

1 次元ランダムシュレーディンガー作用素の 準位統計について

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Abstract

This note is based on the works [5], [7]. We study the level statistics of one-dimensional Schrödinger operator with random potential decaying like $x^{-\alpha}$ at infinity. We consider the point process ξ_L consisting of the rescaled eigenvalues and show that : (i)(ac spectrum case) for $\alpha > \frac{1}{2}$, ξ_L converges to a clock process, and the fluctuation of the eigenvalue spacing converges to Gaussian. (ii)(critical case) for $\alpha = \frac{1}{2}$, ξ_L converges to the limit of the β -ensemble.

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1 Introduction

1.1 Background

We consider the following Schrödinger operator

$$H := -\frac{d^2}{dt^2} + a(t)F(X_t) \quad \text{on } L^2(\mathbf{R})$$

where $a \in C^\infty$ is real valued, $a(-t) = a(t)$, non-increasing for $t \geq 0$, and satisfies

$$C_1 t^{-\alpha} \leq a(t) \leq C_2 t^{-\alpha}$$

for some positive constants C_1, C_2 and $\alpha > 0$. F is a real-valued, smooth, and non-constant function on a compact Riemannian manifold M such that

$$\langle F \rangle := \int_M F(x) dx = 0.$$

$\{X_t\}$ is a Brownian motion on M . Since the potential $a(t)F(X_t)$ is $-\frac{d^2}{dt^2}$ -compact, we have $\sigma_{\text{ess}}(H) = [0, \infty)$. Kotani-Ushiroya[3] proved that the

spectrum of H in $[0, \infty)$ is

- (1) for $\alpha < \frac{1}{2}$: pure point with exponentially decaying eigenfunctions,
- (2) for $\alpha = \frac{1}{2}$: pure point on $[0, E_c]$ and purely singular continuous on $[E_c, \infty)$ with some explicitly computable E_c ,
- (3) for $\alpha > \frac{1}{2}$: purely absolutely continuous.

In what follows we discuss the level statistics of this operator. For that purpose, let $H_L := H|_{[0, L]}$ be the local Hamiltonian with Dirichlet boundary condition and let $\{E_n(L)\}_{n=1}^\infty$ be its eigenvalues in the increasing order. Let $n(L) \in \mathbb{N}$ be s.t. $\{E_n(L)\}_{n \geq n(L)}$ coincides with the set of positive eigenvalues of H_L . We arbitrary take the reference energy $E_0 > 0$ and consider the following point process

$$\xi_L := \sum_{n \geq n(L)} \delta_{L(\sqrt{E_n(L)} - \sqrt{E_0})}$$

in order to study the local fluctuation of eigenvalues near E_0 . Our aim is to identify the limit of ξ_L as $L \rightarrow \infty$ ¹.

As for the related works, Killip-Stoiciu [2] studied the CMV matrices whose matrix elements decay like $n^{-\alpha}$. They showed that, ξ_L converges to (i) $\alpha > \frac{1}{2}$: the clock process, (ii) $\alpha = \frac{1}{2}$: the limit of the circular β -ensemble, (iii) $0 < \alpha < \frac{1}{2}$: the Poisson process. Krichevski-Valko-Virag[6] studied the one-dimensional discrete Schrödinger operator with the random potential decaying like $n^{-1/2}$, and proved that ξ_L converges to the Sine $_\beta$ -process.

Our aim is to do the analogue of their works for the one-dimensional Schrödinger operator in the continuum.

In subsection 1.2 (resp. subsection 1.3), we state our results for ac-case : $\alpha > \frac{1}{2}$ (resp. critical-case : $\alpha = \frac{1}{2}$)².

¹Here we consider the scaling of $\sqrt{E_n(L)}$'s instead of $E_n(L)$'s. This corresponds to the unfolding with respect to the density of states.

²We have not obtained results for pp-case : $\alpha < \frac{1}{2}$.

1.2 AC-case

Definition 1.1 Let μ be a probability measure on $[0, \pi)$. We say that ξ is the clock process with spacing π with respect to μ if and only if

$$\mathbf{E}[e^{-\xi(f)}] = \int_0^\pi d\mu(\phi) \exp \left(- \sum_{n \in \mathbf{Z}} f(n\pi - \phi) \right)$$

where $f \in C_c(\mathbf{R})$ and $\xi(f) := \int_{\mathbf{R}} f d\xi$.

