

# Margulis Decomposition and Translation Lengths of Discrete Möbius Groups

Katsumi Inoue

井上 克巳 (金沢大学医短)

## 1 Introduction

For any integer  $n \geq 2$ , let  $R^n$  denote the  $n$ -dimensional Euclidean space and  $\overline{R^n} = R^n \cup \{\infty\}$  its one-point compactification. Any point  $x \in R^n$  is represented as  $x = (x_1, \dots, x_n)$  and when matrices act on  $x$ ,  $x$  is treated as a column vector. The subspace  $H^n = \{x \in R^n | x_n > 0\}$  of  $R^n$  with metric  $\rho(\cdot, \cdot)$  induced by the line element  $ds^2 = |dx|^2/dx_n^2$  is a model of the  $n$ -dimensional hyperbolic space and we call  $H^n$  the  $n$ -dimensional upper half-space.

A Möbius transformation of  $\overline{R^n}$  is a finite product of reflections in  $(n-1)$ -dimensional spheres or hypersurfaces. A group of Möbius transformation of  $\overline{R^n}$  is denoted by  $M(\overline{R^n})$  and call the (full) Möbius group. Möbius transformations are classified by their conjugacy class in  $M(\overline{R^n})$ . The canonical forms are as follows. An element in  $M(\overline{R^n})$  is said to be loxodromic if it is conjugate to a transformation of the form

$$\gamma(x) = \lambda Tx$$

where  $\lambda > 0, \lambda \neq 1$ , and  $T \in O(n)$ , the group of  $n \times n$ -orthogonal matrices, and parabolic if it is conjugate to the transformation of the form

$$\gamma(x) = Tx + a$$

where  $T \in O(n), a \in R^n$  and  $Ta = a \neq 0$ . A non-trivial element is said to be elliptic if it is neither loxodromic nor parabolic.

For  $\gamma \in M(\overline{R^n})$  we denote the Jacobian matrix of  $\gamma$  at  $x \in R^n$  by  $\gamma'(x)$ . Then chain rule implies that  $\gamma'(x) = \nu Ux$  with  $\nu > 0, U \in O(n)$ . We call the positive number  $\nu$  the linear magnification of  $\gamma$  at  $x$  and denote by  $|\gamma'(x)|$ . If  $\gamma \in M(\overline{R^n})$  does not fix  $\infty$ , the set  $I(\gamma) = \{x \in R^n | |\gamma'(x)| = 1\}$  becomes an  $(n-1)$ -sphere centered at  $\gamma^{-1}(\infty)$ . We call  $I(\gamma)$  the isometric sphere of  $\gamma$ . The action of  $\gamma$  on  $\overline{R^n}$  is the composition of an inversion in  $I(\gamma)$ , followed by a Euclidean isometry. For  $x \in \overline{R^n}$  denote  $x^*$  by the image of the reflection of  $x$  in the unit sphere centered at the origin. Let  $\gamma \in M(\overline{R^n})$  be an arbitrary element which does not fix  $\infty$ . Then  $\gamma$  can be represented uniquely in the form

$$\gamma(x) = \lambda T(x - a)^* + b$$

where  $\lambda > 0, T \in O(n)$  and  $a, b \in R^n$ . In this expression  $\lambda^{1/2}$  is the radius of  $I(\gamma)$  and  $a = \gamma^{-1}(\infty)$  ( resp.  $b = \gamma(\infty)$ ) is the center of  $I(\gamma)$  ( resp.  $I(\gamma^{-1})$  ). If  $\gamma \in M(\overline{R^n})$  fixes  $\infty$ , then  $\gamma$  can be written as a similarity in the form

$$\gamma(x) = \lambda T x + a$$

where  $\lambda > 0, T \in O(n)$  and  $a \in R^n$ .

