

Irregularities on hyperlanes of holonomic $\mathcal{D}$ -module (especially defined by confluent hypergeometric partial differential equations)

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1 Introduction

In this expository paper, I will explain the irregularity at a singular point of differential equation. At first, I will give you a review of study on ordinary linear differential equations. Secondly, I will talk about holonomic $\mathcal{D}$ -modules, especially, Humbert confluent hypergeometric differential modules in m variables.

2 Irregularity of holonomic $\mathcal{D}$ -module defined by an ordinary differential operator.

Consider a linear ordinary differential operator with coefficients in holomorphic functions at the origin in the Riemann Sphere:

$$Pu = \left(\sum_{i=0}^m a_i(x) (d/dx)^i \right) u.$$

where a_m is supposed not to be identically zero. Let $\mathcal{O}$ and $\hat{\mathcal{O}}$ be the ring of convergent power-series and the ring of formal power-series in x , respectively. Then, we see the following isomorphism of linear spaces due to Deligne (cf. [23], etc.) :

$$H^1(S^1, \text{Ker}(P : \mathcal{A}_0)) \simeq \text{Ker}(P; \hat{\mathcal{O}}/\mathcal{O}),$$

for, from the existence theorem of asymptotic solutions due to Hukuhara (cf. [26]) (and other many contributors), we have the short exact sequence

$$0 \rightarrow \text{Ker}(P : \mathcal{A}_0) \rightarrow \mathcal{A}_0 \xrightarrow{P} \mathcal{A}_0 \rightarrow 0,$$

from which, we get the exact sequence,

$$0 \rightarrow H^1(S^1, \text{Ker}(P : \mathcal{A}_0)) \rightarrow H^1(S^1, \mathcal{A}_0)(= \hat{\mathcal{O}}/\mathcal{O}) \xrightarrow{P} H^1(S^1, \mathcal{A}_0)(= \hat{\mathcal{O}}/\mathcal{O}) \rightarrow 0.$$

The dimension is finite and is equal to

$$\begin{aligned} i_0(P) &= \sup\{i - v(a_i) : i = 0, \dots, m\} - (m - v(a_m)) \\ &= (v(a_m) - m) - \inf\{v(a_i) - i : i = 0, \dots, m\}, \end{aligned}$$

which is called the irregularity by Malgrange [17], [18], the invariant of Fuchs by Gérard-Levelt [3], [4] or the irregular index by Komatsu (in a private communication), where,

$$v(a) = \sup\{p : x^{-p}a(x) \text{ is holomorphic at the origin.}\}.$$

Remark 0: Let $\mathcal{K}$, $\hat{\mathcal{K}}$ and $\mathcal{E}$ be the ring of the ring of convergent Laurent series with finite negative order terms, the ring of formal, the ring of formal Laurent series with finite negative order terms and the ring of convergent Laurent series, respectively. Denote by F one of $\mathcal{O}$, $\hat{\mathcal{O}}$, $\mathcal{K}$, $\hat{\mathcal{K}}$ and $\mathcal{E}$. We consider P as an operator from F to itself. Then, $\text{Ker}(P; F)$ and $\text{Coker}(P; F)$ are finite dimensional, and has a index $\chi(P; F) = \dim_{\mathbb{C}} \text{Ker}(P; F) - \dim_{\mathbb{C}} \text{Coker}(P; F)$, which can be calculated as follows:

$$\begin{aligned} \chi(P; \mathcal{O}) &= m - v(a_m), \\ \chi(P; \hat{\mathcal{O}}) &= \sup\{i - v(a_i) : i = 1, \dots, m\}, \\ \chi(P; \mathcal{K}) &= m - v(a_m) - \sup\{i - v(a_i) : i = 1, \dots, m\}, \\ \chi(P; \hat{\mathcal{K}}) &= 0, \\ \chi(P; \mathcal{E}) &= 0. \end{aligned}$$

