

ON CERTAIN INTEGRAL TRANSFORMATIONS

SHIGEYOSHI OWA (近畿大・理工 尾和重義)

ABSTRACT

The object of the present paper is to derive some subordination properties of certain integral transformations of functions which are analytic in the open unit disk.

I. INTRODUCTION

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathbb{U} = \{z: |z| < 1\}$. For functions $f(z) \in \mathcal{A}$ and $g(z) \in \mathcal{A}$, we say that $f(z)$ is subordinate to $g(z)$ if there exists an analytic function $w(z)$ in $\mathbb{U}$ which satisfies $w(0) = 0$, $|w(z)| < 1$ ($z \in \mathbb{U}$), and $f(z) = g(w(z))$. We denote this subordination by $f(z) \prec g(z)$. If $g(z)$ is univalent in $\mathbb{U}$, then this subordination $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

For a function $f(z)$ belonging to $\mathcal{A}$, we define the following integral transformation $I(f(z))$ by

$$I(f(z)) = \left\{ \frac{\alpha+\beta}{z^\beta} \int_0^z t^{\beta-1} f(t)^\alpha dt \right\} \quad (z \in \mathbb{U}),$$

where $\alpha \in \mathbb{C}$, $\alpha \neq 0$, and $\beta \in \mathbb{C}$.

To derive some subordination properties of the integral transformations $I(f(z))$, we have to recall here the following lemmas.

LEMMA 1 ([1]). Let $f(z) \in A$, $g(z) \in A$, and $g(z)$ be univalent in $\mathbb{U} = \mathbb{U} \setminus \partial\mathbb{U}$. If there exists a point $z_0 \in \mathbb{U}$ such that $f(|z| < |z_0|) \subset g(\mathbb{U})$ and $f(z_0) = g(\zeta_0)$ for $\zeta_0 \in \partial\mathbb{U}$, then $z_0 f'(z_0) = m \zeta_0 g'(\zeta_0)$, where m is real and $m \geq 1$.

LEMMA 2 ([2]). Let $p(z) = 1 + p_1 z + p_2 z^2 + \dots$ be analytic in $\mathbb{U}$ with $p(z) \neq 1$. Let the function $\Psi(u, v): \mathbb{C}^2 \rightarrow \mathbb{C}$ ($u = u_1 + iu_2$, $v = v_1 + iv_2$) satisfy the following conditions

- (i) $\Psi(u, v)$ is continuous in $\mathbb{D} \subset \mathbb{C}^2$,
- (ii) $(1, 0) \in \mathbb{D}$ and $\operatorname{Re}(\Psi(1, 0)) > 0$,
- (iii) for all $(iu_2, v_1) \in \mathbb{D}$ such that $v_1 \leq -(1+u_2^2)/2$, $\operatorname{Re}(\Psi(iu_2, v_1)) \leq 0$.

If $(p(z), zp'(z)) \in \mathbb{D}$ for $z \in \mathbb{U}$ and $\operatorname{Re}(\Psi(p(z), zp'(z))) > 0$ for $z \in \mathbb{U}$, then $\operatorname{Re}(p(z)) > 0$ for all $z \in \mathbb{U}$.

A function $L(z, t)$ defined on $\mathbb{U} \times [0, \infty)$ is said to be a subordination chain (or Loewner chain) if it satisfies

- (i) $L(z, t)$ is analytic and univalent in $\mathbb{U}$ for all $t \geq 0$,
- (ii) $L(z, t)$ is continuously differentiable on $t \geq 0$ for all $z \in \mathbb{U}$,
- (iii) $L(z, s) \prec L(z, t)$ for $0 \leq s \leq t$.

LEMMA 3 ([4]). The function $L(z, t)$ given by

$$L(z, t) = a_1(t)z + a_2(t)z^2 + \dots \quad (a_1(t) \neq 0; t \geq 0)$$

is a subordination chain if and only if

$$\operatorname{Re} \left\{ \frac{z \frac{\partial L(z, t)}{\partial z}}{\frac{\partial L(z, t)}{\partial t}} \right\} > 0 \quad (z \in \mathbb{U}; t \geq 0).$$

Further, we need

LEMMA 4 ([3]). Let $\alpha \in \mathbb{C}$, $\alpha \neq 0$, $\beta \in \mathbb{C}$, and let the function

$$h(z) = c + h_1 z + h_2 z^2 + \dots$$

be analytic in $\mathbb{U}$. If the function $h(z)$ satisfies

$$\operatorname{Re}(\alpha h(z) + \beta) > 0 \quad (z \in \mathbb{U}),$$

then the solution of the Briot-Bouquet differential equation

$$q(z) + \frac{zq'(z)}{\alpha q(z) + \beta} = h(z) \quad (h(0) = q(0) = c)$$

is analytic in $\mathbb{U}$ and $\operatorname{Re}(\alpha q(z) + \beta) > 0$ ($z \in \mathbb{U}$).

