u^1, u^2 \rangle_{U^2} \neq 0, \quad (2.4)$$

for $\mathcal{Y}^3(\mathbf{1}, z)u^1$ is a nonzero element of the dual space $U^2(0)^*$ of $U^2(0)$. The weight of u^2 coincides with that of u^1 . It follows from (2.2), (2.3) and (2.4) that

$$\langle e^{zL(-1)}\mathcal{Y}^1(u^1, -z)\psi(\mathbf{1}), u^2 \rangle_{U^2} \neq 0. \quad (2.5)$$

Since $u^2 \in U^2(0)$, we have $L(1)u^2 = 0$ and

$$\langle \mathcal{Y}^1(u^1, -z)\psi(\mathbf{1}), u^2 \rangle_{U^2} \neq 0 \quad (2.6)$$

by (2.5) and [30, (5.2.10)]. Recall that $u^1 \in U^1(0)$ and the weight of u^1 is h . Then

$$e^{-zL(1)}(-z^{-2})^{L(0)}u^1 = (-z^{-2})^h u^1$$

and so

$$\langle \mathcal{Y}^1(u^1, -z)\psi(\mathbf{1}), u^2 \rangle_{U^2} = (-z^{-2})^h \langle \psi(\mathbf{1}), \mathcal{Y}(u^1, -z^{-1})u^2 \rangle_V,$$

which is nonzero by (2.6). Since $\psi(\mathbf{1})$ is an element of the dual space $V_{(0)}^*$ of $V_{(0)}$, we conclude that $u_{2h-1}^1 u^2$ is a nonzero element of $V_{(0)}$. $\square$

2.2. Simple current extensions. A simple module M of a vertex operator algebra is called a simple current if the tensor product $M \boxtimes N$ exists and it is a simple module for every simple module N . We review some known results about simple current extensions of vertex operator algebras for later use.

We assume the following hypothesis.

Hypothesis 2.2. (1) V is a simple, self-dual, rational and C_2 -cofinite vertex operator algebra of CFT-type.

(2) The top weight of any simple V -module M is non-negative, and is zero only if $M = V$.

(3) U^i , $i \in D$ is a set of simple current V -modules with integral weight, where D is a finite abelian group and $U^0 = V$. The fusion rules among U^i 's are

$$U^i \times U^j = U^{i+j}, \quad i, j \in D.$$

Let $\mathcal{I}_{i,j}^{i+j} = I_V \left(\begin{smallmatrix} U^{i+j} \\ U^i \ U^j \end{smallmatrix} \right)$ be the space of intertwining operators of type $\left(\begin{smallmatrix} U^{i+j} \\ U^i \ U^j \end{smallmatrix} \right)$ for $i, j \in D$. Let $\mathcal{Y}_{i,j}^{i+j}$ be a nonzero element of $\mathcal{I}_{i,j}^{i+j}$. Since $\mathcal{I}_{i,j}^{i+j}$ is one dimensional, $\mathcal{Y}_{i,j}^{i+j}$ is unique up to a nonzero scalar multiple. However it is far from trivial whether there are $\lambda_{i,j} \in \mathbb{C}^\times$ such that $\{\lambda_{i,j} \mathcal{Y}_{i,j}^{i+j}\}_{i,j \in D}$ gives a vertex operator algebra structure on a direct sum $\bigoplus_{i \in D} U^i$.

For $i, j \in D$, define $\Omega(i, j) \in \mathbb{C}^\times$ by

$$\mathcal{Y}_{i,j}^{i+j}(u, z)v = \Omega(i, j)e^{zL(-1)}\mathcal{Y}_{j,i}^{i+j}(v, -z)u \quad (2.7)$$

for $u \in U^i$, $v \in U^j$ [12, Definition 2.2.4] (see also [13, 23]). Such a constant $\Omega(i, j)$ exists by [30, Proposition 5.4.7].

The following fact has been shown in the proof of [23, Theorem 4.1] (see also [13, Theorem 3.12]).

Proposition 2.3. $\Omega(i, i) = 1$, $i \in D$.

Indeed, the categorical dimension of U^i coincides with the quantum dimension in the sense of [15, Definition 3.1] by [15, Eq. (4.1)] under Hypothesis 2.2. Since U^i is a simple current, the quantum dimension of U^i is 1 [15, Lemma 4.15]. Then the condition that U^i has integral weight implies $\Omega(i, i) = 1$ by the equation $e(q_\Omega(\alpha)) = e(-q_\Delta(\alpha))$ in the proof of [23, Theorem 4.1], for $q_\Delta(\alpha) = 0$ with $\alpha = i$.

The condition that $\Omega(i, i) = 1$, $i \in D$ is called evenness in [12]. The evenness implies the next theorem [12, Theorem 3.2.12], [13, Theorem 3.12], [23, Theorem 4.2].

Theorem 2.4. *There exists a choice of nonzero intertwining operators $\mathcal{Y}_{i,j}^{i+j} \in \mathcal{I}_{i,j}^{i+j}$, $i, j \in D$ which gives a vertex operator algebra structure on $\bigoplus_{i \in D} U^i$ as an extension of V .*

Such a vertex operator algebra structure on $\bigoplus_{i \in D} U^i$ is unique up to isomorphism [20, Proposition 5.3]. The vertex operator algebra $\bigoplus_{i \in D} U^i$ is called a simple current extension of V . It is a simple, self-dual, rational and C_2 -cofinite vertex operator algebra of CFT-type [42, Theorem 2.14].

2.3. Lattice vertex operator algebra $V_{\mathbb{Z}\gamma}$. We recall some basic properties of a vertex operator algebra associated with a positive definite even rank one lattice. Let $\mathbb{Z}\gamma$ be a positive definite even rank one lattice generated by γ , where the square norm of γ is $\langle \gamma, \gamma \rangle = 2k$. Let $V_{\mathbb{Z}\gamma} = M(1) \otimes \mathbb{C}[\mathbb{Z}\gamma]$ be a vertex operator algebra associated with the lattice $\mathbb{Z}\gamma$ [31]. Thus $M(1)$ is a simple highest weight module for the Heisenberg algebra generated by γ with highest weight 0. It is isomorphic to a polynomial algebra $\mathbb{C}[\gamma(-n) \mid n \in \mathbb{Z}_{>0}]$ as a vector space. Since $\mathbb{Z}\gamma$ is a rank one lattice, the twisted group algebra $\mathbb{C}\{\mathbb{Z}\gamma\}$ considered in [31] is isomorphic to an ordinary group algebra $\mathbb{C}[\mathbb{Z}\gamma]$. Its standard basis is $\{e^{n\gamma} \mid n \in \mathbb{Z}\}$ with multiplication $e^\alpha e^\beta = e^{\alpha+\beta}$. The conformal vector is

$$\omega_\gamma = \frac{1}{4k}\gamma(-1)^2\mathbf{1} \quad (2.8)$$

and its central charge is 1.

The vertex operator algebra $V_{\mathbb{Z}\gamma}$ is simple, self-dual, rational, C_2 -cofinite and of CFT-type. The simple modules for $V_{\mathbb{Z}\gamma}$ were classified [14]. Since the dual lattice of $\mathbb{Z}\gamma$ is $(1/2k)\mathbb{Z}\gamma$, any simple $V_{\mathbb{Z}\gamma}$ -module is isomorphic to one of $V_{\mathbb{Z}\gamma+i\gamma/2k}$, $0 \leq i \leq 2k-1$. The top level of $V_{\mathbb{Z}\gamma+i\gamma/2k}$ is $\mathbb{C}e^{i\gamma/2k}$ with weight $i^2/4k$ if $0 \leq i < k$ and $\mathbb{C}e^{(2k-i)\gamma/2k}$ with weight $(2k-i)^2/4k$ if $k < i \leq 2k-1$. In the case $i = k$, the top level is $\mathbb{C}e^{\gamma/2} + \mathbb{C}e^{-\gamma/2}$ with weight $k/4$.

The fusion rules among these simple modules are also known and intertwining operators were constructed by using vertex operators [18, Chapter 12]. In fact, the fusion rules are

$$V_{\mathbb{Z}\gamma+i\gamma/2k} \times V_{\mathbb{Z}\gamma+j\gamma/2k} = V_{\mathbb{Z}\gamma+(i+j)\gamma/2k}. \quad (2.9)$$

In particular, all the simple modules are simple currents.

Since the commutator map $(\cdot, \cdot)$ of [18, (12.5)] is trivial for the rank one lattice $(1/k)\mathbb{Z}\gamma$ and since $\langle \gamma, i\gamma/k \rangle = 2i \in 2\mathbb{Z}$ implies $(-1)^{\langle \bar{a}, \bar{b} \rangle} = 1$ in [18, (12.5)], the vertex operator $Y(\cdot, z)$ on $V_{(1/k)\mathbb{Z}\gamma} = M(1) \otimes \mathbb{C}[(1/k)\mathbb{Z}\gamma]$ defined in [31, Section 8.4] (see also [18, Chapter 3]) itself can be taken as the intertwining operator $\mathcal{Y}_{\lambda_j}(\cdot, z)$ of [18, (12.3)] for $V_{\mathbb{Z}\gamma+j\gamma/k}$, $0 \leq j \leq k-1$. That is, $(-1)^{\langle \bar{a}, \bar{b} \rangle} c(\bar{a}, \bar{b}) = 1$ in [18, (12.5)] and so $Y(v, z)w$ with $v \in V_{\mathbb{Z}\gamma+i\gamma/k}$ and $w \in V_{\mathbb{Z}\gamma+j\gamma/k}$ satisfies the Jacobi identity for intertwining operators [18, (12.8)].

Let $\mathcal{Y}_{L,i,j}^{i+j}(\cdot, z)$ be the vertex operator $Y(\cdot, z)$ for $V_{(1/k)\mathbb{Z}\gamma}$ restricted to $V_{\mathbb{Z}\gamma-i\gamma/k}$ and acting on $V_{\mathbb{Z}\gamma-j\gamma/k}$, so that $\mathcal{Y}_{L,i,j}^{i+j}(u, z)v = Y(u, z)v$ for $u \in V_{\mathbb{Z}\gamma-i\gamma/k}$ and $v \in V_{\mathbb{Z}\gamma-j\gamma/k}$. It is an intertwining operator of type

$$\begin{pmatrix} V_{\mathbb{Z}\gamma-(i+j)\gamma/k} \\ V_{\mathbb{Z}\gamma-i\gamma/k} & V_{\mathbb{Z}\gamma-j\gamma/k} \end{pmatrix}$$

for the vertex operator algebra $V_{\mathbb{Z}\gamma}$. The action of $Y(e^{\pm\gamma/k}, z)$ on the top level of $V_{\mathbb{Z}\gamma \pm \gamma/k}$ will be used later. By the definition

$$Y(e^{\pm\gamma/k}, z) = E^-(\mp\gamma/k, z)E^+(\mp\gamma/k, z)e^{\pm\gamma/k}z^{\pm\gamma/k},$$

where $E^\pm(\alpha, z) = \exp(\sum_{n \in \pm\mathbb{Z}_{>0}} \frac{\alpha(n)}{n} z^{-n})$. Since $e^{\gamma/k}z^{\gamma/k}e^{-\gamma/k} = \mathbf{1}z^{-2/k}$, we have

$$Y(e^{\gamma/k}, z)e^{-\gamma/k} = \mathbf{1}z^{-2/k} + \frac{1}{k}\gamma(-1)\mathbf{1}z^{1-2/k} + \dots \quad (2.10)$$

2.4. Parafermion vertex operator algebra $K(\mathfrak{sl}_2, k)$. We recall from [7, 16, 17] the properties of parafermion vertex operator algebra associated with $\mathfrak{sl}_2$. Let $k \geq 3$ be an integer. Let $\{h, e, f\}$ be a standard Chevalley basis of the Lie algebra $\mathfrak{sl}_2$, so that $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$ for the bracket and $(h|h) = 2$, $(e|f) = 1$, $(h|e) = (h|f) = (e|e) = (f|f) = 0$ for the normalized invariant inner product.