The quantity $i_0(P)$ is also equal to the followings [17], [18] :

$$\begin{aligned} &\chi(P; \hat{\mathcal{O}}) - \chi(P; \mathcal{O}), \\ &\chi(P; \hat{\mathcal{K}}) - \chi(\mathcal{K}), \\ &-\chi(P; \mathcal{K}), \\ &\chi(P; \hat{\mathcal{K}}/\mathcal{K}), \\ &\chi(P; \mathcal{E}) - \chi(P; \mathcal{K}), \\ &\chi(P; \mathcal{E}/\mathcal{K}), \\ &\chi(P; \mathcal{E}/\mathcal{O}) - \chi(P; \mathcal{K}/\mathcal{O}), \\ &\dim_{\mathbb{C}} \text{Ker}(P; \hat{\mathcal{O}}/\mathcal{O}), \\ &\dim_{\mathbb{C}} \text{Ker}(P; \hat{\mathcal{K}}/\mathcal{K}), \\ &\dim_{\mathbb{C}} \text{Ker}(P; \mathcal{E}/\mathcal{K}), \\ &\dim_{\mathbb{C}} \text{Ker}(P; (\mathcal{E}/\mathcal{O})/(\mathcal{K}/\mathcal{O})). \end{aligned}$$

Remark 1: If we consider a linear ordinary differential operator with coefficients in holomorphic functions at the infinity in the Riemann Sphere and we do not use the variable $t = \frac{1}{x}$, the quantity is equal to

$$\begin{aligned} i_\infty(P) &= \sup\{v'(a_i) - i : i = 0, \dots, m\} - (v'(a_m) - m) \\ &= (m - v'(a_m)) - \inf\{i - v'(a_i) : i = 0, \dots, m\}, \end{aligned}$$

where

$$v'(a) = \sup\{p : x^{-p}a(x) \text{ is holomorphic at the infinity.}\}.$$

Remark 2: We have also another important quantity associated with the linear ordinary differential operator $P = (\sum_{i=0}^m a_i(x)(d/dx)^i)$. At the origin, we set

$$k = \sup\{0, \frac{(v(a_m) - m) - (v(a_i) - i)}{m - i} : i = 0, \dots, m - 1\},$$

and at the infinity, we set

$$k = \sup\{0, \frac{(m - v'(a_m)) - (i - v'(a_i))}{m - i} : i = 0, \dots, m - 1\},$$

which is called the invariant of Katz by Gérard-Levelt [3], [4] or the order by Sibuya [28], and $k + 1$ is called the irregularity by Komatsu [9], [10]. In order to understand the importance of this quantity, see the above references and also Ramis [24], [25], Komatsu [11]. Malgrange [21]. In adding a word,

$$i_0(P) \geq k \geq \frac{i_0(P)}{m}, \quad mk \geq i_0(P) \geq k.$$

Let $\mathcal{D}_0$ be the stalk of germs of linear ordinary differential operators with holomorphic coefficients, and put $\mathcal{M}_0 = \mathcal{D}_0/\mathcal{D}_0P$. Then, $\mathcal{M}_0$ has a projective resolution

$$0 \leftarrow \mathcal{M}_0 \leftarrow \mathcal{D}_0 \xleftarrow{P} \mathcal{D}_0 \leftarrow 0,$$

from which, by operating the functor $\mathcal{H}om_{\mathcal{D}_0}(\cdot, \mathcal{F}_0)$, we have the solution complex with values in $\mathcal{F}$ at the origin,

$$\mathcal{S}ol(\mathcal{M}_0, \mathcal{F}_0) : \mathcal{F}_0 \xrightarrow{P} \mathcal{F}_0 \rightarrow 0.$$

We have the isomorphism:

$$\text{Ext}^0(\mathcal{M}_0, \mathcal{F}_0) \simeq \text{Ker}(\mathcal{F}_0; P), \quad \text{Ext}^1(\mathcal{M}_0, \mathcal{F}_0) \simeq \text{Coker}(\mathcal{F}_0; P).$$

Therefore, the index as $\mathcal{D}$ -module at the origin,

$$\chi(\mathcal{M}; \mathcal{F})_0 = \dim_{\mathbb{C}} \text{Ext}^0(\mathcal{M}_0, \mathcal{F}_0) - \dim_{\mathbb{C}} \text{Ext}^1(\mathcal{M}_0, \mathcal{F}_0),$$

is equal to the index $\chi(P; F)$, and the irregularity as $\mathcal{D}$ -module at the origin,

$$\text{Irr}(\mathcal{M})_0 = \chi(\mathcal{M}_0; \hat{\mathcal{O}}) - \chi(\mathcal{M}_0; \mathcal{O}),$$

is equal to the irregularity $\text{Irr}(P)_0$ and

$$\text{Irr}(\mathcal{M})_0 = \chi(\mathcal{M}_0; \hat{\mathcal{K}}) - \chi(\mathcal{M}_0; \mathcal{K}),$$

$$\text{Irr}(\mathcal{M})_0 = \chi(\mathcal{M}_0; \mathcal{E}) - \chi(\mathcal{M}_0; \mathcal{K}),$$

$$\text{Irr}(\mathcal{M})_0 = \chi(\mathcal{M}_0; \mathcal{E}/\mathcal{O}) - \chi(\mathcal{M}_0; \mathcal{K}/\mathcal{O}).$$

