

A Lefschetz trace formula for p^n -torsion étale cohomology: a resume

By

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This is a resume of our results ([5]) on a Lefschetz trace formula on varieties defined over a finite field $\mathbb{F}_q$ of characteristic p . It is a p^n -torsion version of a conjecture of Deligne which was originally formulated with ℓ -adic étale cohomology ($\ell \neq p$) and has been proved by Fujiwara in full generality ([2]).

We introduce some notations to state our results. We fix an algebraic closure k of $\mathbb{F}_q$. For an object $\mathcal{X}_0$ (e.g. scheme, sheaf on a scheme, morphism of schemes) over $\mathbb{F}_q$, $\mathcal{X}$ denotes the base change of $\mathcal{X}_0$ by the injection $\mathbb{F}_q \hookrightarrow k$. Let S be a scheme. For a morphism of S -schemes $b: V \rightarrow U \times_S U$, we put $b_1 = \text{pr}_1 \circ b$ and $b_2 = \text{pr}_2 \circ b$, where pr_1 (resp. pr_2) is the first (resp. second) projection of $U \times_S U$. The S -scheme $\text{Fix}(b) = V \times_{U \times_S U} U$ is defined by the following cartesian diagram

$$\begin{array}{ccc} \text{Fix}(b) & \longrightarrow & U \\ \downarrow & & \downarrow \Delta_{U/S} \\ V & \xrightarrow{b} & U \times_S U, \end{array}$$

where $\Delta_{U/S}$ is the diagonal morphism. Remark that if U and V are smooth over S , $db_1: b_1^* \Omega_{U/S} \rightarrow \Omega_{V/S}$ is zero and b_2 is étale, then $\text{Fix } b$ is étale over S ([7, Cor. 17.13.6]). For an S -endomorphism $f: U \rightarrow U$, we put $\text{Fix } f = \text{Fix}(f \times_S \text{id}_U)$. Let U_0 and V_0 be $\mathbb{F}_q$ -schemes and $b_0: V_0 \rightarrow U_0 \times_{\mathbb{F}_q} U_0$ an $\mathbb{F}_q$ -morphism of schemes. We put $b^{(m)} = (\text{Fr}_U^m \circ b_1, b_2)$, where Fr_U is the relative q -th power Frobenius morphism of U i.e. the base change of the absolute q -th power Frobenius morphism of U_0 by $\mathbb{F}_q \rightarrow k$.

First, we state a p -torsion version of Fujiwara's trace formula.

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Theorem 1 ([5, Corollary 3.2], [5, Theorem 6.1]). *Let U_0 and V_0 be separated $\mathbb{F}_q$ -schemes of finite type and $\mathcal{F}_0$ a constructible étale $\mathbb{Z}/p$ -sheaf on U_0 .*

- (1) *Let $f_0: U_0 \rightarrow U_0$ be an automorphism of finite order and $m \geq 1$ an integer. Let $u_0: (\mathrm{Fr}_{U_0}^m \circ f_0)^* \mathcal{F}_0 \rightarrow \mathcal{F}_0$ be an isomorphism of sheaves whose order is the same as that of f_0 . Then we have the following equality*

$$\sum_i (-1)^i \mathrm{Tr}(u_0 \circ (\mathrm{Fr}_U^m \circ f)^* \mid H_c^i(U, \mathcal{F})) = \sum_{P \in \mathrm{Fix}(\mathrm{Fr}_U^m \circ f)} \mathrm{Tr}(u_P \mid \mathcal{F}_P).$$

- (2) *Let $b_0: V_0 \rightarrow U_0 \times_{\mathbb{F}_q} U_0$ be a morphism of $\mathbb{F}_q$ -schemes. We assume that $\mathcal{F}_0$ is smooth. Further we assume that there exist proper smooth $\mathbb{F}_q$ -schemes X_0 and Y_0 , and an $\mathbb{F}_q$ -morphism $a_0: Y_0 \rightarrow X_0 \times_{\mathbb{F}_q} X_0$ such that*

- (a) *U_0 (resp. V_0) is an open $\mathbb{F}_q$ -subscheme of X_0 (resp. Y_0), the diagram*

$$\begin{array}{ccc} V_0 & \xrightarrow{b_0} & U_0 \times_{\mathbb{F}_q} U_0 \\ \downarrow & & \downarrow \\ Y_0 & \xrightarrow{a_0} & X_0 \times_{\mathbb{F}_q} X_0 \end{array}$$

is cartesian,

- (b) *b_1 is proper, a_2 is étale, a is a closed immersion,*
(c) *$X \setminus U$ is a Cartier divisor, and*
(d) *there exists a smooth constructible étale $\mathbb{Z}/p$ -sheaf $\mathcal{G}_0$ on X_0 such that $\mathcal{G}_0|_{U_0} = \mathcal{F}_0$.*