Let $V(k, 0) = V_{\widehat{\mathfrak{sl}_2}}(k, 0)$ be a Weyl module for the affine Lie algebra $\widehat{\mathfrak{sl}_2} = \mathfrak{sl}_2 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}C$ at level k . Denote by $\mathbf{1}$ its canonical highest weight vector, which is called the vacuum vector. Then $\mathfrak{sl}_2 \otimes \mathbb{C}[t]$ acts as 0 and C acts as k on $\mathbf{1}$, and $V(k, 0)$ is the induced module of the $\mathfrak{sl}_2 \otimes \mathbb{C}[t] \oplus \mathbb{C}C$ -module $\mathbb{C}\mathbf{1}$. We write $a(n)$ for the action of $a \otimes t^n$ on $V(k, 0)$. The Weyl module $V(k, 0)$ is a vertex operator algebra with the conformal vector

$$\omega_{\text{aff}} = \frac{1}{2(k+2)} \left(\frac{1}{2}h(-1)^2\mathbf{1} + e(-1)f(-1)\mathbf{1} + f(-1)e(-1)\mathbf{1} \right),$$

whose central charge is $3k/(k+2)$ [32], [38, Section 6.2].

Let $M_{\widehat{\mathfrak{h}}}(k, 0)$ be the vertex operator subalgebra of $V(k, 0)$ generated by $h(-1)\mathbf{1}$. That is, $M_{\widehat{\mathfrak{h}}}(k, 0)$ is a Heisenberg vertex operator algebra. The conformal vector of $M_{\widehat{\mathfrak{h}}}(k, 0)$ is

$$\omega_h = \frac{1}{4k}h(-1)^2\mathbf{1}$$

and its central charge is 1. As a module for $M_{\widehat{\mathfrak{h}}}(k, 0)$, we have a decomposition

$$V(k, 0) = \bigoplus_{\lambda \in 2\mathbb{Z}} M_{\widehat{\mathfrak{h}}}(k, \lambda) \otimes N_\lambda,$$

where $M_{\widehat{\mathfrak{h}}}(k, \lambda)$ is a simple highest weight module for $M_{\widehat{\mathfrak{h}}}(k, 0)$ with a highest weight vector v_λ such that $h(0)v_\lambda = \lambda v_\lambda$ and

$$N_\lambda = \{v \in V(k, 0) \mid h(m)v = \lambda \delta_{m,0}v \text{ for } m \geq 0\}. \quad (2.11)$$

In particular, N_0 is the commutant [32, Theorem 5.1] of $M_{\widehat{\mathfrak{h}}}(k, 0)$ in $V(k, 0)$, which is a vertex operator algebra with the conformal vector $\omega_{\text{para}} = \omega_{\text{aff}} - \omega_h$. The central charge of N_0 is $2(k-1)/(k+2)$. The character of N_0 is $\text{ch } N_0 = 1 + q^2 + 2q^3 + \dots$. It is known [17, Section 2] that

$$\begin{aligned} W^3 &= k^2 h(-3)\mathbf{1} + 3kh(-2)h(-1)\mathbf{1} + 2h(-1)^3\mathbf{1} - 6kh(-1)e(-1)f(-1)\mathbf{1} \\ &\quad + 3k^2 e(-2)f(-1)\mathbf{1} - 3k^2 e(-1)f(-2)\mathbf{1}. \end{aligned} \quad (2.12)$$

is a unique, up to a scalar multiple, Virasoro singular vector in the weight 3 subspace $(N_0)_{(3)}$. The vertex operator algebra N_0 is generated by the conformal vector ω_{para} and the weight 3 vector W^3 [16, Theorem 3.1].

The vertex operator algebra $V(k, 0)$ has a unique maximal ideal $\mathcal{J}$, which is generated by a single element $e(-1)^{k+1}\mathbf{1}$ [34]. Let $L(k, 0) = L_{\widehat{\mathfrak{sl}}_2}(k, 0) = V(k, 0)/\mathcal{J}$. Since $M_{\widehat{\mathfrak{h}}}(k, 0) \cap \mathcal{J} = 0$, $M_{\widehat{\mathfrak{h}}}(k, 0)$ can be considered as a subalgebra of $L(k, 0)$ and we have a decomposition

$$L(k, 0) = \bigoplus_{\lambda \in 2\mathbb{Z}} M_{\widehat{\mathfrak{h}}}(k, \lambda) \otimes K_\lambda$$

of $M_{\widehat{\mathfrak{h}}}(k, 0)$ -modules, where

$$K_\lambda = \{v \in L(k, 0) \mid h(m)v = \lambda \delta_{m,0}v \text{ for } m \geq 0\}.$$

Note that K_0 is the commutant of $M_{\widehat{\mathfrak{h}}}(k, 0)$ in $L(k, 0)$. We use the same symbols $a(-1)\mathbf{1}$ for $a \in \{h, e, f\}$, ω_{aff} , ω_h , ω_{para} and W^3 to denote their images in $L(k, 0)$.

We call K_0 a parafermion vertex operator algebra associated with $\widehat{\mathfrak{sl}}_2$ and denote it by $K(\widehat{\mathfrak{sl}}_2, k)$. It is a simple vertex operator algebra of central charge $2(k-1)/(k+2)$ and generated by ω_{para} and W^3 . The character is $\text{ch } K_0 = 1 + q^2 + 2q^3 + \dots$.

The vertex operator algebra K_0 can be embedded in a vertex operator algebra V_L associated with a rank k lattice $L = \mathbb{Z}\alpha_1 + \dots + \mathbb{Z}\alpha_k$ with $\langle \alpha_p, \alpha_q \rangle = 2\delta_{p,q}$ [17, Section 4], [18, Chapter 14]. In fact, let $\gamma = \alpha_1 + \dots + \alpha_k$ and set

$$H = \gamma(-1)\mathbf{1}, \quad E = e^{\alpha_1} + \dots + e^{\alpha_k}, \quad F = e^{-\alpha_1} + \dots + e^{-\alpha_k}.$$

Then $\langle \gamma, \gamma \rangle = 2k$ and the component operators $H_n, E_n, F_n, n \in \mathbb{Z}$ give a level k representation of $\widehat{\mathfrak{sl}}_2$ under the correspondence $h(n) \leftrightarrow H_n, e(n) \leftrightarrow E_n, f(n) \leftrightarrow F_n$. In particular, the vertex operator subalgebra V^{aff} of V_L generated by H, E and F is isomorphic to $L(k, 0)$. We also consider the vertex operator subalgebra V^γ of V_L generated by e^γ and $e^{-\gamma}$. Note that $V^\gamma \cong V_{\mathbb{Z}\gamma}$. We identify V^{aff} with $L(k, 0)$ and V^γ with $V_{\mathbb{Z}\gamma}$. We also identify H_n with $h(n)$, E_n with $e(n)$ and F_n with $f(n)$. Then we have [17, Lemma 4.2]

$$L(k, 0) = \bigoplus_{j=0}^{k-1} V_{\mathbb{Z}\gamma-j\gamma/k} \otimes M^j \quad (2.13)$$

as $V_{\mathbb{Z}\gamma} \otimes M^0$ -modules, where

$$M^j = \{v \in L(k, 0) \mid \gamma(m)v = -2j\delta_{m,0}v \text{ for } m \geq 0\}. \quad (2.14)$$

That is, $M^j = K_{-2j}$ for $0 \leq j \leq k-1$. In particular,

$$M^0 = K_0 = K(\mathfrak{sl}_2, k).$$

Remark 2.5. M^j is denoted by $M^{0,j}$ in [17, Lemma 4.2]. The index j of M^j is considered to be modulo k .

Those M^j 's are simple M^0 -modules [17, Theorem 4.4]. Hence (2.13) is a decomposition of $L(k, 0)$ as a direct sum of simple $V_{\mathbb{Z}\gamma} \otimes M^0$ -modules $V_{\mathbb{Z}\gamma-j\gamma/k} \otimes M^j$, $0 \leq j \leq k-1$.

Let $L^\perp = \frac{1}{2}L$ be the dual lattice of L . The simple module $L(k, i) = L_{\widehat{\mathfrak{sl}_2}}(k, i)$ for the simple affine vertex operator algebra $L(k, 0)$ with $i+1$ dimensional top level of weight $\frac{i(i+2)}{4(k+2)}$ can be constructed in the simple V_L -module $V_{L^\perp}$, $1 \leq i \leq k$. Let

$$M^{i,j} = \{v \in L(k, i) \mid \gamma(m)v = (i-2j)\delta_{m,0}v \text{ for } m \geq 0\} \quad (2.15)$$

for $0 \leq i \leq k$, $0 \leq j \leq k-1$. Then $M^{i,j}$'s are simple M^0 -modules and

$$L(k, i) = \bigoplus_{j=0}^{k-1} V_{\mathbb{Z}\gamma+(i-2j)\gamma/2k} \otimes M^{i,j} \quad (2.16)$$

as $V_{\mathbb{Z}\gamma} \otimes M^0$ -modules [17, Lemma 4.3].

Remark 2.6. The decomposition (2.15) implies that $M^{i,j}$ is the multiplicity of $V_{\mathbb{Z}\gamma+(i-2j)\gamma/2k}$ in $L(k, i)$. Since for a fixed i , $0 \leq i \leq k$, the simple $V_{\mathbb{Z}\gamma}$ -module $V_{\mathbb{Z}\gamma+(i-2j)\gamma/2k}$ is determined by $j \pmod{k}$, we consider the second index j of $M^{i,j}$ to be modulo k .

The -1 isometry of the lattice L lifts to an automorphism θ of the vertex operator algebra V_L of order 2. Actually, $\theta(H) = -H$, $\theta(E) = F$ and $\theta(F) = E$. Both $V^{\text{aff}} = L(k, 0)$ and $M^0 = K(\mathfrak{sl}_2, k)$ are invariant under θ . In fact, the automorphism group $\text{Aut } M^0$ of M^0 is generated by θ and we have $\theta(W^3) = -W^3$.

We summarize the properties of M^0 and $M^{i,j}$ for $0 \leq i \leq k$, $0 \leq j \leq k-1$ (see [7, Theorem 4.4, Proposition 4.5], [16, Theorem 4.1], [17, Theorem 8.2]).

- (1) M^0 is a simple vertex operator algebra of central charge $2(k-1)/(k+2)$.
- (2) $\text{ch } M^0 = 1 + q^2 + 2q^3 + \dots$.
- (3) M^0 is generated by $M_{(2)}^0$ and $M_{(3)}^0$.
- (4) The simple M^0 -modules $M^{i,j}$'s are not always inequivalent. In fact,

$$M^{i,j} \cong M^{k-i,j-i} \quad (2.17)$$

for $0 \leq i \leq k$, $0 \leq j \leq k-1$.

(5) $M^{i,j}$, $0 \leq j < i \leq k$ form a complete set of representatives of the isomorphism classes of simple M^0 -modules. There are exactly $k(k+1)/2$ inequivalent simple M^0 -modules.