3 Indices of holonomic $\mathcal{D}$ -modules and their irregularities

Let $\mathcal{D}$ be the sheaf of germs of linear partial differential operators with coefficients of holomorphic functions on a manifold M and let $\mathcal{M}$ be a holonomic $\mathcal{D}$ -module. The module $\mathcal{M}$ has a projective resolution

$$0 \leftarrow \mathcal{M} \leftarrow \mathcal{D}^{m_0} \xleftarrow{P_0} \mathcal{D}^{m_1} \xleftarrow{P_1} \mathcal{D}^{m_2} \xleftarrow{P_2} \dots \xleftarrow{P_{2n-1}} \mathcal{D}^{m_{2n}} \leftarrow 0$$

from which, by operating the functor $\mathcal{H}om_{\mathcal{D}}(\cdot, \mathcal{F})$, we have the solution complex with values in $\mathcal{F}$,

$$\text{Sol}(\mathcal{M}, \mathcal{F}) : \mathcal{F}^{m_0} \xrightarrow{P_0^t} \mathcal{F}^{m_1} \xrightarrow{P_1^t} \dots \xrightarrow{P_{2n-1}^t} \mathcal{F}^{m_{2n}} \rightarrow 0.$$

For a point p , the index of holonomic $\mathcal{D}$ -module $\mathcal{M}$ with values in $\mathcal{F}$ is defined by

$$\chi(\mathcal{M}; \mathcal{F})_p = \sum_{i=0}^{2n} \dim_{\mathbb{C}}(-1)^i \mathcal{E}xt^i(\mathcal{M}, \mathcal{F})_p.$$

For the point p , the irregularity of holonomic $\mathcal{D}$ -module $\mathcal{M}$ is defined by

$$\text{Irr}(\mathcal{M})_p = \chi(\mathcal{M}; \mathcal{O}_{\hat{M}|H})_p - \chi(\mathcal{M}; \mathcal{O}_{M|H})_p,$$

where $\mathcal{O}$ is the sheaf of germs of holomorphic functions on M , H is the set of singular points of $\mathcal{M}$, $\mathcal{O}_{M|H}$ is the zero-extension of the restriction of $\mathcal{O}$ on H and $\mathcal{O}_{\hat{M}|H}$ is the Zariski completion of $\mathcal{O}$ along H .

4 Holonomic $\mathcal{D}$ -module defined by Humbert confluent hypergeometric partial differential equations Φ_D

In the sequel, we consider the solution complexes of holonomic $\mathcal{D}$ -module defined by Humbert confluent hypergeometric partial differential equations Φ_D (derived from Lauricella F_D) and give the calculation of the cohomology groups.

We put $M = (P_C^1)^m$ and $H = \bigcup_{k=1}^m H_k$, where $H_k = P_C^1 \times \cdots \times \{\infty\} \times \cdots \times P_C^1$. For a domain U included in H_k , we define

$$\mathcal{O}_{\widehat{M|H},s,A}(U) = \left\{ \sum_{j \geq 0} f_j(y_k)(x_k)^{-j}; \exists C > 0, \forall n, s.t. \sup_{\hat{y}_k \in U} |f_n(y_k)| < CA^n \{(n-1)!\}^{s-1} \right\},$$

where $y_k = (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_m)$, $\hat{y}_k = (x_1, \dots, x_{k-1}, \infty, x_{k+1}, \dots, x_m)$. For a point $p \in H \setminus \bigcup_{k \neq l} (H_k \cap H_l)$, if $p \in H_k$ then we put

$$(\mathcal{O}_{\widehat{M|H},s,A})_p = \text{Ind} \lim_{p \in U \subset H_k} \mathcal{O}_{\widehat{M|H},s,A}(U).$$