2. MAIN THEOREM

We begin with the statement and the proof of the following main result.

THEOREM. Let $f(z) \in \mathcal{A}$ and $g(z) \in \mathcal{A}$. If

$$(i) \operatorname{Re}(\alpha + \beta) > 0,$$

$$(ii) g(z)/z \neq 0 \quad (z \in \mathbb{U}), \text{ and } I(g(z))/z \neq 0 \quad (z \in \mathbb{U}) \text{ for } \alpha \neq 1,$$

$$(iii) \phi(z) = (g(z)/z)^\alpha \text{ satisfies}$$

$$\operatorname{Re}\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta \quad (z \in \mathbb{U})$$

with

$$-1 < \delta \leq \frac{1 + |\alpha + \beta|^2 - \sqrt{(1 + |\alpha + \beta|^2)^2 - 4(\operatorname{Re}(\alpha + \beta))^2}}{4\operatorname{Re}(\alpha + \beta)},$$

then the subordination

$$\frac{f(z)}{z} \prec \frac{g(z)}{z}$$

implies

$$\frac{I(f(z))}{z} \prec \frac{I(g(z))}{z}.$$

PROOF. We define $F(z) = (I(f(z))/z)^\alpha$ and $G(z) = (I(g(z))/z)^\alpha$.

Without loss of generality, we may assume that $G(z)$ is analytic and univalent in $\overline{\mathbb{U}} = \mathbb{U} \cup \partial\mathbb{U}$. Otherwise, we consider $F(rz)/r$ and $G(rz)/r$ ($0 < r < 1$) instead of $F(z)$ and $G(z)$, respectively.

We first prove that if $q(z) = 1 + zG''(z)/G'(z)$, then $\operatorname{Re}(q(z)) > 0$ ($z \in \mathbb{U}$).

Since

$$I(g(z)) = \left\{ \frac{\alpha+\beta}{z^\beta} \int_0^z t^{\beta-1} g(t)^\alpha dt \right\}^{1/\alpha},$$

we have

$$\alpha \frac{z(I(g(z)))'}{I(g(z))} = -\beta + (\alpha+\beta) \frac{\phi(z)}{G(z)}.$$

Also we have

$$\alpha \frac{z(I(g(z)))'}{I(g(z))} = \alpha + \frac{zG'(z)}{G(z)}.$$

Thus

$$(\alpha+\beta)\phi(z) = (\alpha+\beta)G(z) + zG'(z).$$

Differentiating both sides the above, we see that

$$\begin{aligned} \beta z\phi'(z) &= zG'(z) \left\{ \alpha + \beta + 1 + \frac{zG''(z)}{G'(z)} \right\} \\ &= zG'(z)(q(z) + \alpha + \beta). \end{aligned}$$

Further, making the logarithmic differentiation of the above, we get

$$\begin{aligned} 1 + \frac{z\phi''(z)}{\phi'(z)} &= 1 + \frac{zG''(z)}{G'(z)} + \frac{zq'(z)}{q(z) + \alpha + \beta} \\ &= q(z) + \frac{zq'(z)}{q(z) + \alpha + \beta} \\ &\equiv h(z). \end{aligned}$$

Note that $q(0) = h(0) = 1$ and $2\delta \leq \operatorname{Re}(\alpha+\beta)$. Therefore,

$$\begin{aligned} \operatorname{Re}(h(z)+\alpha+\beta) &= \operatorname{Re} \left\{ 1 + \frac{z\phi''(z)}{\phi'(z)} + \alpha + \beta \right\} \\ &\geq -\delta + \operatorname{Re}(\alpha+\beta) \\ &\geq \frac{1}{2} \operatorname{Re}(\alpha+\beta) \\ &> 0. \end{aligned}$$

Using Lemma 4, we have that $q(z)$ is analytic in $\mathbb{U}$ and $\operatorname{Re}(q(z)+\alpha+\beta) > 0$ ($z \in \mathbb{U}$).