(6) The top level of $M^{i,j}$ is one dimensional and its weight is

$$\frac{1}{2k(k+2)} \left(k(i-2j) - (i-2j)^2 + 2k(i-j+1)j \right) \quad (2.18)$$

for $0 \leq j < i \leq k$.

(7) The automorphism θ of M^0 induces a permutation

$$M^{i,j} \mapsto M^{i,j} \circ \theta = M^{i,i-j} \quad (2.19)$$

on the simple M^0 -modules for $0 \leq i \leq k$, $0 \leq j \leq k-1$.

Remark 2.7. It follows from (2.17) that $M^{k,j}$ is isomorphic to $M^{0,j-k} = M^{0,j}$ for $0 \leq j \leq k-1$, even though the simple $L(k,0)$ -module $L(k,k)$ is not isomorphic to $L(k,0)$ in the decomposition (2.16). We also note that M^j is equal to $M^{0,j} = M^{k,j}$ and its top weight is $j(k-j)/k$ for $0 \leq j \leq k-1$ by (2.18). The top weight (2.18) of $M^{i,j}$ is non-negative, and is zero only if $i = k$ and $j = 0$.

3. A CHARACTERIZATION OF $L_{\widehat{\mathfrak{sl}_2}}(k, 0)$

Let $k \geq 3$ be an integer. In this section we argue that a simple vertex operator algebra satisfying the following hypothesis is isomorphic to the affine vertex operator algebra $L_{\widehat{\mathfrak{sl}_2}}(k, 0)$.

Hypothesis 3.1. (1) $(V, Y, \mathbf{1}, \omega)$ is a simple vertex operator algebra of CFT-type with central charge $3k/(k+2)$.

(2) V contains a vertex operator subalgebra $(T^0, Y, \mathbf{1}, \omega^1)$ isomorphic to $V_{\mathbb{Z}\gamma}$, where $\langle \gamma, \gamma \rangle = 2k$. We identify T^0 with $V_{\mathbb{Z}\gamma}$. Then $\omega^1 = \frac{1}{4k}\gamma(-1)^2\mathbf{1}$ is the conformal vector of T^0 with central charge 1. We assume that $\omega^1 \in V_{(2)}$ and $\omega_2\omega^1 = 0$.

(3) Let N^0 be the commutant of T^0 in V and set $\omega^2 = \omega - \omega^1$. Thus $(N^0, Y, \mathbf{1}, \omega^2)$ is a vertex operator subalgebra of V with central charge $2(k-1)/(k+2)$. We assume that $\text{ch } N^0 = 1 + q^2 + 2q^3 + \dots$ and that N^0 is generated by $N_{(2)}^0$ and $N_{(3)}^0$ as a vertex operator algebra.

(4) We assume that V is isomorphic to $\bigoplus_{j=0}^{k-1} V_{\mathbb{Z}\gamma-j\gamma/k} \otimes N^j$ as a T^0 -module, where

$$N^j = \{v \in V \mid \gamma(m)v = -2j\delta_{m,0}v \text{ for } m \geq 0\}.$$

We also assume that as a module for N^0 , the top weight of N^j is $j(k-j)/k$.

Under Hypothesis 3.1 we shall show that V is isomorphic to $L(k, 0) = L_{\widehat{\mathfrak{sl}_2}}(k, 0)$. The proof is divided into several steps. First, we shall introduce some notation. From the hypothesis we may assume that $T^0 \otimes N^0$ is a vertex operator subalgebra of V [32, Theorem 5.1]. Then the vacuum vector and the conformal vector of V are given as $\mathbf{1} = \mathbf{1}^1 \otimes \mathbf{1}^2$ and $\omega = \omega^1 \otimes \mathbf{1}^2 + \mathbf{1}^1 \otimes \omega^2$, where $\mathbf{1}^1$ and $\mathbf{1}^2$ are the vacuum vectors of T^0 and N^0 , respectively. For simplicity, we usually do not distinguish between $T^0 \otimes \mathbf{1}^2$ and T^0 (resp. $\mathbf{1}^1 \otimes N^0$ and N^0) and so $\omega^1 \otimes \mathbf{1}^2$ and ω^1 (resp. $\mathbf{1}^1 \otimes \omega^2$ and ω^2). The weight of $v \in V$ as a module for T^0 (resp. N^0) or ω^1 -weight (resp. ω^2 -weight) means the eigenvalue for the operator $L^1(0)$ (resp. $L^2(0)$), where $L^i(n) = \omega_{n+1}^i$.

By our hypothesis, V decomposes into a direct sum of simple T^0 -modules and each simple direct summand is isomorphic to one of $V_{\mathbb{Z}\gamma-j\gamma/k}$, $0 \leq j \leq k-1$. Moreover, N^j is the sum of top levels of all simple T^0 -submodules of V isomorphic to $V_{\mathbb{Z}\gamma-j\gamma/k}$. We examine the action of $\gamma(0) = (\gamma(-1)\mathbf{1})_0$ on the top level of each direct summand.

Let $\sigma = \exp(2\pi\sqrt{-1}\gamma(0)/2k)$, which is an automorphism of the vertex operator algebra V of order k . We consider its eigenspace

$$V^j = \{v \in V \mid \sigma v = \exp(-2\pi j\sqrt{-1}/k)v\} \quad (3.1)$$

with eigenvalue $\exp(-2\pi j\sqrt{-1}/k)$. Then $V = \bigoplus_{j=0}^{k-1} V^j$. By [19, Theorem 3], V^0 is a simple vertex operator algebra and V^j , $1 \leq j \leq k-1$ are simple V^0 -modules. For convenience, we understand the index j of V^j to be modulo k . Since σ is an automorphism

of V , we have

$$u_nv \in V^{i+j} \quad \text{for } u \in V^i, v \in V^j, n \in \mathbb{Z}. \quad (3.2)$$

In fact, V^j is the sum of all simple T^0 -submodules of V isomorphic to $V_{\mathbb{Z}\gamma-j\gamma/k}$, for the operator $\gamma(0)$ acts on $e^{n\gamma-j\gamma/k} \in V_{\mathbb{Z}\gamma-j\gamma/k}$ as a scalar $\langle \gamma, n\gamma - j\gamma/k \rangle = 2kn - 2j$ and commutes with $\gamma(m)$, $m \in \mathbb{Z}$. Hence $V^j \cong T^j \otimes N^j$, where T^j is a simple T^0 -module isomorphic to $V_{\mathbb{Z}\gamma-j\gamma/k}$. In particular, $V^0 \cong T^0 \otimes N^0$ as vertex operator algebras. Since V^0 is simple and V^j , $1 \leq j \leq k-1$ are simple V^0 -modules, the following lemma holds.

Lemma 3.2. *$V^0 \cong T^0 \otimes N^0$ as vertex operator algebras and N^0 is a simple vertex operator algebra. Moreover, N^j , $1 \leq j \leq k-1$ are simple N^0 -modules.*

The weight 1 subspace of V^0 is $V_{(1)}^0 = \mathbb{C}\gamma(-1)\mathbf{1}$ and we have $\omega_2 V_{(1)}^0 = 0$. Hence, V^0 possesses a nonzero invariant bilinear form by [39, Theorem 3.1] and so the following lemma holds.

Lemma 3.3. *The vertex operator algebra V^0 is self-dual.*

The top level $N^j(0)$ of N^j is of weight $j(k-j)/k$ by our hypothesis. Hence the weight of the top level $V^j(0) = T^j(0) \otimes N^j(0)$ of V^j is j if $0 \leq j < k/2$ and $k-j$ if $k/2 < j \leq k-1$. In the case k is even and $j = k/2$, the weight of $V^j(0)$ is $k/2$.

Now, $V_{(n)} = 0$ for $n < 0$ and $V_{(0)} = V_{(0)}^0 = \mathbb{C}\mathbf{1}$. Moreover, $V_{(1)} = V_{(1)}^0 + V^{k-1}(0) + V^1(0)$ and $V_{(1)}^0 = \mathbb{C}\gamma(-1)\mathbf{1}$, for we are assuming that $k \geq 3$. Note that $V^{k-1}(0) = \mathbb{C}e^{\gamma/k} \otimes N^{k-1}(0)$ and $V^1(0) = \mathbb{C}e^{-\gamma/k} \otimes N^1(0)$. Also, $u_nv \in V^0$ for $u \in V^{k-1}$ and $v \in V^1$ by (3.2). By Lemmas 2.1 and 3.3, we can choose $E \in V^{k-1}(0)$ and $F \in V^1(0)$ such that $E_1F = k\mathbf{1}$. Then (2.10) implies that $E_0F = \gamma(-1)\mathbf{1}$. Let $H = \gamma(-1)\mathbf{1}$.

Lemma 3.4. (1) $H_0H = 0$, $H_1H = 2k\mathbf{1}$.

(2) $H_0E = 2E$, $H_1E = 0$, $H_0F = -2F$, $H_1F = 0$.

(3) $E_0F = H$, $E_1F = k\mathbf{1}$.

(4) $E_0E = E_1E = F_0F = F_1F = 0$.

Proof. Since $H_n = \gamma(n) \otimes 1$ for $n \in \mathbb{Z}$ as an operator on the $T^0 \otimes N^0$ -module $T^j \otimes N^j$ and since $\gamma(n)e^{\pm\gamma/k} = \pm 2\delta_{n,0}e^{\pm\gamma/k}$ if $n \geq 0$, (1) and (2) hold. We have chosen E and F so that (3) holds. By [31, (8.9.9)], $E_0E = F_0F = 0$. We also have $E_1E \in V^{k-2} \cap V_{(0)} = 0$ and $F_1F \in V^2 \cap V_{(0)} = 0$. Hence (4) holds. $\square$

We want to show that the vertex operator algebra V is generated by H , E and F . Let U be the vertex operator subalgebra of V generated by H , E and F . Note that $A_nB = 0$ with $A, B \in \{H, E, F\}$ and $n \geq 2$, for the weight of A_nB is $-n+1$. Then Lemma 3.4 implies that the component operators H_n, E_n, F_n , $n \in \mathbb{Z}$ give a level k representation of $\widehat{\mathfrak{sl}}_2$ under the correspondence

$$h(n) \leftrightarrow H_n, \quad e(n) \leftrightarrow E_n, \quad f(n) \leftrightarrow F_n.$$

Since $A_n\mathbf{1} = 0$ if $n \geq 0$ and since $A_{-1}\mathbf{1} = A$ for $A \in \{H, E, F\}$, the map

$$h(-1)\mathbf{1} \mapsto H, \quad e(-1)\mathbf{1} \mapsto E, \quad f(-1)\mathbf{1} \mapsto F \quad (3.3)$$

lifts to a surjective homomorphism $\varphi : V(k, 0) \rightarrow U$ of vertex operator algebras by the universality of the Weyl module $V(k, 0)$. The image $\varphi(\mathcal{J})$ of the maximal ideal $\mathcal{J}$ of $V(k, 0)$ is a maximal ideal of U and the quotient vertex operator algebra $U/\varphi(\mathcal{J})$ is

isomorphic to $L(k, 0) = V(k, 0)/\mathcal{J}$. Hence there is a surjective homomorphism $\psi : U \rightarrow L(k, 0)$ of vertex operator algebras such that $\psi(H) = h(-1)\mathbf{1}$, $\psi(E) = e(-1)\mathbf{1}$, and $\psi(F) = f(-1)\mathbf{1}$. Recall that we use the same symbols to denote elements of $V(k, 0)$ and their images in $L(k, 0)$.