Let denote by  $M(H^n)$  the subgroup of  $M(\overline{R^n})$  consisting of elements which keep  $H^n$  invariant. Then  $M(H^n)$  is the full group of hyperbolic isometries of  $H^n$ . For any subgroup  $\Gamma$  of  $M(H^n)$ ,  $\Gamma$  is discrete if and only if  $\Gamma$  acts discontinuously on  $H^n$ . Also  $\Gamma$  acts on  $\partial H^n = \overline{R^{n-1}}$  as a group of conformal automorphisms. For a discrete subgroup  $\Gamma$  of  $M(H^n)$ , the region of discontinuity  $\Omega(\Gamma)$  of  $\Gamma$  is the subset of  $\overline{R^{n-1}}$  on which  $\Gamma$  acts discontinuously. The limit set  $\Lambda(\Gamma)$  of  $\Gamma$  is the complement of  $\Omega(\Gamma)$  in  $\overline{R^{n-1}}$ . A discrete subgroup  $\Gamma$  of  $M(H^n)$  whose limit set consists of at most two points is called elementary. If  $\Gamma$  is not elementary,  $\Lambda(\Gamma)$  is a perfect, uncountable set.

Let  $\gamma$  be a loxodromic transformation. Then  $\gamma$  has exactly two fixed points on  $\overline{R^{n-1}}$ . The geodesic  $A_\gamma$  joining these two points is called the axis of  $\gamma$ . The axis  $A_\gamma$  is kept invariant under the action of  $\gamma$ . For a loxodromic transformation  $\gamma \in M(H^n)$  we set

$$l_\gamma = \inf_{x \in H^n} \rho(x, \gamma(x)).$$

We know that  $l_\gamma$  is positive and attained at any point of  $A_\gamma$ . This constant  $l_\gamma$  is called the translation length of  $\gamma$ . We denote  $L(\Gamma)$  by the set of translation lengths of all loxodromic transformations of  $\Gamma$ .

For a discrete subgroup  $\Gamma$  of  $M(H^n)$ , let  $E_\Gamma$  be the set of all geodesics in  $H^n$  whose end points belong to  $\Lambda(\Gamma)$ . The convex hull  $Hull(\Lambda(\Gamma))$  is the intersection of all hyperbolically convex sets in  $H^n$  which contain  $E_\Gamma$ . Let  $N_\Gamma = H^n/\Gamma$  be a quotient orbifold for  $\Gamma$  and  $M_\Gamma = (H^n \cup \Omega(\Gamma))/\Gamma$  its closure. The quotient  $C_\Gamma = Hull(\Lambda(\Gamma))/\Gamma$  is a subset of  $N_\Gamma$  and is called the Nielsen convex core for  $\Gamma$ .

## 2 The Margulis decomposition for quotient orbifolds

For a discrete subgroup  $\Gamma$  of  $M(H^n)$ , let  $\tilde{\Gamma}$  be the subset of  $\Gamma$  consisting of all elements of infinite orders. For  $\epsilon > 0$  and  $x \in H^n$ , we define

$$I_\epsilon(x) = \{\gamma \in \tilde{\Gamma} \mid \rho(x, \gamma(x)) < \epsilon\}$$

and

$$\Gamma_\epsilon(x) = \langle \Gamma \cap I_\epsilon(x) \rangle.$$

For our argument, the following result is essential ( see [1], [3] ).

PROPOSITION 1.( MARGULIS LEMMA ) *For each  $n$ , there exists a positive number  $\epsilon(n)$  such that for any discrete subgroup  $\Gamma$  of  $M(H^n)$ ,  $x \in H^n$  and  $\epsilon \leq \epsilon(n)$ ,  $\Gamma_\epsilon(x)$  is a finite extension of an abelian group.*

We call  $\epsilon(n)$  the Margulis constant in dimension  $n$ .