We set

$$(x)_{\pi\mathbf{Z}} := x - [x]_{\pi\mathbf{Z}}, \quad [x]_{\pi\mathbf{Z}} := \max\{y \in \pi\mathbf{Z} \mid y \leq x\}.$$

We study the limit of ξ_L under the following assumption

(A)

(1) $\alpha > \frac{1}{2}$,

(2) A sequence $\{L_j\}_{j=1}^\infty$ satisfies $\lim_{j \rightarrow \infty} L_j = \infty$ and

$$(\sqrt{E_0}L_j)_{\pi\mathbf{Z}} = \beta + o(1), \quad j \rightarrow \infty$$

for some $\beta \in [0, \pi)$.

The condition A(2) ensures that ξ_L converges to a point process. If $a \equiv 0$ for instance, A(2) is indeed necessary.

Theorem 1.1 Assume (A). Then ξ_{L_j} converges in distribution to the clock process with spacing π with respect to a probability measure μ_β on $[0, \pi)$.

Remark 1.1 Let x_t be the solution to the eigenvalue equation : $H_L x_t = \kappa^2 x_t$ ($\kappa > 0$). If we set

$$\begin{pmatrix} x_t \\ x'_t/\kappa \end{pmatrix} = \begin{pmatrix} r_t \sin \theta_t \\ r_t \cos \theta_t \end{pmatrix}, \quad \theta_t(\kappa) = \kappa t + \tilde{\theta}_t(\kappa),$$

then $\tilde{\theta}_t(\kappa)$ has a limit as t goes to infinity[3] : $\lim_{t \rightarrow \infty} \tilde{\theta}_t(\kappa) = \tilde{\theta}_\infty(\kappa)$, a.s. ; μ_β is the distribution of the random variable $(\beta + \tilde{\theta}_\infty(\sqrt{E_0}))_{\pi\mathbf{Z}}$. In some special cases, we can show that $(\tilde{\theta}_\infty(\sqrt{E_0}))_{\pi\mathbf{Z}}$ is not uniformly distributed for large E_0 , implying that μ_β really depends on β .

Remark 1.2 We can consider point processes with respect to two reference energies E_0, E'_0 ($E_0 \neq E'_0$) simultaneously : suppose a sequence $\{L_j\}_{j=1}^\infty$ satisfies

$$(\sqrt{E_0}L_j)_{\pi\mathbf{Z}} = \beta + o(1), \quad (\sqrt{E'_0}L_j)_{\pi\mathbf{Z}} = \beta' + o(1), \quad j \rightarrow \infty$$

for some $\beta, \beta' \in [0, \pi)$. We set

$$\xi_L := \sum_{n \geq n(L)} \delta_{L(\sqrt{E_n(L)} - \sqrt{E_0})}, \quad \xi'_L := \sum_{n \geq n(L)} \delta_{L(\sqrt{E_n(L)} - \sqrt{E'_0})}.$$

Then the joint distribution of ξ_{L_j}, ξ'_{L_j} converges, for $f, g \in C_c(\mathbf{R})$,

$$\begin{aligned} & \lim_{j \rightarrow \infty} \mathbf{E} [\exp(-\xi_{L_j}(f) - \xi'_{L_j}(g))] \\ &= \int_0^\pi d\mu(\phi, \phi') \exp \left(- \sum_{n \in \mathbf{Z}} (f(n\pi - \phi) + g(n\pi - \phi')) \right) \end{aligned}$$

where $\mu(\phi, \phi')$ is the joint distribution of $(\beta + \tilde{\theta}_\infty(\sqrt{E_0}))_{\pi\mathbf{Z}}$ and $(\beta' + \tilde{\theta}_\infty(\sqrt{E'_0}))_{\pi\mathbf{Z}}$. We are unable to identify $\mu(\phi, \phi')$ but it may be possible that ϕ and ϕ' are correlated.

Remark 1.3 Suppose we rearrange eigenvalues near the reference energy E_0 so that

$$\dots < E'_{-2}(L) < E'_{-1}(L) < E_0 \leq E'_0(L) < E'_1(L) < E'_2(L) < \dots$$

Then an argument similar to the proof of Theorem 2.4 in [4] proves the following fact : for any $n \in \mathbf{Z}$ we have

$$\lim_{L \rightarrow \infty} L(\sqrt{E'_{n+1}(L)} - \sqrt{E'_n(L)}) = \pi, \quad \text{a.s.} \quad (1.1)$$

which is called the strong clock behavior [1]. We note that the integrated density of states is equal to $\sqrt{E}/\pi$.