Since $H_n = \gamma(n) \otimes 1$, we have

$$\gamma(-n_1)\gamma(-n_2)\cdots\gamma(-n_r)\mathbf{1} \in U$$

for $n_1 \geq n_2 \geq \cdots \geq n_r \geq 1$, $r = 0, 1, 2, \dots$

Lemma 3.5. $N^0 \subset U$.

Proof. The image of the conformal vector ω_{aff} of $V(k, 0)$ under φ is

$$\varphi(\omega_{\text{aff}}) = \frac{1}{2(k+2)} \left(\frac{1}{2} H_{-1} H + E_{-1} F + F_{-1} E \right),$$

which is contained in V^0 by (3.2). We also have

$$\varphi(\omega_h) = \frac{1}{4k} H_{-1} H \in T^0.$$

Note that $N^0 = \{v \in V^0 \mid H_m v = 0 \text{ for } m \geq 0\}$. Since ω_{para} is the conformal vector of N_0 (cf. (2.11)) of central charge $2(k-1)/(k+2)$, its image $\varphi(\omega_{\text{para}})$ is a Virasoro element of the same central charge. Moreover, $h(m)\omega_{\text{para}} = 0$ implies $H_m \varphi(\omega_{\text{para}}) = 0$, $m \geq 0$. Thus $\varphi(\omega_{\text{para}})$ is contained in N^0 . Since $N_{(2)}^0 = \mathbb{C}\omega^2$ by our hypothesis, we conclude that $\varphi(\omega_{\text{para}}) = \omega^2$. In particular, $\omega^2 \in U$.

Next, we consider the image of W^3 (2.12) under the homomorphism φ ,

$$\begin{aligned} \varphi(W^3) &= k^2 H_{-3} \mathbf{1} + 3k H_{-2} H_{-1} \mathbf{1} + 2(H_{-1})^3 \mathbf{1} - 6k H_{-1} E_{-1} F_{-1} \mathbf{1} \\ &\quad + 3k^2 E_{-2} F_{-1} \mathbf{1} - 3k^2 E_{-1} F_{-2} \mathbf{1}. \end{aligned}$$

As in the case of ω_{para} , we have $\varphi(W^3) \in V^0$ by (3.2) and furthermore, $\varphi(W^3) \in N^0$, for $W^3 \in N_0$. Recall that W^3 is a Virasoro singular vector with respect to the conformal vector ω_{para} of N_0 . Hence $\varphi(W^3)$ is a Virasoro singular vector with respect to the conformal vector ω^2 of N^0 . Now, the weight 3 subspace $N_{(3)}^0$ is of dimension 2 by our hypothesis. Thus $\omega_0^2 \omega^2$ and $\varphi(W^3)$ form a basis of $N_{(3)}^0$. Hence the lemma holds, for we are assuming that the vertex operator algebra N^0 is generated by $N_{(2)}^0$ and $N_{(3)}^0$. $\square$

Lemma 3.6. $e^{\pm\gamma} \otimes N^0 \subset U$.

Proof. Since $H_n = \gamma(n) \otimes 1$ and since $V = \bigoplus_{j=0}^{k-1} V^j$ with $V^j \cong T^j \otimes N^j$, it follows from (3.1) that

$$e^{\pm\gamma} \otimes N^0 = \{v \in V \mid H_n v = \pm 2k \delta_{n,0} v \text{ for } n \geq 0\}.$$

Now, $e(-1)^k \mathbf{1} \notin \mathcal{J}$ and so $\varphi(e(-1)^k \mathbf{1}) = (E_{-1})^k \mathbf{1}$ is a nonzero element of U . Note also that

$$h(n)e(-1)^k \mathbf{1} = 2k \delta_{n,0} e(-1)^k \mathbf{1}, \quad n \geq 0$$

in the Weyl module $V(k, 0)$. Taking the image under the homomorphism φ , we have

$$H_n (E_{-1})^k \mathbf{1} = 2k \delta_{n,0} (E_{-1})^k \mathbf{1}, \quad n \geq 0.$$

This implies that $(E_{-1})^k \mathbf{1}$ is a nonzero element of $e^\gamma \otimes N^0$. Then we have $e^\gamma \otimes N^0 \subset U$ by Lemma 3.5. Replacing $e(-1)$ with $f(-1)$ and E with F in the above argument, we can also show that $e^{-\gamma} \otimes N^0 \subset U$. $\square$

Lemma 3.7. $U = V$.

Proof. The vertex operator algebra $V_{\mathbb{Z}\gamma}$ is generated by e^γ and $e^{-\gamma}$. Hence Lemma 3.6 implies that U contains V^0 . Recall that V^j is a simple V^0 -module. Since $F \in V^1$, it follows that $V^1 \subset U$. Then $V^j \subset U$ for all j by [18, Proposition 11.9], and we have $U = V$ as desired. $\square$

Since V is a simple vertex operator algebra, Lemma 3.7 implies the following theorem.

Theorem 3.8. $V \cong L_{\widehat{\mathfrak{sl}_2}}(k, 0)$.

4. A CHARACTERIZATION OF $K(\mathfrak{sl}_2, k)$

In this section we apply the results of Section 3 to obtain a characterization of the parafermion vertex operator algebra $K(\mathfrak{sl}_2, k)$ associated with $\mathfrak{sl}_2$. Let $k \geq 3$ be an integer. Throughout this section we assume the following hypothesis.

Hypothesis 4.1. (1) N^0 is a simple, self-dual, rational and C_2 -cofinite vertex operator algebra of CFT-type with central charge $2(k-1)/(k+2)$.

(2) $\text{ch } N^0 = 1 + q^2 + 2q^3 + \dots$.

(3) N^0 is generated by $N_{(2)}^0$ and $N_{(3)}^0$.

(4) There exist simple current N^0 -modules N^j , $1 \leq j \leq k-1$ such that the top weight of N^j is $j(k-j)/k$ and the fusion rules among N^j 's are

$$N^i \times N^j = N^{i+j}, \quad 0 \leq i, j \leq k-1.$$

Here the indices i, j are considered to be modulo k .

(5) Any simple N^0 -module except N^0 itself has positive top weight.

Let $V_{\mathbb{Z}\gamma-j\gamma/k}$ be as in Section 2.3. Thus $\langle \gamma, \gamma \rangle = 2k$. Let

$$V^j = V_{\mathbb{Z}\gamma-j\gamma/k} \otimes N^j, \quad 0 \leq j \leq k-1$$

be a tensor product of vector spaces $V_{\mathbb{Z}\gamma-j\gamma/k}$ and N^j . Then $V^0 = V_{\mathbb{Z}\gamma} \otimes N^0$ carries a structure of vertex operator algebra. In fact, V^0 is a simple, self-dual, rational and C_2 -cofinite vertex operator algebra of CFT-type with central charge $3k/(k+2)$ by our assumption on N^0 . Moreover, any simple V^0 -module except V^0 itself has positive top weight. The V^j , $0 \leq j \leq k-1$ are simple V^0 -modules [30, Section 4.7]. These simple modules are simple current V^0 -modules and the fusion rules among them are

$$V^i \times V^j = V^{i+j}, \quad 0 \leq i, j \leq k-1 \tag{4.1}$$

by (2.9) and Hypothesis 4.1.

The top weight of the V^0 -module V^j is a sum of those of the $V_{\mathbb{Z}\gamma}$ -module $V_{\mathbb{Z}\gamma-j\gamma/k}$ and the N^0 -module N^j , so that it is j if $0 \leq j < k/2$, $k-j$ if $k/2 < j \leq k-1$ and $k/2$ if k is even and $j = k/2$. In particular, V^j has integral weight.

Therefore, $\bigoplus_{j=0}^{k-1} V^j$ has a vertex operator algebra structure by Theorem 2.4. It is a $\mathbb{Z}_k$ -graded simple current extension of V^0 .

Now, we can apply Theorem 3.8 to conclude that the vertex operator algebra $\bigoplus_{j=0}^{k-1} V^j$ is isomorphic to $L_{\widehat{\mathfrak{sl}_2}}(k, 0)$. Thus the following theorem holds.

Theorem 4.2. *The space $\bigoplus_{j=0}^{k-1} V^j$ is a $\mathbb{Z}_k$ -graded simple current extension of V^0 and it is isomorphic to the vertex operator algebra $L_{\widehat{\mathfrak{sl}_2}}(k, 0)$, where $V^j = V_{\mathbb{Z}\gamma - j\gamma/k} \otimes N^j$ with $\langle \gamma, \gamma \rangle = 2k$ and N^j being as in Hypothesis 4.1.*

Since N^0 is the commutant of $V_{\mathbb{Z}\gamma}$ in the vertex operator algebra $\bigoplus_{j=0}^{k-1} V^j$ and since the parafermion vertex operator algebra $K(\mathfrak{sl}_2, k)$ is the commutant of $V_{\mathbb{Z}\gamma}$ in $L_{\widehat{\mathfrak{sl}_2}}(k, 0)$, the following theorem is a consequence of Theorem 4.2.

Theorem 4.3. *Let N^0 be as in Hypothesis 4.1. Then N^0 is isomorphic to the parafermion vertex operator algebra $K(\mathfrak{sl}_2, k)$ associated with $\mathfrak{sl}_2$ for $k \geq 3$.*

5. W-ALGEBRAS

In this section we recall some results on W -algebras.

5.1. W -algebra $\mathcal{W}_\ell(\mathfrak{sl}_k)$. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra, $(\cdot | \cdot)$ the normalized invariant inner product of $\mathfrak{g}$, and let $\widehat{\mathfrak{g}}$ be the affine Lie algebra associated with $\mathfrak{g}$ and $(\cdot | \cdot)$:

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}C,$$

where C is the central element. Let $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}C$ be the Cartan subalgebra of $\widehat{\mathfrak{g}}$, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Let $\widehat{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C}\Lambda_0$ be the dual of $\widehat{\mathfrak{h}}$, where $\Lambda_0(C) = 1$, $\Lambda_0(\mathfrak{h}) = 0$.

Let $V_{\widehat{\mathfrak{g}}}(\ell, 0) = U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}C)} \mathbb{C}_\ell$ be the *universal affine vertex algebra* associated with $\mathfrak{g}$ at level $\ell \in \mathbb{C}$. Here $\mathfrak{g} \otimes \mathbb{C}[t]$ acts as 0 and C acts as ℓ on $\mathbb{C}_\ell$.

For a weight λ of $\mathfrak{g}$ denote by $L_{\widehat{\mathfrak{g}}}(\ell, \lambda)$ the simple highest weight module for $\widehat{\mathfrak{g}}$ with highest weight $\widehat{\lambda}_\ell = \lambda + \ell\Lambda_0$. The vacuum simple module $L_{\widehat{\mathfrak{g}}}(\ell, 0)$ is a quotient vertex algebra of $V_{\widehat{\mathfrak{g}}}(\ell, 0)$ and called the simple *affine vertex algebra associated with $\mathfrak{g}$ at level ℓ* .

Let $\mathcal{W}^\ell(\mathfrak{g})$ be the W -algebra associated with $\mathfrak{g}$ and its principal nilpotent element at a non-critical level ℓ defined by the quantized Drinfeld-Sokolov reduction [25]:

$$\mathcal{W}^\ell(\mathfrak{g}) = H_{DS}^0(V_{\widehat{\mathfrak{g}}}(\ell, 0)),$$

where $H_{DS}^\bullet(M)$ is the cohomology of the BRST complex for the quantized Drinfeld-Sokolov reduction with coefficient in a $\widehat{\mathfrak{g}}$ -module M ([25]). Denote by $\mathcal{W}_\ell(\mathfrak{g})$ the unique simple quotient of $\mathcal{W}^\ell(\mathfrak{g})$.