Let us define the function $\Psi(u,v)$ by

$$\Psi(u,v) = u + \frac{v}{u+\alpha+\beta} + \delta$$

with $u = u_1 + iu_2$ and $v = v_1 + iv_2$. Then $\Psi(u,v)$ satisfies

(i) $\Psi(u,v)$ is continuous in $\mathbb{D} = (\mathbb{C} - \{-\alpha-\beta\}) \times \mathbb{C} \subset \mathbb{C}^2$,

(ii) $(1,0) \in \mathbb{D}$ and $\operatorname{Re}(\Psi(1,0)) = 1 + \delta > 0$,

(iii) for all $(iu_2, v_1) \in \mathbb{D}$ such that $v_1 \leq -(1+u_2^2)/2$,

$$\begin{aligned} \operatorname{Re}(\Psi(iu_2, v_1)) &= \operatorname{Re}\left\{\frac{v_1}{iu_2 + \alpha + \beta}\right\} + \delta \\ &= \delta - \frac{v_1 \operatorname{Re}(\alpha+\beta)}{|\alpha+\beta|^2 + 2\operatorname{Im}(\alpha+\beta)u_2 + u_2^2} \\ &\leq \delta - \frac{(1+u_2^2)\operatorname{Re}(\alpha+\beta)}{2(|\alpha+\beta|^2 + 2\operatorname{Im}(\alpha+\beta)u_2 + u_2^2)} \\ &= \frac{(2\delta - \operatorname{Re}(\alpha+\beta))u_2^2 + 4\delta\operatorname{Im}(\alpha+\beta)u_2 + 2\delta|\alpha+\beta|^2 - \operatorname{Re}(\alpha+\beta)}{2(|\alpha+\beta|^2 + 2\operatorname{Im}(\alpha+\beta)u_2 + u_2^2)} \end{aligned}$$

Define $E_\delta(u_2)$ by

$$E_\delta(u_2) = (2\delta - \operatorname{Re}(\alpha+\beta))u_2^2 + 4\delta\operatorname{Im}(\alpha+\beta)u_2 + 2\delta|\alpha+\beta|^2 - \operatorname{Re}(\alpha+\beta).$$

Then, it is easy to see that $2\delta - \operatorname{Re}(\alpha+\beta) < 0$ and $2\delta|\alpha+\beta|^2 - \operatorname{Re}(\alpha+\beta) \leq 0$.

The discrimination Δ of $E_\delta(u_2)$ is

$$\begin{aligned} \Delta &= 4(\operatorname{Im}(\alpha+\beta))^2\delta^2 - (2\delta - \operatorname{Re}(\alpha+\beta))(2\delta|\alpha+\beta|^2 - \operatorname{Re}(\alpha+\beta)) \\ &= -4(\operatorname{Re}(\alpha+\beta))^2\delta^2 + 2\operatorname{Re}(\alpha+\beta)(1+|\alpha+\beta|^2)\delta - (\operatorname{Re}(\alpha+\beta))^2 \\ &\leq 0 \end{aligned}$$

because

$$-1 < \delta \leq \frac{1 + |\alpha+\beta|^2 - \sqrt{(1+|\alpha+\beta|^2)^2 - 4(\operatorname{Re}(\alpha+\beta))^2}}{4\operatorname{Re}(\alpha+\beta)}$$

This implies that $E_\delta(u_2) \leq 0$, that is, that $\operatorname{Re}(\Psi(iu_2, v_1)) \leq 0$.

Further, we see that

$$\begin{aligned}\operatorname{Re}(\Psi(q(z), zq'(z))) &= \operatorname{Re}\left\{q(z) + \frac{zq'(z)}{q(z)+\alpha+\beta} + \delta\right\} \\ &= \operatorname{Re}\left\{1 + \frac{z\phi''(z)}{\phi'(z)} + \delta\right\} \\ &> 0.\end{aligned}$$

Therefore, an application of Lemma 2 gives us that $\operatorname{Re}(q(z)) > 0$ ($z \in \mathbb{U}$).