We define as follow:

$$\begin{aligned} (\mathcal{O}_{\widehat{M|H},s})_p &= \text{Ind} \lim_{A > 0} (\mathcal{O}_{\widehat{M|H},s,A})_p, \\ (\mathcal{O}_{\widehat{M|H},(s)})_p &= \text{Proj} \lim_{A > 0} (\mathcal{O}_{\widehat{M|H},s,A})_p, \\ (\mathcal{O}_{\widehat{M|H},s,A-})_p &= \text{Ind} \lim_{0 < B < A} (\mathcal{O}_{\widehat{M|H},s,B})_p, \\ (\mathcal{O}_{\widehat{M|H},(s,A+)}_p &= \text{Proj} \lim_{B > A} (\mathcal{O}_{\widehat{M|H},s,B})_p. \end{aligned}$$

The system of Humbert confluent hypergeometric partial differential equations Φ_D [1] is as follows:

$$\Phi_D : x_k \frac{\partial^2 u}{\partial x_k^2} + \sum_{l \neq k} x_l \frac{\partial^2 u}{\partial x_k \partial x_l} + (c - x_k) \frac{\partial u}{\partial x_k} - b_k u = 0 \quad (\text{denoted by } L_k u = 0 \text{ for } k = 1, \dots, m).$$

where $b_k (k = 1, \dots, m)$ and c are not non-negative integers. Note that L_k 's commute with each other. We consider the $\mathcal{D}$ -module $\mathcal{M}_D$ defined by Φ_D , namely we put

$$\mathcal{M}_D = \mathcal{D} / (\mathcal{D}L_1 + \cdots + \mathcal{D}L_m).$$

We have a projective resolution like Koszul complex

$$0 \longleftarrow \mathcal{M}_D \longleftarrow \mathcal{D}^{n_0} \xleftarrow{\nabla_0^m} \mathcal{D}^{n_1} \xleftarrow{\nabla_1^m} \cdots \xleftarrow{\nabla_{q-1}^m} \mathcal{D}^{n_q} \xleftarrow{\nabla_q^m} \cdots \xleftarrow{\nabla_{m-1}^m} \mathcal{D}^{n_m} \longleftarrow 0$$

and we have the solution complex $Sol(\mathcal{M}_D, \mathcal{F})$ with values in $\mathcal{F}$

$$\mathcal{F}^{n_0} \xrightarrow{\nabla_0^m} \mathcal{F}^{n_1} \xrightarrow{\nabla_1^m} \cdots \xrightarrow{\nabla_{q-1}^m} \mathcal{F}^{n_q} \xrightarrow{\nabla_q^m} \cdots \xrightarrow{\nabla_{m-1}^m} \mathcal{F}^{n_m} \longrightarrow 0,$$

where $n_q = \frac{m!}{q!(m-q)!}$ and ∇_q^m 's are defined by the following recursive manner:

$$\nabla_0^1 = L_1, \dots, \nabla_0^m = \begin{pmatrix} L_1 \\ \vdots \\ L_m \end{pmatrix},$$

$$\begin{aligned}\nabla_1^2 &= ((-1)^{2-1}L_2, L_1), \dots, \nabla_1^m = \begin{pmatrix} \nabla_1^{m-1} & 0 \\ (-1)^1 L_m I_{m-1} & \nabla_0^{m-1} \end{pmatrix}. \\ \nabla_q^m &= \begin{pmatrix} \nabla_1^{m-1} & 0 \\ (-1)^q L_m I_{\frac{(m-1)!}{q!(m-1-q)!}} & \nabla_{q-1}^{m-1} \end{pmatrix}. \\ &\dots, \\ \nabla_{m-1}^m &= ((-1)^{m-1}L_m, \nabla_{m-2}^{m-1}),\end{aligned}$$

and we have the following

Theorem 1 . *Let $M = (P_C^1)^m$, $H = \bigcup_{k=1}^m H_k$, $p \in H \setminus \bigcup_{k,l}(H_k \cap H_l)$ be as above. The dimensions of cohomology groups of the solution complexes for the $\mathcal{D}$ -module defined by Φ_D are as follow:*

(1) *If $1 \leq s < 2$,*

$$\text{for } \mathcal{F} = \mathcal{O}_{\widehat{M|H,(s)}}, \mathcal{O}_{\widehat{M|H,s,A-}}, \mathcal{O}_{\widehat{M|H,(s,A+)}}, \mathcal{O}_{\widehat{M|H,s}},$$

$$\dim_C \text{Ext}^j((\mathcal{M}_D)_p, \mathcal{F}_p) = \begin{cases} 0, & (j = 0, 2, \dots, m) \\ 1, & (j = 1) \end{cases}$$