Later we shall set $\mathfrak{g} = \mathfrak{sl}_k$, in which case $\mathcal{W}^\ell(\mathfrak{g})$ is isomorphic to the W_k -algebra defined by Fateev and Lukyanov [24] (cf. [8]).

The level ℓ is called *admissible* if $L_{\widehat{\mathfrak{g}}}(\ell, 0)$ is an admissible representation [35] of $\widehat{\mathfrak{g}}$. If this is the case, the associated variety $X_{L_{\widehat{\mathfrak{g}}}(\ell, 0)}$ of $L_{\widehat{\mathfrak{g}}}(\ell, 0)$ is contained in the nullcone $\mathcal{N}$ of $\mathfrak{g}$ ([2]), where $X_{L_{\widehat{\mathfrak{g}}}(\ell, 0)}$ is the maximal spectrum of Zhu's C_2 -algebra of $L_{\widehat{\mathfrak{g}}}(\ell, 0)$ ([2]). An admissible number ℓ is called *non-degenerate* if $X_{L_{\widehat{\mathfrak{g}}}(\ell, 0)} = \mathcal{N}$ ([29, 2]). For an admissible level ℓ , we have ([3])

$$H_{DS}^\bullet(L_{\widehat{\mathfrak{g}}}(\ell, 0)) \neq 0 \iff \ell \text{ is non-degenerate.}$$

The following assertion was conjectured by Frenkel, Kac and Wakimoto [29].

Theorem 5.1 ([1, 3, 4]). *Let ℓ be a non-degenerate admissible level. Then we have the isomorphism*

$$\mathcal{W}_\ell(\mathfrak{g}) \cong H_{DS}^0(L_{\widehat{\mathfrak{g}}}(\ell, 0))$$

of vertex algebras. Moreover, $\mathcal{W}_\ell(\mathfrak{g})$ is self-dual, rational and C_2 -cofinite.

The rational and C_2 -cofinite W -algebras appearing in Theorem 5.1 are called *minimal series W -algebras*. In the case that $\mathfrak{g} = \mathfrak{sl}_2$, they are exactly the Virasoro vertex operator algebras which belong to minimal series representations [9] of the Virasoro algebra.

The conjectural classification [29] of simple modules for minimal series W -algebras was also established in [4]. In [6] it was shown that this together with Theorem 5.1 verifies the conjectural fusion rules of minimal series W -algebras obtained in [29]. In the next subsection we shall describe these results more precisely in the cases that we are interested in this article.

5.2. The case $\mathfrak{g} = \mathfrak{sl}_k$. Now, we set $\mathfrak{g} = \mathfrak{sl}_k$. One knows [27] that $\mathcal{W}^\ell(\mathfrak{g})$ is freely generated by homogeneous elements of weight $d_1 + 1, \dots, d_{\text{rank } \mathfrak{g}} + 1$, where $d_1, \dots, d_{\text{rank } \mathfrak{g}}$ is the exponents of $\mathfrak{g}$. In particular $\mathcal{W}^\ell(\mathfrak{sl}_k)$ is freely generated by homogeneous elements of weight $2, 3, \dots, k$. Hence

$$\text{ch } \mathcal{W}^\ell(\mathfrak{sl}_k) = \prod_{i=1}^{k-1} \prod_{j=1}^{\infty} (1 - q^{i+j+1})^{-1} = 1 + q^2 + 2q^3 + \dots$$

Proposition 5.2 ([24]). *For any non-critical level ℓ , $\mathcal{W}^\ell(\mathfrak{sl}_k)$ is generated by its weight 2 and weight 3 subspaces as a vertex algebra.*

We include a proof of this fact in Appendix A.

Note that we have

$$\mathcal{W}^\ell(\mathfrak{sl}_k) \cong \mathcal{W}^{\ell'}(\mathfrak{sl}_k) \quad \text{if } (\ell + k)(\ell' + k) = 1$$

([24], see also [8]), which is the special case of the Feigin-Frenkel duality [26].

The level ℓ is a non-degenerate admissible number for $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_k$ if and only if

$$\ell + k = \frac{p}{q} \quad \text{with } p, q \in \mathbb{N}, (p, q) = 1, p, q \geq k.$$

Set

$$\mathcal{W}_{p,q}(\mathfrak{sl}_k) = \mathcal{W}_{p/q-k}(\mathfrak{sl}_k) = \mathcal{W}_{q/p-k}(\mathfrak{sl}_k)$$

for $p/q - k$ with $p, q \in \mathbb{N}$, $(p, q) = 1$, $p, q \geq k$. The central charge of $\mathcal{W}_{p,q}(\mathfrak{sl}_k)$ is given by

$$c_{p,q} = -\frac{(k-1)((k+1)p - kq)(kp - (k+1)q)}{pq}.$$

Note that

$$c_{k+1,k+2} = \frac{2(k-1)}{k+2}. \tag{5.1}$$

The simple modules of $\mathcal{W}_{p,q}(\mathfrak{sl}_k)$ are parametrized by the set

$$I_{p,q} = (\hat{P}_+^{p-k} \times \hat{P}_+^{q-k}) / \widetilde{W}_+,$$

where $\widehat{P}_+^m$ denotes the set of integral dominant weights of $\widehat{\mathfrak{g}}$ of level m and $\widetilde{W}_+$ is the subgroup of the extended affine Weyl group consisting of elements of length zero which acts diagonally on the set $\widehat{P}_+^{p-k} \times \widehat{P}_+^{q-k}$. By [4, Theorem 10.4] and [1, Remark 9.1.8],

$$\{H_{DS}^0(L_{\widehat{\mathfrak{g}}}(\ell, \lambda - \frac{p}{q}\mu)) \mid [(\hat{\lambda}_{p-k}, \hat{\mu}_{q-k})] \in I_{p,q}, \lambda, \mu \in \mathfrak{h}^*\}$$

gives a complete set of representatives of the isomorphism classes of simple $\mathcal{W}_{p,q}(\mathfrak{sl}_k)$ -modules.

5.3. Simple modules and fusion rules of $\mathcal{W}_{k+1,k+2}(\mathfrak{sl}_k)$. Consider the special case that $q = k + 1$. Then we have a bijection

$$\widehat{P}_+^{p-k} \xrightarrow{\sim} I_{p,k+1}, \quad \hat{\lambda}_{p-k} \mapsto [(\hat{\lambda}_{p-k}, \hat{0}_1)].$$

Therefore, by putting

$$\mathbb{L}(\Lambda) = H_{BRST}^0(L_{\widehat{\mathfrak{sl}_k}}(\ell, \bar{\Lambda})),$$

where $\bar{\Lambda}$ is the restriction of Λ to $\mathfrak{h}$, the set $\{\mathbb{L}(\Lambda) \mid \Lambda \in \widehat{P}_+^{p-k}\}$ gives a complete set of representatives of the isomorphism classes of simple $\mathcal{W}_{p,k+1}(\mathfrak{sl}_k)$ -modules.

Let $\mathcal{A}_{\mathfrak{g}}^m = \{[L_{\widehat{\mathfrak{sl}_k}(m,\lambda)}] \mid \lambda \in P_+^m\}$ be the fusion algebra for $\mathfrak{sl}_k$ at level m , and let $\mathcal{A}_{\mathcal{W}}^{p,q}$ be the fusion algebra of $\mathcal{W}_{p,q}(\mathfrak{sl}_k)$. Note that $\mathcal{W}_{p,q}(\mathfrak{sl}_k) = \mathcal{W}_{q,p}(\mathfrak{sl}_k)$ and so $\mathcal{A}_{\mathcal{W}}^{p,q} \cong \mathcal{A}_{\mathcal{W}}^{q,p}$.

The following assertion is the special case of the fusion rule of $\mathcal{W}_{p,q}(\mathfrak{sl}_k)$ computed in [29].

Theorem 5.3. *Let p be an integer such that $p \geq k$, $(p, k+1) = 1$. Then the assignment*

$$[L_{\widehat{\mathfrak{sl}_k}}(\ell, \lambda)] \mapsto [\mathbb{L}(\hat{\lambda}_{\ell})]$$

gives the isomorphism of fusion algebras $\mathcal{A}_{\mathfrak{g}}^{p-k} \xrightarrow{\sim} \mathcal{A}_{\mathcal{W}}^{p,k+1}$.

Now we set $p = k + 2$. Then

$$\{\mathbb{L}(\Lambda_i + \Lambda_j) \mid 0 \leq i \leq j \leq k-1\}$$

gives a complete set of representatives of the isomorphism classes of simple $\mathcal{W}_{k+2,k+1}(\mathfrak{sl}_k)$ -modules. The top level of $\mathbb{L}(\Lambda_i + \Lambda_j)$ is one dimensional with weight given by

$$\frac{-i^2 + i(k(2k+3) - 2j(k+1)) + j(k-j)}{2k(k+2)}. \quad (5.2)$$

Corollary 5.4. $\mathbb{L}(\Lambda_i + \Lambda_j)$ is a simple current module for $\mathcal{W}_{k+1,k+2}(\mathfrak{sl}_k)$ if and only if $i = j$. We have

$$\mathbb{L}(2\Lambda_p) \times \mathbb{L}(\Lambda_j + \Lambda_i) = \mathbb{L}(\Lambda_{j+p}, \Lambda_{i+p}) \quad (5.3)$$

for $0 \leq p \leq k-1$ and $0 \leq i, j \leq k-1$, where the index is considered to be modulo k . In particular,

$$\mathbb{L}(2\Lambda_p) \times \mathbb{L}(2\Lambda_q) = \mathbb{L}(2\Lambda_{p+q}). \quad (5.4)$$

We remark that

$$\mathbb{L}(\Lambda_i + \Lambda_j)' \cong \mathbb{L}(\Lambda_{-i} + \Lambda_{-j}),$$

where $\mathbb{L}(\Lambda_i + \Lambda_j)'$ is the dual module of $\mathbb{L}(\Lambda_i + \Lambda_j)$ (see [1, Theorem 5.5.4]).

Here we summarize some of the properties of $\mathbb{L}(2\Lambda_0) = \mathcal{W}_{k+1,k+2}(\mathfrak{sl}_k)$.

- (1) $\mathbb{L}(2\Lambda_0)$ is a simple, self-dual, rational and C_2 -cofinite vertex operator algebra of CFT-type with central charge $2(k-1)/(k+2)$.
- (2) $\text{ch } \mathbb{L}(2\Lambda_0) = 1 + q^2 + 2q^3 + \cdots$.
- (3) $\mathbb{L}(2\Lambda_0)$ is generated by $\mathbb{L}(2\Lambda_0)_{(2)}$ and $\mathbb{L}(2\Lambda_0)_{(3)}$.
- (4) $\mathbb{L}(2\Lambda_j)$, $0 \leq j \leq k-1$ are simple current $\mathbb{L}(2\Lambda_0)$ -modules and the fusion rules among them are $\mathbb{L}(2\Lambda_i) \times \mathbb{L}(2\Lambda_j) = \mathbb{L}(2\Lambda_{i+j})$.
- (5) The top weight of $\mathbb{L}(2\Lambda_j)$ is $j(k-j)/k$.
- (6) Any simple $\mathbb{L}(2\Lambda_0)$ -module except $\mathbb{L}(2\Lambda_0)$ itself has positive top weight.