Then, for any integer $m \geq 1$ and any $u_0 \in \mathrm{Hom}(b_{01}^{(m)} \mathcal{F}_0, b_{02}^* \mathcal{F}_0)$, we have the following equality*

$$\sum_i (-1)^i \mathrm{Tr}(u_1 \mid H_c^i(U, \mathcal{F})) = \sum_{P \in \mathrm{Fix}(b^{(m)})} \mathrm{Tr}(u_P \mid \mathcal{F}_P),$$

where u_1 is the composition

$$H_c^i(U, \mathcal{F}) \xrightarrow{b_1^{(m)*}} H_c^i(V, b_1^{(m)*} \mathcal{F}) \xrightarrow{u} H_c^i(V, b_2^* \mathcal{F}) \xrightarrow{b_{21}} H_c^i(U, \mathcal{F}).$$

Remark that $\mathrm{Fix}(\mathrm{Fr}_U^m \circ f)$ is finite over k by Zink's lemma [6, Lemma 2.3] and $\mathrm{Fix}(b^{(m)})$ is finite étale over k since U and V are smooth over k , the differential of $b_1^{(m)}$ is zero and b_2 is étale.

Theorem 1 (1) is proved by using the Lefschetz trace formula for the Frobenius correspondence ([8, Fonct. L mod. ℓ Théorème 4.1]) and Deligne-Lusztig's method ([1,

Section 3]). We sketch the proof of Theorem 1 (2). This is a generalization of the proof of [8, Fonct. L mod. ℓ^n , Théorème 4.1]. We put $\mathcal{G}'_0 = \mathcal{I}_0(\mathcal{G}_0 \otimes \mathcal{O}_{X_0})$, where $\mathcal{I}_0$ is the ideal sheaf of definition of $X_0 \setminus U_0$. Since $X \setminus U$ is a Cartier divisor, $\mathcal{G}'$ is a locally free $\mathcal{O}_X$ -module of finite rank and sits in the exact sequence

$$0 \longrightarrow j_{!}\mathcal{F} \longrightarrow \mathcal{G}' \xrightarrow{1-\Phi} \mathcal{G}' \longrightarrow 0,$$

where $\Phi: \mathcal{G}' \rightarrow \mathcal{G}'$ is the morphism induced by the p -th power map on $\mathcal{O}_X$. Then we can reduce the calculation of the trace of the endomorphism of the cohomology group of $\mathcal{F}$ to that of $\mathcal{G}'$. By applying the following trace formula to the trace, we obtain Theorem 1 (2).

Theorem 2 (Woods Hole formula, [5, Theorem 4.1]). *Let S be the spectrum of an artinian local ring, X and Y proper smooth schemes over S , and $a: Y \hookrightarrow X \times_S X$ a closed immersion over S . We assume that a_2 is étale and the homomorphism $da_1: a_1^*\Omega_{X/S} \rightarrow \Omega_{Y/S}$ is zero. Then, for any perfect complex $\mathcal{K}$ of $\mathcal{O}_X$ -modules and any $u \in \text{Hom}(a_1^*\mathcal{K}, a_2^*\mathcal{K})$, we have*

$$\sum_i (-1)^i \text{Tr}(u_* \mid H^i(X, \mathcal{K})) = \sum_{\beta \in \pi_0(\text{Fix}(a))} \text{Tr}_{\beta/S}(\text{Tr}(u_\beta \mid \mathcal{K}_\beta)),$$

where u_* is the composition

$$H^i(X, \mathcal{K}) \xrightarrow{a_1^*} H^i(Y, a_1^*\mathcal{K}) \xrightarrow{u} H^i(Y, a_2^*\mathcal{K}) \xrightarrow{a_{2*}} H^i(X, \mathcal{K}),$$

$\pi_0(\text{Fix}(a))$ is the set of connected components of $\text{Fix}(a)$, $\mathcal{K}_\beta$ (resp. u_β) is the pull-back of $\mathcal{K}$ (resp. u) by the immersion $i_\beta: \beta \hookrightarrow Y$ and $\text{Tr}_{\beta/S}$ is the trace map $\Gamma(\beta, \mathcal{O}_\beta) \rightarrow \Gamma(S, \mathcal{O}_S)$.

Remark that $\text{Fix}(a)$ is finite étale over S . Theorem 2 is a generalization of [9, Exp. III, Corollaire 6.12], and can be proved by using the Lefschetz-Verdier trace formula ([9, Exp. III, Théorème. 6.10]) and properties of residue symbols in [3, Ch. III, §9].

Secondly, we state a p^n -torsion version of Fujiwara's trace formula. At present, this requires more assumptions than Theorem 1.

For a perfect field K of characteristic p , we denote by $W_n(K)$ the ring of Witt vectors of K of length n . We write σ_0 for the Frobenius automorphism of $W_n(\mathbb{F}_q)$. For a scheme S , we write $\mathcal{O}_S$ for the structure sheaf of S . If S is of characteristic p , denote by $\Phi_{\mathcal{O}_S}$ the p -th power map on $\mathcal{O}_S$.