We next study the finer structure of the eigenvalue spacing, under the following assumption.

(B)

- (1) $\frac{1}{2} < \alpha < 1$,
 (2) A sequence $\{L_j\}_{j=1}^{\infty}$ satisfies $\lim_{j \rightarrow \infty} L_j = \infty$ and

$$\sqrt{E_0}L_j = m_j\pi + \beta + \epsilon_j, \quad j \rightarrow \infty$$

for some $\{m_j\}_{j=1}^{\infty} (\subset \mathbf{N})$, $\beta \in [0, \pi)$ and $\{\epsilon_j\}_{j=1}^{\infty}$ with $\lim_{j \rightarrow \infty} \epsilon_j = 0$.

- (3) $a(t) = t^{-\alpha}(1 + o(1))$, $t \rightarrow \infty$.

Roughly speaking, $E_{m_j}(L_j)$ is the eigenvalue closest to E_0 . In view of (1.1), we set

$$X_j(n) := \left\{ \left(\sqrt{E_{m_j+n+1}(L_j)} - \sqrt{E_{m_j+n}(L_j)} \right) L_j - \pi \right\} L_j^{\alpha-\frac{1}{2}}, \quad n \in \mathbf{Z}.$$

Theorem 1.2 *Assume (B). Then $\{X_j(n)\}_{n \in \mathbf{Z}}$ converges in distribution to the Gaussian system with covariance*

$$C(n, n') = \frac{C(E_0)}{8E_0} \operatorname{Re} \int_0^1 s^{-2\alpha} e^{2i(n-n')\pi s} 2(1 - \cos 2\pi s) ds, \quad n, n' \in \mathbf{Z},$$

where $C(E) := \int_M \left| \nabla (L + 2i\sqrt{E})^{-1} F \right|^2 dx$ and L is the generator of (X_t) .

Remark 1.4 *By using the results in [2] we have*

$$\sqrt{E_{m_j}(L_j)} = \sqrt{E_0} - \frac{\beta + \tilde{\theta}_{\infty}(\sqrt{E_0})}{L_j} + Y_j$$

where $Y_j = O(L_j^{-\alpha-\frac{1}{2}+\epsilon}) + O(\epsilon_j L_j^{-1})$, a.s. for any $\epsilon > 0$. Furthermore by the definition of $\{X_j(n)\}$ we have

$$\sqrt{E_{m_j+n}(L_j)} = \begin{cases} \sqrt{E_{m_j}(L_j)} + \frac{n\pi}{L_j} + \frac{1}{L_j^{\alpha+\frac{1}{2}}} \sum_{l=0}^{n-1} X_j(l) & (n \geq 1) \\ \sqrt{E_{m_j}(L_j)} + \frac{n\pi}{L_j} - \frac{1}{L_j^{\alpha+\frac{1}{2}}} \sum_{l=n}^{-1} X_j(l) & (n \leq -1) \end{cases}$$

and Theorem 1.2 thus describes the behavior of eigenvalues near $E_{m_j}(L_j)$ in the second order.

Remark 1.5 *Suppose we consider two reference energies E_0, E'_0 ($E_0 \neq E'_0$) simultaneously and suppose a sequence $\{L_j\}_{j=1}^{\infty}$ satisfies $\lim_{j \rightarrow \infty} L_j = \infty$ and*

$$\sqrt{E_0}L_j = m_j\pi + \beta + o(1), \quad \sqrt{E'_0}L_j = m'_j\pi + \beta' + o(1), \quad j \rightarrow \infty$$

for some $m_j, m'_j \in \mathbf{N}$, and $\beta, \beta' \in [0, \pi)$. Then $\{X_j(n)\}_n$ and $\{X'_j(n)\}_n$ converge jointly to the mutually independent Gaussian systems.

1.3 Critical Case

We set the following assumption.