(2) *If $s > 2$,*

$$\text{for } \mathcal{F} = \mathcal{O}_{\widehat{M|H,(s)}}, \mathcal{O}_{\widehat{M|H,s,A-}}, \mathcal{O}_{\widehat{M|H,(s,A+)}}, \mathcal{O}_{\widehat{M|H,s}},$$

$$\dim_C \text{Ext}^j((\mathcal{M}_D)_p, \mathcal{F}_p) = 0, \quad (j = 0, 1, 2, \dots, m).$$

(3) *In the case of $s = 2$,*

(i) *if $A > 1$,*

$$\text{for } \mathcal{F} = \mathcal{O}_{\widehat{M|H,2,A-}}, \mathcal{O}_{\widehat{M|H,(2,A+)}},$$

$$\dim_C \text{Ext}^j((\mathcal{M}_D)_p, \mathcal{F}_p) = 0, \quad (j = 0, 1, 2, \dots, m).$$

(ii) *if $0 < A < 1$,*

$$\text{for } \mathcal{F} = \mathcal{O}_{\widehat{M|H,2,A-}}, \mathcal{O}_{\widehat{M|H,(2,A+)}},$$

$$\dim_C \text{Ext}^j((\mathcal{M}_D)_p, \mathcal{F}_p) = \begin{cases} 0, & (j = 0, 2, \dots, m) \\ 1, & (j = 1) \end{cases}$$

(iii) *if $A = 1$,*

$$\dim_C \text{Ext}^j((\mathcal{M}_D)_p, (\mathcal{O}_{\widehat{M|H,2,1-}})_p) = \begin{cases} 0, & (j = 0, 2, \dots, m) \\ 1, & (j = 1) \end{cases}.$$

$$\dim_C \operatorname{Ext}^j((\mathcal{M}_D)_p, (\mathcal{O}_{\widehat{M|H},(2,1+)})_p) = 0, \quad (j = 0, 1, 2, \dots, m).$$

$$(iv) \dim_C \operatorname{Ext}^j((\mathcal{M}_D)_p, (\mathcal{O}_{\widehat{M|H},(2)})_p) = \begin{cases} 0, & (j = 0, 2, \dots, m) \\ 1, & (j = 1) \end{cases}.$$

$$\dim_C \operatorname{Ext}^j((\mathcal{M}_D)_p, (\mathcal{O}_{\widehat{M|H},2})_p) = 0, \quad (j = 0, 1, 2, \dots, m).$$

$$(4) \dim_C \operatorname{Ext}^j((\mathcal{M}_D)_p, (\mathcal{O}_{\widehat{M|H}})_p) = 0, \quad (j = 0, 1, 2, \dots, m).$$

Corollary 1 . *The indexes of $\mathcal{D}$ -module defined by Φ_2 are as follow:*

(1) *If $1 \leq s < 2$,*

$$\text{for } \mathcal{F} = \mathcal{O}_{\widehat{M|H},(s)}, \mathcal{O}_{\widehat{M|H},s,A-}, \mathcal{O}_{\widehat{M|H},(s,A+)}, \mathcal{O}_{\widehat{M|H},s},$$

$$\mathcal{X}((\mathcal{M}_D)_p, \mathcal{F}_p) = -1.$$

(2) *If $s > 2$,*

$$\text{for } \mathcal{F} = \mathcal{O}_{\widehat{M|H},(s)}, \mathcal{O}_{\widehat{M|H},s,A-}, \mathcal{O}_{\widehat{M|H},(s,A+)}, \mathcal{O}_{\widehat{M|H},s},$$

$$\mathcal{X}((\mathcal{M}_D)_p, \mathcal{F}_p) = 0.$$

(3) *In the case of $s = 2$*

(i) *if $A > 1$,*

$$\text{for } \mathcal{F} = \mathcal{O}_{\widehat{M|H},2,A-}, \mathcal{O}_{\widehat{M|H},(2,A+)},$$

$$\mathcal{X}((\mathcal{M}_D)_p, \mathcal{F}_p) = 0.$$

(ii) *if $0 < A < 1$,*

$$\text{for } \mathcal{F} = \mathcal{O}_{\widehat{M|H},2,A-}, \mathcal{O}_{\widehat{M|H},(2,A+)},$$