Remark 5.5. *The pair (i, j) of the indices i and j for a complete set of representatives of the isomorphism classes of simple M^0 -modules $M^{i,j}$ runs over the range $0 \leq j < i \leq k$, while that for $\mathbb{L}(2\Lambda_0)$ -modules $\mathbb{L}(\Lambda_j + \Lambda_i)$ runs over the range $0 \leq j \leq i \leq k-1$. Let $i' = k - i + j$. Then $0 \leq j \leq i' \leq k-1$ if and only if $0 \leq j < i \leq k$. Thus these two sets of parameters (i, j) 's are related as*

$$\{(i, j) | 0 \leq j \leq i \leq k-1\} = \{(k-i+j, j) | 0 \leq j < i \leq k\}.$$

The following lemma will be used in Section 6.

Lemma 5.6. *The top weight of the simple $K(\mathfrak{sl}_2, k)$ -module $M^{i,j}$ is equal to that of the simple $\mathbb{L}(2\Lambda_0)$ -module $\mathbb{L}(\Lambda_j + \Lambda_{j-i})$ for $0 \leq i \leq k$, $0 \leq j \leq k-1$.*

Proof. By (2.18) and (5.2) we see that the top weight of the simple $K(\mathfrak{sl}_2, k)$ -module $M^{i,j}$ is equal to that of the simple $\mathbb{L}(2\Lambda_0)$ -module $\mathbb{L}(\Lambda_j + \Lambda_{j-i})$ for $0 \leq j < i \leq k$. For a pair (i, j) with $0 \leq i \leq j \leq k-1$, we have $0 \leq j-i < k-i \leq k$. Hence the top weight of $M^{k-i, j-i}$ coincides with that of

$$\mathbb{L}(\Lambda_{j-i} + \Lambda_{(j-i)-(k-i)}) = \mathbb{L}(\Lambda_{j-i} + \Lambda_{j-k}).$$

Since $M^{i,j}$ is isomorphic to $M^{k-i, j-i}$ as M^0 -modules by (2.17) and $\mathbb{L}(\Lambda_{j-i} + \Lambda_j) = \mathbb{L}(\Lambda_{j-k} + \Lambda_{j-i})$ the assertion holds for such a pair (i, j) also. $\square$

6. IDENTIFICATION OF $K(\mathfrak{sl}_2, k)$ AND $\mathcal{W}_{k+1, k+2}(\mathfrak{sl}_k)$

In this section we use the results of Section 4 to show that $K(\mathfrak{sl}_2, k)$ is isomorphic to the $(k+1, k+2)$ -minimal series W -algebra $\mathcal{W}_{k+1, k+2}(\mathfrak{sl}_k)$. We also discuss a correspondence of the simple modules for $K(\mathfrak{sl}_2, k)$ with those for $\mathcal{W}_{k+1, k+2}(\mathfrak{sl}_k)$. If $k=2$, it is well-known that both $K(\mathfrak{sl}_2, 2)$ and $\mathcal{W}_{3,4}(\mathfrak{sl}_2)$ are isomorphic to the simple Virasoro vertex operator algebra with central charge $1/2$. So let us assume that $k \geq 3$.

By the properties of $\mathcal{W}_{k+1, k+2}(\mathfrak{sl}_k)$ described in Section 5.3, we see that $\mathcal{W}_{k+1, k+2}(\mathfrak{sl}_k)$ satisfies the five conditions of Hypothesis 4.1 for N^0 together with the simple current modules $\mathbb{L}(2\Lambda_j)$ for N^j , $0 \leq j \leq k-1$. Therefore, Theorem 4.3 implies the next theorem.

Theorem 6.1. *The $(k+1, k+2)$ -minimal series W -algebra $\mathcal{W}_{k+1, k+2}(\mathfrak{sl}_k)$ is isomorphic to the parafermion vertex operator algebra $K(\mathfrak{sl}_2, k)$ of type $\mathfrak{sl}_2$.*

Furthermore, it follows from Theorem 4.2 that $\bigoplus_{j=0}^{k-1} V_{\mathbb{Z}\gamma - j\gamma/k} \otimes \mathbb{L}(2\Lambda_j)$ is a $\mathbb{Z}_k$ -graded simple current extension of $V_{\mathbb{Z}\gamma} \otimes \mathbb{L}(2\Lambda_0)$ and it is isomorphic to $L_{\widehat{\mathfrak{sl}_2}}(k, 0)$. That is,

$$L_{\widehat{\mathfrak{sl}_2}}(k, 0) = \bigoplus_{j=0}^{k-1} V_{\mathbb{Z}\gamma - j\gamma/k} \otimes \mathbb{L}(2\Lambda_j) \quad (6.1)$$

as $V_{\mathbb{Z}\gamma} \otimes M(2\Lambda_0)$ -modules.

Corollary 6.2. *The parafermion vertex operator algebra $K(\mathfrak{sl}_2, k)$ is rational.*

It is known that $K(\mathfrak{sl}_2, k)$ is isomorphic to $\text{Com}_{L_{\widehat{\mathfrak{sl}_k}}(1,0) \otimes L_{\widehat{\mathfrak{sl}_k}}(1,0)}(L_{\widehat{\mathfrak{sl}_k}}(2,0))$ ([37]). Therefore Theorem 6.1 immediately gives the following assertion that has been conjectured in [36, 11].

Corollary 6.3. *We have the isomorphism*

$$\mathcal{W}_{k+1,k+2}(\mathfrak{sl}_k) \cong \text{Com}_{L_{\widehat{\mathfrak{sl}_k}}(1,0) \otimes L_{\widehat{\mathfrak{sl}_k}}(1,0)}(L_{\widehat{\mathfrak{sl}_k}}(2,0)).$$

For simplicity of notation we identify M^0 with $\mathbb{L}(2\Lambda_0)$, so that $M^0 = \mathbb{L}(2\Lambda_0)$. Then it follows from (2.13), (2.14) and (6.1) that $M^j = \mathbb{L}(2\Lambda_j)$ for $0 \leq j \leq k-1$. This in particular implies that M^j , $0 \leq j \leq k-1$ are the simple current modules for $K(\mathfrak{sl}_2, k)$. The fusion rules among them are

$$M^i \times M^j = M^{i+j} \quad (6.2)$$

by (5.4), which is compatible with (2.9).

We have another description of $L_{\widehat{\mathfrak{sl}_2}}(k, 0)$ as a $\mathbb{Z}_k$ -graded simple current extension of $V_{\mathbb{Z}\gamma} \otimes \mathbb{L}(2\Lambda_0)$. Indeed, let N^j be as in Section 4. Then

$$N^{k-i} \times N^{k-j} = N^{k-(i+j)},$$

and so N^{k-j} , $0 \leq j \leq k-1$ satisfy the conditions of Hypothesis 4.1 for N^j , $0 \leq j \leq k-1$. Therefore, we can apply the argument in Section 4 to N^{k-j} in place of N^j to conclude that $\bigoplus_{j=0}^{k-1} V_{\mathbb{Z}\gamma-j\gamma/k} \otimes N^{k-j}$ is a simple current extension of V^0 isomorphic to $L_{\widehat{\mathfrak{sl}_2}}(k, 0)$. Thus

$$\begin{aligned} \bigoplus_{j=0}^{k-1} V_{\mathbb{Z}\gamma-j\gamma/k} \otimes N^j &\cong \bigoplus_{j=0}^{k-1} V_{\mathbb{Z}\gamma-j\gamma/k} \otimes N^{k-j} \\ &= \bigoplus_{j=0}^{k-1} V_{\mathbb{Z}\gamma+j\gamma/k} \otimes N^j \end{aligned} \quad (6.3)$$

as vertex operator algebras and we have

$$L_{\widehat{\mathfrak{sl}_2}}(k, 0) = \bigoplus_{j=0}^{k-1} V_{\mathbb{Z}\gamma-j\gamma/k} \otimes \mathbb{L}(2\Lambda_{k-j}) \quad (6.4)$$

as $V_{\mathbb{Z}\gamma} \otimes \mathbb{L}(2\Lambda_0)$ -modules. In (6.4) we have $M^j = \mathbb{L}(2\Lambda_{k-j})$ for $0 \leq j \leq k-1$.

Recall the automorphism θ of the vertex operator algebra $L_{\widehat{\mathfrak{sl}_2}}(k, 0)$ of order 2 discussed in Section 2.4. The automorphism θ transforms $V_{\mathbb{Z}\gamma-j\gamma/k}$ to $V_{\mathbb{Z}\gamma+j\gamma/k}$ and induces an automorphism of $V_{\mathbb{Z}\gamma}$. Hence $\theta \otimes 1$ gives the isomorphism (6.3). Note also that $1 \otimes \theta$ is an automorphism of $V_{\mathbb{Z}\gamma} \otimes M^0$ and that θ transforms M^j to $M^j \circ \theta = M^{k-j}$ by (2.19). Thus the isomorphism (6.3) is afforded by $1 \otimes \theta$ also (cf. [41, Lemmas 3.14 and 3.15]).

The next proposition implies that there are only two ways of describing $L_{\widehat{\mathfrak{sl}_2}}(k, 0)$ as a $\mathbb{Z}_k$ -graded simple current extension of $V_{\mathbb{Z}\gamma} \otimes \mathbb{L}(2\Lambda_0)$, namely (6.1) and (6.4).

Proposition 6.4. *One of the following two cases occurs.*

- (1) $M^j = \mathbb{L}(2\Lambda_j)$ for all $0 \leq j \leq k-1$.
- (2) $M^j = \mathbb{L}(2\Lambda_{k-j})$ for all $0 \leq j \leq k-1$.

Proof. Among the simple current $K(\mathfrak{sl}_2, k)$ -modules M^j , $1 \leq j \leq k-1$ (resp. $\mathcal{W}_{k+1, k+2}(\mathfrak{sl}_k)$ -modules $\mathbb{L}(2\Lambda_j)$, $1 \leq j \leq k-1$), only M^1 and M^{k-1} (resp. $\mathbb{L}(2\Lambda_1)$ and $\mathbb{L}(2\Lambda_{k-1})$) have top weight $(k-1)/k$. Hence $M^1 = \mathbb{L}(2\Lambda_1)$ or $M^1 = \mathbb{L}(2\Lambda_{k-1})$. By the fusion rules $M^p \times M^1 = M^{p+1}$ and $\mathbb{L}(2\Lambda_p) \times \mathbb{L}(2\Lambda_1) = \mathbb{L}(2\Lambda_{p+1})$, (1) holds if $M^1 = \mathbb{L}(2\Lambda_1)$, and (2) holds if $M^1 = \mathbb{L}(2\Lambda_{k-1})$. $\square$

Next, we study the correspondence of the remaining simple modules for $K(\mathfrak{sl}_2, k)$ with those for $\mathcal{W}_{k+1, k+2}(\mathfrak{sl}_k)$. For this purpose we first inspect $M^{i,j}$, $0 \leq i \leq k$, $0 \leq j \leq k-1$. Recall that the second index j of $M^{i,j}$ is considered to be modulo k (cf. Remark 2.6). By Lemma 5.6, the top weight of $M^{i,j}$ is equal to that of $\mathbb{L}(\Lambda_j + \Lambda_{j-i})$ for $0 \leq i \leq k$, $0 \leq j \leq k-1$.