Next, we prove that if $f(z)/z \prec g(z)/z$, then $F(z) \prec G(z)$.

Let us define the function $L(z, t)$ by

$$L(z, t) = G(z) + \frac{1+t}{\alpha+\beta} zG'(z) \quad (z \in \mathbb{U}; t \geq 0).$$

Noting that $G'(0) = 1$, we see that

$$\begin{aligned}\left. \frac{\partial L(z, t)}{\partial z} \right|_{z=0} &= G'(0) \left\{ 1 + \frac{1+t}{\alpha+\beta} \right\} \\ &= 1 + \frac{1+t}{\alpha+\beta} \\ &\neq 0.\end{aligned}$$

This implies that if

$$L(z, t) = a_1(t)z + a_2(t)z^2 + \dots \quad (z \in \mathbb{U}, t \geq 0),$$

then $a_1(t) \neq 0$ for all $t \geq 0$. Further, we know that

$$\begin{aligned}\operatorname{Re}\left\{ \frac{z \frac{\partial L(z, t)}{\partial z}}{\frac{\partial L(z, t)}{\partial t}} \right\} &= \operatorname{Re}\left\{ \alpha + \beta + (1+t) \left(1 + \frac{zG''(z)}{G'(z)} \right) \right\} \\ &= \operatorname{Re}(q(z) + \alpha + \beta) + t\operatorname{Re}(q(z)) \\ &> 0\end{aligned}$$

for all $z \in \mathbb{U}$. Therefore, it follows from Lemma 3 that $L(z, t)$ is the

subordination chain. Thus we have

$$\phi(z) = L(z, 0) \prec L(z, t) \quad (t \geq 0)$$

by the definition of the subordination chain.

Suppose that $F(z) \not\prec G(z)$. Then there exists a point $z_0 \in \mathbb{U}$ such that $F(|z| < |z_0|) \subset G(\mathbb{U})$ and $F(z_0) = G(\zeta_0)$ ($\zeta_0 \in \partial\mathbb{U}$). This means that $L(\zeta_0, t) \notin L(\mathbb{U}, t)$. Since, by Lemma 1,

$$z_0 F'(z_0) = (1+t) \zeta_0 G'(\zeta_0) \quad (t \geq 0),$$

we have

$$\begin{aligned} L(\zeta_0, t) &= G(\zeta_0) + \frac{1+t}{\alpha+\beta} \zeta_0 G'(\zeta_0) \\ &= F(z_0) + \frac{1}{\alpha+\beta} z_0 F'(z_0) \\ &= \left[\frac{f(z_0)}{z_0} \right]^\alpha, \end{aligned}$$

so $L(\zeta_0, t) \in \phi(\mathbb{U})$, for $f(z)/z \prec g(z)/z$. This contradicts that $L(\zeta_0, t) \notin L(\mathbb{U}, t)$. Thus we prove $F(z) \prec G(z)$.

Finally, note that

$$\frac{I(g(z))}{z} = 1 + c_1 z + c_2 z^2 + \dots \neq 0 \quad (z \in \mathbb{U})$$

for $\alpha \neq 1$. This proves that if $F(z) \prec G(z)$, then $I(f(z))/z \prec I(g(z))/z$. Thus we complete the proof of our main theorem.

Making $\alpha+\beta = 1$ in Theorem, we have

COROLLARY I. Let $f(z)$ and $g(z)$ be in the class $\mathcal{A}$. If

(i) $g(z)/z \neq 0$ ($z \in \mathbb{U}$), and $I(g(z))/z \neq 0$ ($z \in \mathbb{U}$) when $\alpha \neq 1$,

(ii) $\phi(z) = (g(z)/z)^\alpha$ satisfies

$$\operatorname{Re} \left\{ 1 + \frac{z \phi''(z)}{\phi'(z)} \right\} > -\frac{1}{2} \quad (z \in \mathbb{U}),$$

then $f(z)/z \prec g(z)/z$ implies $I(f(z))/z \prec I(g(z))/z$, where

$$I(f(z)) = \left\{ \frac{1}{z^{1-\alpha}} \int_0^z \left(\frac{f(t)}{t} \right)^\alpha dt \right\}^{1/\alpha}.$$