For any  $\epsilon \in (0, \epsilon(n)]$  and a discrete subgroup  $\Gamma$  of  $M(H^n)$ , we write

$$R_\epsilon(\Gamma) = \{x \in H^n \mid \rho(x, \gamma(x)) < \epsilon \text{ for some } \gamma \in \tilde{\Gamma}\}.$$

We can easily see that  $R_\epsilon(\Gamma)$  is a  $\Gamma$ -invariant set of  $H^n$ . The quotient  $R_\epsilon(\Gamma)/\Gamma \subset N_\Gamma$  is called the thin part of  $N_\Gamma$  and is denoted by  $N_{(0,\epsilon)}$ . The complement of  $N_{(0,\epsilon)}$  in  $N_\Gamma$  is denoted by  $N_{[\epsilon,\infty)}$  and is called the thick part of  $N_\Gamma$ . The decomposition

$$N_\Gamma = N_{(0,\epsilon)} \cup N_{[\epsilon,\infty)}$$

is called the Margulis decomposition for  $N_\Gamma$ .

A discrete subgroup  $\Gamma$  of  $M(H^n)$  is said to be geometrically finite if there exists  $\epsilon \in (0, \epsilon(n)]$  so that  $C_\Gamma \cap N_{[\epsilon,\infty)}$  is compact.

Let  $\Gamma'$  be a subgroup of  $\Gamma$ . A set  $X \subset H^n$  is precisely invariant under  $\Gamma'$  in  $\Gamma$  if  $\gamma(X) = X$  for any  $\gamma \in \Gamma'$  and  $\gamma(X) \cap X = \emptyset$  for any  $\gamma \in \Gamma - \Gamma'$ . Let  $\Lambda_P(\Gamma)$  denote the set of parabolic fixed points of  $\Gamma$ . For  $p \in \Lambda_P(\Gamma)$ , we write  $\Gamma_p = \{\gamma \in \Gamma \mid \gamma(p) = p\}$  and call the stabilizer of  $p$ .

The following is an immediate consequence of Margulis lemma.

PROPOSITION 2. ( [2], [3] ) *Let  $\Gamma$  be a discrete subgroup of  $M(H^n)$ . Then there exists a constant  $\epsilon \in (0, \epsilon(n)]$  so that the following holds:*

- (1) *For any  $p \in \Lambda_P(\Gamma)$  there exists an open region  $T_p$  in  $H^n$  which contains a component of  $R_\epsilon(\Gamma)$  so that  $T_p$  is precisely invariant under  $\Gamma_p$  in  $\Gamma$ .*
- (2) *For any distinct points  $p, q \in \Lambda_P(\Gamma)$ ,  $T_p$  and  $T_q$  are mutually disjoint to each other.*

We say that  $T = \bigcup_{p \in \Lambda_P(\Gamma)} T_p$  is a strictly invariant system of parabolic neighborhoods for  $\Gamma$ .

A parabolic fixed point  $p$  of  $\Gamma$  is called a bounded parabolic fixed point if there exists a compact subset of  $\overline{R^{n-1}} - \{p\}$  whose translates by  $\Gamma_p$  cover  $\Lambda(\Gamma) - \{p\}$ . We say that a limit point  $y$  of  $\Gamma$  is a conical limit point of  $\Gamma$  if for some geodesic ray  $I$  in  $H^n$  ending at  $y$ , there is a compact set  $K$  in  $H^n$  so that  $\{\gamma \in \Gamma \mid \gamma(I) \cap K \neq \emptyset\}$  is an infinite set.

The following is well known.

PROPOSITION 3. ([3],[4]) *Let  $\Gamma$  be a discrete subgroup of  $M(H^n)$ . Then the following statements are equivalent.*

- (1)  $\Gamma$  is geometrically finite.
- (2)  $\Lambda(\Gamma)$  consists of conical limit points or bounded parabolic fixed points.
- (3) There exist  $p_1, \dots, p_r \in \Lambda_P(\Gamma)$  with respective horoball neighborhoods  $B_1, \dots, B_r$  such that the set  $B = \bigcup_{\gamma \in \Gamma} \gamma(B_1 \cup \dots \cup B_r)$  forms a strictly invariant system of parabolic neighborhoods for  $\Gamma$  and  $(\text{Hull}(\Lambda(\Gamma)) - B)/\Gamma$  is compact.