$$(C) \quad a(t) = t^{-\frac{1}{2}}(1 + o(1)), \quad t \rightarrow \infty.$$

Theorem 1.3 *Assume (C). Then*

$$\lim_{L \rightarrow \infty} \mathbf{E}[e^{-\xi_L(f)}] = \mathbf{E} \left[\int_0^{2\pi} \frac{d\theta}{2\pi} \exp \left(- \sum_{n \in \mathbf{Z}} f(\Psi_1^{-1}(2n\pi + \theta)) \right) \right]$$

where $\{\Psi_t(\cdot)\}_{t \geq 0}$ is the strictly-increasing function valued process such that for any $c_1, \dots, c_m \in \mathbf{R}$, $\{\Psi_t(c_j)\}_{j=1}^m$ is the unique solution of the following SDE :

$$\begin{aligned} d\Psi_t(c_j) &= 2c_j dt + D \operatorname{Re} \left\{ (e^{i\Psi_t(c_j)} - 1) \frac{dZ_t}{\sqrt{t}} \right\} \\ \Psi_0(c_j) &= 0, \quad j = 1, 2, \dots, m \end{aligned}$$

where $C(E_0) := \int_M |\nabla(L + 2i\sqrt{E_0})^{-1}F|^2 dx$, $D := \sqrt{\frac{C(E_0)}{2E_0}}$ and Z_t is a complex Brownian motion.

Definition 1.2 *For $\beta > 0$, the circular β -ensemble with n -points is given by*

$$\mathbf{E}_n^\beta[G] := \frac{1}{Z_{n,\beta}} \int_{-\pi}^{\pi} \frac{d\theta_1}{2\pi} \cdots \int_{-\pi}^{\pi} \frac{d\theta_n}{2\pi} G(\theta_1, \dots, \theta_n) |\Delta(e^{i\theta_1}, \dots, e^{i\theta_n})|^\beta$$

where $Z_{n,\beta}$ is the normalization constant, $G \in C(\mathbf{T}^n)$ is bounded and Δ is the Vandermonde determinant. The limit ξ_β of the circular β -ensemble is defined

$$\mathbf{E}[e^{-\xi_\beta(f)}] = \lim_{n \rightarrow \infty} \mathbf{E}_n^\beta \left[\exp \left(- \sum_{j=1}^n f(n\theta_j) \right) \right], \quad f \in C_c^+(\mathbf{R})$$

whose existence and characterization is given by [2]. The result in [2] together with Theorem 1.3 imply the limit of ξ_L coincides with that of the circular β -ensemble modulo a scaling.

Corollary 1.4 Assume (C). Writing $\xi_\beta = \sum_n \delta_{\lambda_n}$, let $\xi'_\beta := \sum_n \delta_{\lambda_n/2}$. Then $\xi_L \xrightarrow{d} \xi'_\beta$ with $\beta = \beta(E_0) := \frac{8E_0}{C(E_0)}$.

Remark 1.6 The corresponding $\beta = \beta(E_0) = \frac{8E_0}{C(E_0)}$ depends on the reference energy E_0 , so that the spacing distribution may change if we look at the different region in the spectrum. To see how β changes, we recall some results in [3]. Let $\sigma_F(\lambda)$ be the spectral measure of the generator L of $\{X_t\}$ with respect to F . Then

$$\gamma(E) := -\frac{1}{4E} \int_{-\infty}^0 \frac{\lambda}{\lambda^2 + 4E} d\sigma_F(\lambda), \quad E > 0$$

is the Lyapunov exponent in the sense that any generalized eigenfunction ψ_E of H satisfies

$$\lim_{|t| \rightarrow \infty} (\log t)^{-1} \log \{ \psi_E(t)^2 + \psi'_E(t)^2 \}^{1/2} = -\gamma(E), \quad a.s.$$

Moreover $E < E_c$ (resp. $E > E_c$) if and only if $\gamma(E) > \frac{1}{2}$ (resp. $\gamma(E) < \frac{1}{2}$) and $\gamma(E_c) = \frac{1}{2}$. Since $C(E) = 8E \cdot \gamma(E)$, we have $\beta(E) = \frac{1}{\gamma(E)}$. It then follows that $E < E_c$ (resp. $E > E_c$) if and only if $\beta(E) < 2$ (resp. $\beta(E) > 2$) and $\beta(E_c) = 2$. This is consistent with our general belief that in the point spectrum (resp. in the continuous spectrum) the level repulsion is weak (resp. strong). We also note that if $\beta = 2$, the circular β -ensemble with n -points coincides with the eigenvalue distribution of the unitary ensemble with the Haar measure on $U(n)$.

Remark 1.7 If we consider two reference energies E_0, E'_0 ($E_0 \neq E'_0$), then the corresponding point process ξ_L, ξ'_L converges jointly to the independent $\xi_\beta, \xi'_{\beta'}$.

Remark 1.8 We can also prove that ξ_L converges to the Sine_β -process [7], which is the bulk scaling limit of the Gaussian beta ensemble [8]. Together with Corollary 1.4, we have that the scaling limits of these two beta-ensembles coincide.

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