Since M^j is a simple current M^0 -module, we see from (2.13), (2.16) and the fusion rules (2.9) for simple $V_{\mathbb{Z}\gamma}$ -modules that the fusion rule of M^p and $M^{i,j}$ is

$$M^p \times M^{i,j} = M^{i,j+p}, \quad 0 \leq p, j \leq k-1, 0 \leq i \leq k. \quad (6.5)$$

Let

$$P(i, j) = k(i-2j) - (i-2j)^2 + 2k(i-j+1)j \quad (6.6)$$

for $0 \leq j \leq i \leq k$. Then the top weight of $M^{i,j}$ is $P(i, j)/2k(k+2)$. Since $M^{i,j} \cong M^{k-i, j-i}$ as M^0 -modules by (2.17), the top weight of $M^{i,j}$ for $0 \leq i \leq j \leq k-1$ is given by $P(k-i, j-i)/2k(k+2)$. We have $P(i, 0) = P(i, i) = i(k-i)$ and

$$P(i, j) - i(k-i) = 2(k+2)j(i-j) \geq 0$$

for $0 \leq j \leq i \leq k$. Moreover,

$$P(k-i, j-i) - i(k-i) = 2(k+2)(k-j)(j-i) > 0$$

for $0 \leq i < j \leq k-1$. Thus the following Lemma holds.

Lemma 6.5. *Let $1 \leq i \leq k$. Then the top weight of $M^{i,j}$ for $0 \leq j \leq k-1$ is at least $i(k-i)/2k(k+2)$ and it is equal to $i(k-i)/2k(k+2)$ if and only if $j = 0, i$.*

We also note that $i(k-i)$ is monotone increasing with respect to i for $0 \leq i \leq k/2$.

We shall show that $M^{i,j} = \mathbb{L}(\Lambda_j + \Lambda_{j-i})$ for all $0 \leq j < i \leq k$ in the case (1) of Proposition 6.4. Thus, assume that $M^j = \mathbb{L}(2\Lambda_j)$ for all $0 \leq j \leq k-1$. We consider a decomposition of $L(k, i)$ into a direct sum of simple $V_{\mathbb{Z}\gamma} \otimes \mathbb{L}(2\Lambda_0)$ -modules and compare it with (2.16).

The top weight of $V_{\mathbb{Z}\gamma+(i-2j)\gamma/2k} \otimes M^{i,j}$ is a sum of the top weight of $V_{\mathbb{Z}\gamma+(i-2j)\gamma/2k}$ and that of $M^{i,j}$. Hence we have

$$(\text{top weight of } V_{\mathbb{Z}\gamma+(i-2)\gamma/2k} \otimes M^{i, i+1}) - (\text{top weight of } V_{\mathbb{Z}\gamma+i\gamma/2k} \otimes M^{i, 0}) = \frac{k-2i}{k} \quad (6.7)$$

for $0 \leq i \leq k$. In the range of $1 \leq i \leq k/2$, the difference $(k-2i)/k$ can be an integer only if k is even and $i = k/2$.

Let

$$\begin{aligned} \mathcal{S} &= \{M^{i,j} | 0 \leq j < i \leq k\} \\ &= \{\mathbb{L}(\Lambda_j + \Lambda_i) | 0 \leq j \leq i \leq k-1\} \end{aligned} \quad (6.8)$$

be a complete set of representatives of the isomorphism classes of simple modules for $M^0 = \mathbb{L}(2\Lambda_0)$. Using the isomorphism $M^{i,j} \cong M^{k-i,j-i}$, we can arrange the representatives $M^{i,j}$'s so that

$$\mathcal{S} = \{M^{i,j} | 0 \leq i \leq (k-1)/2, 0 \leq j \leq k-1\} \quad (6.9)$$

if k is odd, and

$$\mathcal{S} = \{M^{i,j} | 0 \leq i \leq k/2 - 1, 0 \leq j \leq k-1\} \cup \{M^{k/2,j} | 0 \leq j \leq k/2 - 1\} \quad (6.10)$$

if k is even. In the case k is even, we note that $M^{k/2,j} \cong M^{k/2,k/2+j}$ for $0 \leq j \leq k/2 - 1$.

Set

$$\mathcal{T}_i = \{M^{i,j} | 0 \leq j \leq k-1\}$$

for $0 \leq i \leq \lfloor (k-1)/2 \rfloor$, where $\lfloor (k-1)/2 \rfloor$ denotes the largest integer which does not exceed $(k-1)/2$.

Moreover, set

$$\mathcal{T}_{k/2} = \{M^{k/2,j} | 0 \leq j \leq k/2 - 1\}$$

if k is even. Set

$$\mathcal{S}_p = \bigcup_{i=p}^{\lfloor k/2 \rfloor} \mathcal{T}_i$$

for $0 \leq p \leq \lfloor k/2 \rfloor$. Then $\mathcal{S}_0 = \mathcal{S}$.

We shall establish an identification of $M^{i,j} \in \mathcal{T}_i$ and $\mathbb{L}(\Lambda_j + \Lambda_{j-i}) \in \mathcal{T}_i$, $0 \leq j \leq k-1$ ($0 \leq j \leq k/2 - 1$ if k is even and $i = k/2$) for $i = 1, 2, \dots, \lfloor k/2 \rfloor$ inductively. Note that $M^j = \mathbb{L}(2\Lambda_j)$, $0 \leq j \leq k-1$ by our assumption, that is, the identification is given for $i = 0$ and (6.1) holds.

First, we discuss the case $i = 1$. By Lemma 6.5, we see that the top weight of a simple module $M \in \mathcal{S}_1$ is at least $(k-1)/2k(k+2)$ and it is equal to $(k-1)/2k(k+2)$ if and only if $M = M^{1,0}$ or $M^{1,1}$. Then Lemma 5.6 implies that one of the following two cases occurs: $M^{1,0} = \mathbb{L}(\Lambda_0 + \Lambda_{k-1})$ and $M^{1,1} = \mathbb{L}(\Lambda_0 + \Lambda_1)$, or $M^{1,0} = \mathbb{L}(\Lambda_0 + \Lambda_1)$ and $M^{1,1} = \mathbb{L}(\Lambda_0 + \Lambda_{k-1})$.

Suppose $M^{1,0} = \mathbb{L}(\Lambda_0 + \Lambda_1)$. Then since the fusion rule

$$(V_{\mathbb{Z}\gamma-\gamma/k} \otimes \mathbb{L}(2\Lambda_1)) \times (V_{\mathbb{Z}\gamma+\gamma/2k} \otimes \mathbb{L}(\Lambda_0 + \Lambda_1)) = V_{\mathbb{Z}\gamma-\gamma/2k} \otimes \mathbb{L}(\Lambda_1 + \Lambda_2) \quad (6.11)$$

holds by (2.9) and (5.3), both $V_{\mathbb{Z}\gamma+\gamma/2k} \otimes \mathbb{L}(\Lambda_0 + \Lambda_1)$ and $V_{\mathbb{Z}\gamma-\gamma/2k} \otimes \mathbb{L}(\Lambda_1 + \Lambda_2)$ appear as direct summands in a decomposition (2.16) of $L(k, 1)$ into a direct sum of simple $V_{\mathbb{Z}\gamma} \otimes \mathbb{L}(2\Lambda_0)$ -modules. However, the top weight of $\mathbb{L}(\Lambda_1 + \Lambda_2)$ coincides with that of $M^{1,2}$ by Lemma 5.6, and so the difference of the top weight of these two direct summands is $(k-2)/k$ by (6.7), which is not an integer. This is a contradiction, for $L(k, 1)$ is a simple $L(k, 0)$ -module. Therefore, $M^{1,0} = \mathbb{L}(\Lambda_0 + \Lambda_{k-1})$. Then by the fusion rules (5.3) and (6.5), we obtain $M^{1,j} = \mathbb{L}(\Lambda_j + \Lambda_{j-1})$ for $0 \leq j \leq k-1$. Thus the identification of simple modules contained in $\mathcal{T}_1$ holds.

Next, let p be an integer such that $2 \leq p \leq \lfloor (k-1)/2 \rfloor$ and assume that the identification of simple modules contained in $\mathcal{T}_i$, $0 \leq i \leq p-1$ holds.

We replace 1 with p in the above argument. By Lemma 6.5, the top weight of a simple module $M \in \mathcal{S}_p$ is at least $p(k-p)/2k(k+2)$ and it is equal to $p(k-p)/2k(k+2)$ if and only if $M = M^{p,0}$ or $M^{p,p}$. Hence one of the following two cases occurs: $M^{p,0} = \mathbb{L}(\Lambda_0 + \Lambda_{k-p})$

and $M^{p,p} = \mathbb{L}(\Lambda_0 + \Lambda_p)$, or $M^{p,0} = \mathbb{L}(\Lambda_0 + \Lambda_p)$ and $M^{p,p} = \mathbb{L}(\Lambda_0 + \Lambda_{k-p})$. Suppose $M^{p,0} = \mathbb{L}(\Lambda_0 + \Lambda_p)$. Then since

$$(V_{\mathbb{Z}\gamma-\gamma/k} \otimes \mathbb{L}(2\Lambda_1)) \times (V_{\mathbb{Z}\gamma+p\gamma/2k} \otimes \mathbb{L}(\Lambda_0 + \Lambda_p)) = V_{\mathbb{Z}\gamma+(p-2)\gamma/2k} \otimes \mathbb{L}(\Lambda_1 + \Lambda_{p+1}) \quad (6.12)$$

by (2.9) and (5.3), both $V_{\mathbb{Z}\gamma+p\gamma/2k} \otimes \mathbb{L}(\Lambda_0 + \Lambda_p)$ and $V_{\mathbb{Z}\gamma+(p-2)\gamma/2k} \otimes \mathbb{L}(\Lambda_1 + \Lambda_{p+1})$ appear as direct summands in a decomposition of $L(k, p)$ into a direct sum of simple $V_{\mathbb{Z}\gamma} \otimes \mathbb{L}(2\Lambda_0)$ -modules. However, the top weight of $\mathbb{L}(\Lambda_1 + \Lambda_{p+1})$ coincides with that of $M^{p,p+1}$, and so the difference of the top weight of these two direct summands is $(k-2p)/k$ by (6.7), which is not an integer. Thus $M^{p,0} = \mathbb{L}(\Lambda_0 + \Lambda_{k-p})$. Hence we have the identification of simple modules contained in $\mathcal{T}_p$ by the fusion rules.

In the case k is even and $p = k/2$, we have $\mathcal{S}_{k/2} = \mathcal{T}_{k/2}$. The minimum of the top weight of the simple modules contained in $\mathcal{T}_{k/2}$ is $k/8(k+2)$ and it is attained only by $M^{2/k,0}$. Hence $M^{2/k,0} = \mathbb{L}(\Lambda_0 + \Lambda_{k/2})$ and the identification of simple modules contained in $\mathcal{T}_{k/2}$ holds by the fusion rules. This completes the induction on i . Therefore, we conclude that $M^{i,j} = \mathbb{L}(\Lambda_j + \Lambda_{j-i})$ for all $0 \leq j < i \leq k$.

For $M^{i,j}$ with $0 \leq i \leq j \leq k-1$, we use the isomorphism $M^{i,j} \cong M^{k-i,j-i}$ (2.18) of M^0 -modules. Since $0 \leq j-i < k-i \leq k$, we apply the above identification to $M^{k-i,j-i}$. Then we have

$$M^{k-i,j-i} = \mathbb{L}(\Lambda_{j-i} + \Lambda_{(j-i)-(k-i)}) = \mathbb{L}(\Lambda_j + \Lambda_{j-i}).$$

Thus the identification $M^{i,j} = \mathbb{L}(\Lambda_j + \Lambda_{j-i})$ holds for all $0 \leq i \leq k$, $0 \leq j \leq k-1$ in the case (1) of Proposition 6.4.