### 3 Translation lengths of discrete Möbius groups

Let  $\Gamma$  be a discrete subgroup of  $M(H^n)$  and  $\epsilon \in (0, \epsilon(n)]$ , be chosen. We define

$$N_{\epsilon,1} = (R_\epsilon(\Gamma) \cap T)/\Gamma,$$

$$N_{\epsilon,2} = N_{(0,\epsilon)} - N_{\epsilon,1}$$

and call  $N_{\epsilon,1}$  ( resp.  $N_{\epsilon,2}$  ) the parabolic part ( resp. the non-parabolic part ) of  $N_{(0,\epsilon)}$ . If  $\Gamma$  is a discrete subgroup of  $M(H^3)$  consisting of orientation-preserving transformations (i.e  $\Gamma$  is a Kleinian group ), then each component of  $N_{(0,\epsilon)}$  is homeomorphic to either  $\{D - \{0\}\} \times S^1$ ,  $\{D - \{0\}\} \times (0, 1)$  or  $D \times S^1$ , where  $D$  is a unit disk.

To investigate the structure of  $N_{(0,\epsilon)}$ , we consider  $L(\Gamma)$ , the set of translation lengths of loxodromic elements of  $\Gamma$ . First we deal with the geometrically finite case.

LEMMA 4. *Let  $\Gamma$  be a geometrically finite subgroup of  $M(H^n)$ . Then  $L(\Gamma)$  is a discrete subset of  $[0, \infty)$ .*

PROOF. Assume the contrary. Then there exist a sequence  $\{\gamma_m\}$  of distinct loxodromic elements of  $\Gamma$  and a constant  $\alpha \geq 0$  such that  $l_m \rightarrow \alpha$  ( $m \rightarrow \infty$ ), where  $l_m$  is a translation length of  $\gamma_m$ .

Let denote by  $D_a$  a Dirichlet region for  $\Gamma$  centered at  $a \in H^n$ , with  $\Gamma_a = \{id\}$ . For any  $m$ , choose a point  $x_m \in A_m$ , the axis of  $\gamma_m$ . Then, for every  $m$ , there exists  $g_m \in \Gamma$  such that  $g_m(x_m) = y_m \in cl(D_a) \cap H^n$ , where  $cl(D_a)$  is the closure of  $D_a$ .

Suppose that  $\{y_m\}$  has an accumulation point  $y_0 \in cl(D_a) \cap H^n$ . Then there exist a subsequence of  $\{\gamma_m\}$  ( use the same notation ) and  $\delta > 0$  so that  $\{x \in H^n | \rho(y_0, x) < \delta\} \cap \tilde{A}_m \neq \emptyset$  for every  $m$ , where  $\tilde{A}_m$  is the axis of  $g_m \circ \gamma_m \circ g_m^{-1}$ . It follows that there exists a positive integer  $m_0$  with  $(g_m \circ \gamma_m \circ g_m^{-1})(y_0) \in \{x \in H^n | \rho(y_0, x) < \delta + 2\alpha\} \subset H^n$  for  $m \geq m_0$ . Then there exist a subsequence of  $\{\gamma_m\}$  ( again use the same notation ) and a point  $y \in \{x \in H^n | \rho(y_0, x) \leq \delta + 2\alpha\}$  such that  $(g_m \circ \gamma_m \circ g_m^{-1})(y_0) \rightarrow y$  ( $m \rightarrow \infty$ ). This

means  $y \in H^n \cap \Lambda(\Gamma) \neq \emptyset$ . It is a contradiction. So there exist a subsequence of  $\{y_m\}$  (use the same notation) and a point  $p \in \partial D_a \cap \overline{R^{n-1}}$  such that  $y_m \rightarrow p$  ( $m \rightarrow \infty$ ).