$$\mathcal{X}((\mathcal{M}_D)_p, \mathcal{F}_p) = -1.$$

(iii) *if $A = 1$,*

$$\mathcal{X}((\mathcal{M}_D)_p, (\mathcal{O}_{\widehat{M|H},2,1-})_p) = -1.$$

$$\mathcal{X}((\mathcal{M}_D)_p, (\mathcal{O}_{\widehat{M|H},(2,1+)})_p) = 0.$$

$$(iv) \mathcal{X}((\mathcal{M}_D)_p, (\mathcal{O}_{\widehat{M|H},(2)})_p) = -1.$$

$$\mathcal{X}((\mathcal{M}_D)_p, (\mathcal{O}_{\widehat{M|H},2})_p) = 0.$$

$$(4) \mathcal{X}((\mathcal{M}_D)_p, (\mathcal{O}_{\widehat{M|H}})_p) = 0.$$

Corollary 2 . *The irregularity $\operatorname{Irr}((\mathcal{M}_D)_p) = 1$.*

The essential parts for $k = 1$ are as follow:

1. We have a formal solution

$$\hat{u}(x) = \sum_{n=1}^{\infty} \frac{(n-1)!(c-n) \cdots (c-2)}{(n-b_1) \cdots (1-b_1)} \Phi_D^{m-1}(b_2, \dots, b_m; c-n; x_2, \dots, x_m) (x_1)^{-n}$$

of the non-homogeneous system of partial differential equations

$$L_1 \hat{u}(x) = \frac{\Phi_D^{m-1}(b_2, \dots, b_m; c-1; x_2, \dots, x_m)}{x_1}, \quad L_l \hat{u}(x) = 0 \quad (l = 2, \dots, m),$$

where $\Phi_D^{m-1}(b_2, \dots, b_m; c-1; x_2, \dots, x_m)$ is the Humbert confluent hypergeometric function in $(m-1)$ variables with the parameter $(b_2, \dots, b_m; c-1)$,

$$\Phi_D^{m-1}(b_2, \dots, b_m; c-1; x_2, \dots, x_m) = \sum_{j_2=0}^{\infty} \cdots \sum_{j_m=0}^{\infty} \frac{(b_2)_{j_2} \cdots (b_m)_{j_m} (x_2)^{j_2} \cdots (x_m)^{j_m}}{(c)_{j_2+\cdots+j_m} j_2! \cdots j_m!},$$

where $(b)_s = (b+1) \cdots (b+s-1)$.

2. If, for

$$v = \begin{pmatrix} \sum_{j=0}^{\infty} P_j^1(y_1) x_1^{-j} \\ \vdots \\ \sum_{j=0}^{\infty} P_j^m(y_1) x_1^{-j} \end{pmatrix}$$

$\nabla_1^m v = 0$, we have $\nabla_0^m (\sum_{j=0}^{\infty} f_j(y_1) x_1^{-j}) = v$, then

$$\begin{aligned} & \frac{(1-b_1) \cdots (n+1-b_1)}{n!(c-n-1) \cdots (c-2)} f_{n+1} \\ &= \sum_{j=1}^{n+1} \frac{(1-b_1) \cdots (j-1-b_1)}{(j-1)!(c-j) \cdots (c-2)} P_j^1 + \sum_{l=2}^m x_l \frac{\partial}{\partial x_l} \sum_{j=1}^n \frac{(1-b_1) \cdots (j-b_1)}{(j-1)!(c-j) \cdots (c-2)} f_j. \end{aligned}$$

Put $F_{n+1} = \frac{(1-b_1) \cdots (n+1-b_1)}{n!(c-n-1) \cdots (c-2)} f_{n+1}$, then, for $l = 2, \dots, m$,

$$\frac{\partial}{\partial x_l} F_{n+1} = \frac{1}{c-n-1} (x_l \frac{\partial}{\partial x_l} F_n + b_l F_n + P_n^l + x_l \frac{\partial}{\partial x_l} P_{n+1}^1),$$

and $\alpha(v) = \lim_{n \rightarrow \infty} F_n$ is well-defined as constant for $v \in (\mathcal{O}_{\widehat{M|H,s,A}}(U))^m$, where $U \subset H_1$ and, $0 < s < 2$ or $(s = 2 \text{ and } 0 < A < 1)$.

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