We have discussed the identification of simple modules for $K(\mathfrak{sl}_2, k)$ and those for $\mathcal{W}_{k+1,k+2}(\mathfrak{sl}_k)$ in the case (1) of Proposition 6.4 so far. As to the case (2) of Proposition 6.4, recall the permutation $M^{i,j} \mapsto M^{i,j} \circ \theta = M^{i,i-j}$ (2.19) on the simple modules induced by the automorphism θ . In the case (1) of Proposition 6.4, $M^{i,i-j} = \mathbb{L}(\Lambda_{-j} + \Lambda_{i-j})$. Therefore $M^{i,j} = \mathbb{L}(\Lambda_{-j} + \Lambda_{i-j})$ for all i, j in the case (2) of Proposition 6.4. In fact, M^p and $M^{i,0}$ are transformed to M^{k-p} and $M^{i,i}$ by the permutation and the fusion rule (6.5) is transformed to $M^{k-p} \times M^{i,i-j} = M^{i,i-(j+p)}$.

We have proved the following theorem.

Theorem 6.6. *There are exactly two ways of identification of the simple modules for $K(\mathfrak{sl}_2, k)$ and those for $\mathcal{W}_{k+1,k+2}(\mathfrak{sl}_k)$, namely,*

- (1) $M^{i,j} = \mathbb{L}(\Lambda_j + \Lambda_{j-i})$ for all $0 \leq i \leq k$, $0 \leq j \leq k-1$.
- (2) $M^{i,j} = \mathbb{L}(\Lambda_{-j} + \Lambda_{i-j})$ for all $0 \leq i \leq k$, $0 \leq j \leq k-1$.

The following corollary is already discussed in the proof of Theorem 6.6.

Corollary 6.7. *The automorphism θ of $K(\mathfrak{sl}_2, k) = \mathcal{W}_{k+1,k+2}(\mathfrak{sl}_k)$ induces a permutation $\mathbb{L}(\Lambda_j + \Lambda_i) \mapsto \mathbb{L}(\Lambda_{-j} + \Lambda_{-i})$ on the simple modules for all i, j .*

APPENDIX A. PROOF OF PROPOSITION 5.2

For a $\mathbb{Z}_{\geq 0}$ -graded vertex operator algebra V , let $F^p V$ be the subspace of V spanned by the vectors

$$a_{-n_1-1}^1 \cdots a_{-n_r-1}^r b$$

with $a_i \in V$, $b \in V$, $n_i \in \mathbb{Z}_{\geq 0}$, $n_1 + \cdots + n_r \geq p$. We have [40]

$$\begin{aligned} V &= F^0 V \supset F^1 V \supset \cdots, \quad TF^p V \subset F^{p+1} V, \quad \bigcap F^p V = 0, \\ a_n F^p V &\subset F^{p+q-n-1} V \text{ for } a \in F^p V, n \in \mathbb{Z}, \\ a_n F^p V &\subset F^{p+q-n} V \text{ for } a \in F^p V, n \geq 0. \end{aligned}$$

Set $\text{gr } V = \bigoplus_{p=0}^{\infty} F^p V / F^{p+1} V$. Let $\sigma_p : F^p V \rightarrow F^p V / F^{p+1} V$ be the symbol map. This induces an linear isomorphism $V \xrightarrow{\sim} \text{gr } V$.

Proposition A.1. *Let r be a non-negative integer such that*

$$a_n F^p V \subset F^{p+q-n+r} V \quad \text{for all } a \in F^p V, n \geq 0. \quad (\text{A.1})$$

Then $\text{gr } V$ is a Poisson vertex algebra by

$$\begin{aligned} \sigma_p(a)\sigma_q(b) &= \sigma_{p+q}(a_{-1}b), \quad T\sigma_p(a) = \sigma_{p+1}(a), \\ \sigma_p(a)_{(n)}\sigma_q(b) &= \sigma_{p+q-n+r}(a_nb) \quad \text{for } n \geq 0. \end{aligned} \quad (\text{A.2})$$

Proof. The assertion was proved by Li [40] for $r = 0$. The same proof applies to the cases that $r > 0$ as well. $\square$

Note that in Proposition A.1 we can always take $r = 0$, and this Poisson vertex algebra structure of $\text{gr } V$ is trivial if and only if $a_n F^q \mathcal{W}^\ell(\mathfrak{g}) \subset F^{p+q-n+1} \mathcal{W}^\ell(\mathfrak{g})$ for all $a \in F^p \mathcal{W}^\ell(\mathfrak{g})$, $n \geq 0$. Therefore, if this is the case we can give $\text{gr } V$ a Poisson vertex algebra structure using Proposition A.1 for $r = 1$.

Example A.2. *The Poisson vertex algebra $\text{gr } V_{\hat{\mathfrak{g}}}(\ell, 0)$ for $r = 0$ in Proposition A.1 is isomorphic to $\mathbb{C}[J_\infty \mathfrak{g}^*]$ equipped with the level 0 Poisson vertex algebra structure induced from the Kirillov-Kostant Poisson structure of $\mathfrak{g}^*$ ([2].) Here $J_\infty X$ denotes the arc space of a scheme X . Let $M_{\hat{\mathfrak{h}}}(\ell, 0) \subset V_{\hat{\mathfrak{g}}}(\ell, 0)$ be the Heisenberg vertex subalgebra generated by $h(-1)\mathbf{1}$ with $h \in \mathfrak{h}$. Then $\text{gr } M_{\hat{\mathfrak{h}}}(\ell, 0) \cong \mathbb{C}[J_\infty \mathfrak{h}^*]$. Here, again, $\mathbb{C}[J_\infty \mathfrak{h}^*]$ equipped with the level 0 Poisson vertex algebra structure induced from the Kirillov-Kostant Poisson structure of $\mathfrak{h}^*$, which is trivial. Therefore, (A.1) holds for $r = 1$. Hence (A.2) for $r = 1$ gives the Poisson vertex algebra structure on $\text{gr } M_{\hat{\mathfrak{h}}}(\ell, 0)$. We have*

$$h_n h' = \begin{cases} \ell(h|h') & \text{for } n = 1 \\ 0 & \text{for } n = 0 \text{ or } n \geq 2, \end{cases}$$

for $h, h' \in \mathfrak{h} \subset \mathbb{C}[\mathfrak{h}^] \subset \mathbb{C}[J_\infty \mathfrak{h}^*]$. In particular the Poisson vertex algebra structure of $\text{gr } M_{\hat{\mathfrak{h}}}(\ell, 0)$ does not depend on the level ℓ provided that $\ell \neq 0$.*

Consider the W -algebra $\mathcal{W}^\ell(\mathfrak{g})$. The vertex Poisson algebra structure of $\text{gr } \mathcal{W}^\ell(\mathfrak{g})$ is trivial if we take r to be 0 in Proposition A.1 ([3]). Therefore, by Proposition A.1 $\text{gr } \mathcal{W}^\ell(\mathfrak{g})$ is the vertex Poisson algebra by $\sigma_p(a)_{(n)}\sigma_q(b) = \sigma_{p+q-n+1}(a_nb)$, $n \geq 0$. Below we regard $\text{gr } \mathcal{W}^\ell(\mathfrak{g})$ as a vertex Poisson algebra with respect to this product.

In order to prove Proposition 5.2 it is sufficient to show the following propositions.

Proposition A.3. *Let ℓ be non-critical. Then the vertex Poisson algebra $\text{gr } \mathcal{W}^\ell(\mathfrak{sl}_k)$ is generated by the weight 2 subspace and the weight 3 subspace.*

Proposition A.4. *For a non-critical ℓ , the vertex Poisson algebra structure of $\text{gr } \mathcal{W}^\ell(\mathfrak{g})$ is independent of $\ell \in \mathbb{C} \setminus \{-h^\vee\}$.*

Proof. We shall use the Miura map

$$\mathcal{W}^\ell(\mathfrak{g}) \hookrightarrow M_{\widehat{\mathfrak{h}}}(\ell + h^\vee, 0),$$

see [5] for the details. This induces the injective Poisson vertex algebra homomorphism

$$\mathrm{gr} \mathcal{W}^\ell(\mathfrak{g}) \hookrightarrow \mathrm{gr} M_{\widehat{\mathfrak{h}}}(\ell + h^\vee, 0),$$

where $\mathrm{gr} M_{\widehat{\mathfrak{h}}}(\ell + h^\vee, 0)$ is equipped with the Poisson vertex algebra structure described in Example A.2. The image of $\mathrm{gr} \mathcal{W}^\ell(\mathfrak{g})$ is generated by symmetric polynomials in $S(\mathfrak{h}) = \mathbb{C}[\mathfrak{h}^*] \subset \mathbb{C}[J_\infty \mathfrak{h}^*]$, and does not depend on ℓ . Hence, the vertex Poisson algebra structure of $\mathrm{gr} \mathcal{W}^\ell(\mathfrak{g})$ is independent of ℓ as long as it is non-critical. $\square$

By Proposition A.4, it is sufficient to show Proposition A.3 for a non-critical ℓ .

Recall the following assertion proved by Frenkel, Kac, Radul and Wang.

Theorem A.5 ([28]). *For $\ell = 1 - k$, $\mathcal{W}^\ell(\mathfrak{gl}_k)$ is isomorphic to the simple quotient of $\mathcal{W}_{1+\infty}$ -algebra $\mathcal{W}_{1+\infty}^k$ of central charge k .*

Proof of Proposition A.3. The vertex algebra $\mathcal{W}_{1+\infty}^c$ is freely generated by fields $J^m(z) = \sum_{n \in \mathbb{Z}} J_n^m z^{-n-1}$, $m = 0, 1, 2, \dots$, satisfying the OPE's

$$\begin{aligned} & J^m(z)J^n(w) \\ & \sim \sum_{a=1}^{m+n} ([n]_a J^{m+n-a}(w) - (-1)^a [m]_a J^{m+n-a}(z)/(z-w)^{a+1}) + \frac{(-1)^m m! n! c}{(z-w)^{m+n+2}}, \end{aligned}$$

where $[n]_a = n(n-1) \cdots (n-a+1)$. The conformal weight of $J^m(z)$ is $m+1$. The image of $J^0(z)$ generates the rank 1 Heisenberg subalgebra π and we have $\mathcal{W}^\ell(\mathfrak{gl}_k) = \mathcal{W}^\ell(\mathfrak{sl}_k) \otimes \pi$.

We have

$$\begin{aligned} J_0^m J^n &\equiv 0 \pmod{F^1 \mathcal{W}_{1+\infty}^c}, & J_1^m J^n &\equiv (m+n) J^{m+n-1} \pmod{F^1 \mathcal{W}_{1+\infty}^c}, \\ J_r^m J^n &= ([n]_r - (-1)^r [m]_r) J^{m+n-r} \pmod{F^1 \mathcal{W}_{1+\infty}^c} \end{aligned}$$

It follows that $\mathrm{gr} \mathcal{W}_{1+\infty}^c$ is generated by J^0 , J^1 and J^2 . Therefore $\mathrm{gr} \mathcal{W}^{1-k}(\mathfrak{sl}_k)$ is generated by the image of J^1 and J^2 . This completes the proof. $\square$

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