Theorem 3 ([5, Theorem 7.1]). *Let U_0 and V_0 be smooth $\mathbb{F}_q$ -schemes, $b_0: V_0 \rightarrow U_0 \times_{\mathbb{F}_q} U_0$ an $\mathbb{F}_q$ -morphism, and $\mathcal{F}_0$ a locally free constructible étale $\mathbb{Z}/p^n$ -sheaf on U_0 . We assume that there exist proper smooth $W_n(\mathbb{F}_q)$ -schemes $\mathcal{X}_0$ and $\mathcal{Y}_0$, a Cartier divisor $\mathcal{D}_0$ of $\mathcal{X}_0$ which is flat over $W_n(\mathbb{F}_q)$, a $W_n(\mathbb{F}_q)$ -morphism $\tilde{a}_0: \mathcal{Y}_0 \rightarrow \mathcal{X}_0 \times_{W_n(\mathbb{F}_q)} \mathcal{X}_0$, and a morphism $\Phi_{\mathcal{O}_{\mathcal{X}_0}}: \mathcal{O}_{\mathcal{X}_0} \rightarrow \mathcal{O}_{\mathcal{X}_0}$ such that*

- (a) when we put $X_0 = \mathcal{X}_0 \times_{W_n(\mathbb{F}_q)} \mathbb{F}_q$, $Y_0 = \mathcal{Y}_0 \times_{W_n(\mathbb{F}_q)} \mathbb{F}_q$ and define a_0 such that the diagram

$$\begin{array}{ccc} \mathcal{Y}_0 & \xrightarrow{\tilde{a}_0} & \mathcal{X}_0 \times_{W_n(\mathbb{F}_q)} \mathcal{X}_0 \\ \uparrow & & \uparrow \\ Y_0 & \xrightarrow{a_0} & X_0 \times_{\mathbb{F}_q} X_0 \end{array}$$

is cartesian, then $(U_0, V_0, b_0, X_0, Y_0, a_0)$ satisfies the condition (a) in Theorem 1 (2),

- (b) b_1 is proper, $\tilde{a}_2$ is étale, $\tilde{a}$ is a closed immersion,
(c) $\mathcal{D}_0$ is a lift of $X_0 \setminus U_0$ to $W_n(\mathbb{F}_q)$,
(d) the diagrams

$$\begin{array}{ccc} W_n(\mathbb{F}_q) & \xrightarrow{\sigma_0} & W_n(\mathbb{F}_q) \\ \downarrow & & \downarrow \\ \mathcal{O}_{\mathcal{X}_0} & \xrightarrow{\Phi_{\mathcal{O}_{\mathcal{X}_0}}} & \mathcal{O}_{\mathcal{X}_0} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{O}_{\mathcal{X}_0} & \xrightarrow{\Phi_{\mathcal{O}_{\mathcal{X}_0}}} & \mathcal{O}_{\mathcal{X}_0} \\ \downarrow & & \downarrow \\ \mathcal{O}_{X_0} & \xrightarrow{\Phi_{\mathcal{O}_{X_0}}} & \mathcal{O}_{X_0} \end{array}$$

commute,

- (e) the inclusion $\Phi_{\mathcal{O}_{\mathcal{X}_0}}(\mathcal{I}_0) \subset \mathcal{I}_0$ holds, where $\mathcal{I}_0$ is the defining ideal of $\mathcal{D}_0$,
(f) there exists a locally free constructible étale $\mathbb{Z}/p^n$ -sheaf $\mathcal{G}_0$ on X_0 such that $\mathcal{G}_0|_{U_0} = \mathcal{F}_0$, and
(g) $H_c^i(U, \mathcal{F})$ (resp. $H^i(\mathcal{X}, \mathcal{G} \otimes_{\mathbb{Z}/p^n} \mathcal{I})$) is free over $\mathbb{Z}/p^n$ (resp. $W_n(k)$) for any i .

Then there exists an integer M such that, for any integer $m \geq M$ and any homomorphism $u_0 \in \text{Hom}(b_{01}^{(m)*} \mathcal{G}_0, b_{02}^* \mathcal{G}_0)$, we have the following equality

$$\sum_i (-1)^i \text{Tr}(u_i \mid H_c^i(U, \mathcal{F})) = \sum_{P \in \text{Fix}(b^{(m)})} \text{Tr}(u_P \mid \mathcal{F}_P).$$

We note that the integer M in Theorem 3 depends on the sheaf $\mathcal{F}_0$. We need the assumption on existence of $\mathcal{X}_0, \mathcal{Y}_0, \mathcal{D}_0, \tilde{a}_0$ and $\Phi_{\mathcal{O}_{\mathcal{X}_0}}$ in order to use the same argument used in the proof of Theorem 1 (2), and the assumption (g) in order to compute the trace in the category of $\mathbb{Z}/p^n$ -modules, not in that of perfect complexes of $\mathbb{Z}/p^n$ -modules. If X_0 is a curve, then the assumption (e) automatically holds ([4, Lemma 1.1.2]).

The proof of Theorem 3 is similar to that of Theorem 1 (2).

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