Letting $\alpha + \beta = 1 + i$ in our theorem,

COROLLARY 2. Let $f(z)$ and $g(z)$ be in the class A . If

(i) $g(z)/z \neq 0$ ($z \in \mathbb{U}$), and $I(g(z))/z \neq 0$ ($z \in \mathbb{U}$) when $\alpha \neq 1$,

(ii) $\phi(z) = (g(z)/z)^\alpha$ satisfies

$$\operatorname{Re} \left\{ 1 + \frac{z\phi''(z)}{\phi'(z)} \right\} > \frac{\sqrt{5} - 3}{4} \quad (z \in \mathbb{U}),$$

then $f(z)/z \prec g(z)/z$ implies $I(f(z))/z \prec I(g(z))/z$, where

$$I(f(z)) = \left\{ \frac{1+i}{z^{1-\alpha+i}} \int_0^z t^{i-\alpha} f(t)^\alpha dt \right\}^{1/\alpha}.$$

COROLLARY 3. If $f(z) \in A$ satisfies

$$\frac{f(z)}{z} \prec \frac{1}{(1-z)^\lambda},$$

then

$$\frac{I(f(z))}{z} \prec {}_2F_1(\alpha + \beta, \lambda\alpha; \alpha + \beta + 1; z)^{1/\alpha},$$

where $\alpha > 0$,

$$1 - \frac{1 + |\alpha + \beta|^2 - \sqrt{(1 + |\alpha + \beta|^2)^2 - 4(\operatorname{Re}(\alpha + \beta))^2}}{2\operatorname{Re}(\alpha + \beta)} \leq \lambda\alpha < 3,$$

and ${}_2F_1(a, b; c; z)$ means the hypergeometric function.

PROOF. Let $g(z) = z/(1-z)^\lambda$ in Theorem, then

$$\begin{aligned} I(g(z)) &= \left\{ \frac{\alpha + \beta}{z^\beta} \int_0^z t^{\beta-1} t^\alpha (1-t)^{-\lambda\alpha} dt \right\}^{1/\alpha} \\ &= \left\{ (\alpha + \beta) z^\alpha \int_0^1 u^{\alpha + \beta - 1} (1-zu)^{-\lambda\alpha} du \right\}^{1/\alpha} \end{aligned}$$

$$= z {}_2F_1(\alpha+\beta, \lambda\alpha; \alpha+\beta+1; z)^{1/\alpha}.$$

Therefore, we have

$$\frac{I(f(z))}{z} \prec {}_2F_1(\alpha+\beta, \lambda\alpha; \alpha+\beta+1; z)^{1/\alpha}.$$

Taking $\alpha+\beta = 1$ in Corollary 3, we have

EXAMPLE 1. If $f(z) \in A$ satisfies

$$\frac{f(z)}{z} \prec \frac{1}{(1-z)^\lambda},$$

then

$$\frac{I(f(z))}{z} \prec {}_2F_1(1, \lambda\alpha; 2; z)^{1/\alpha},$$

where $\alpha > 0$ and $0 \leq \lambda\alpha < 3$.

If we make $\alpha+\beta = 1+i$ in Corollary 3, then we have

EXAMPLE 2. If $f(z) \in A$ satisfies

$$\frac{f(z)}{z} \prec \frac{1}{(1-z)^\lambda},$$

then

$$\frac{I(f(z))}{z} \prec {}_2F_1(1+i, \lambda\alpha; 2+i; z)^{1/\alpha},$$

where $\alpha > 0$ and $(\sqrt{5}-1)/2 \leq \lambda\alpha < 3$.

ACKNOWLEDGMENTS

This work was supported, in part, by the Japanese Ministry of Education, Science and Culture under Grant-in-Aid for General Scientific Research.

REFERENCES

- [1] S. S. Miller and P. T. Mocanu, Differential subordinations and univalent functions, Michigan Math. J. 28(1981), 157 - 171.
- [2] S. S. Miller and P. T. Mocanu, Second order differential inequalities in the complex plane, J. Math. Anal. Appl. 65(1978), 289 - 305.
- [3] S. S. Miller and P. T. Mocanu, Univalent solutions of Briot-Bouquet differential equations, J. Differential Equations 56(1985), 297 - 309.
- [4] C. Pommerenke, Univalent Functions, Vandenhoeck and Ruprecht, Göttingen, 1975.

Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577
Japan