It is well known that conical limit points can not be contained in the boundary of any Dirichlet region. Since  $\Gamma$  is geometrically finite, we conclude that  $p$  is a bounded parabolic fixed point and there exists a horoball neighborhood  $B_p$  which is precisely invariant under  $\Gamma_p$  in  $\Gamma$ .

Note that  $y_m \in \tilde{A}_m$  and the translation length is invariant under the conjugation in  $M(H^n)$ . So there exists a positive integer  $m_1$  such that  $\{x \in |\rho(x, y_{m_1})| < 2\alpha\} \subset B_p$ . Hence we deduce that  $(g_{m_1} \circ \gamma_{m_1} \circ g_{m_1}^{-1})(y_{m_1}) \in (g_{m_1} \circ \gamma_{m_1} \circ g_{m_1}^{-1})(B_p) \cap B_p \neq \emptyset$ . It contradicts the fact that  $B_p$  is precisely invariant under  $\Gamma_p$  in  $\Gamma$ . Therefore we establish this lemma.

q.e.d.

If  $\Gamma$  is geometrically finite, then Lemma 4 yields that the number  $l_\Gamma = \min L(\Gamma)$  is positive. Hence we have the following :

**THEOREM 5.** *Let  $\Gamma$  be a geometrically finite subgroup of  $M(H^n)$ . Then the non-parabolic part  $N_{\epsilon,2}$  of  $N_{(0,\epsilon)}$  is empty for any  $\epsilon \in (0, \min(l_\Gamma, \epsilon(n)))$ .*

**PROOF.** Choose a positive number with  $\epsilon \in (0, \min(l_\Gamma, \epsilon(n)))$ . Take an arbitrary point  $x \in R_\epsilon(\Gamma)$ . Then, from the definition of  $R_\epsilon(\Gamma)$ , there exists  $\gamma \in \tilde{\Gamma}$  such that  $\rho(x, \gamma(x)) < \epsilon$ .

If  $\gamma$  is loxodromic, then  $\rho(x, \gamma(x)) \geq l(\gamma) \geq l_\Gamma > \epsilon$  and it is a contradiction. So  $\gamma$  is parabolic and we have  $x \in R_\epsilon(\Gamma) \cap T$ . It implies  $N_{\epsilon,1} = N_{(0,\epsilon)}$  and  $N_{\epsilon,2} = \emptyset$ .

q.e.d.

Next we consider the general case. The following lemma is essential for our discussion.

**LEMMA 6.** *For any  $\alpha \geq 0$  there exist a non-elementary, discrete subgroup  $\Gamma$  of  $M(H^n)$  and a sequence  $\{\gamma_m\}$  of loxodromic elements of  $\Gamma$  such that  $l_m \searrow \alpha$  ( $m \rightarrow \infty$ ).*

**PROOF.** Let a sequence  $\{r_m\}$  of positive numbers, with  $r_m \searrow e^\alpha$  ( $m \rightarrow \infty$ ), be given. We take hemispheres  $\sigma, \sigma_1, \sigma_2, \dots$  in  $H^n$  as the following:

$$\sigma = \{x \in H^n \mid |x| = 1\},$$

$$\sigma_m = \{x \in H^n \mid |x| = r_m\} \quad (m = 1, 2, \dots).$$

For each  $m$  we define a Möbius transformation  $g_m$  as  $g_m = r_m x$ . It can be easily seen that  $g_m$  is loxodromic,  $g_m(\sigma) = \sigma_m$  and  $\lambda_m$ , the translation length of  $g_m$ , is equal to  $\log r_m$ .

Let  $\{p_m\}$  be a sequence of points in  $\overline{R^{n-1}} (= \partial H^n)$  with  $r_{m+1} < |p_m| < r_m$  ( $m = 1, 2, \dots$ ). We can take a sequence  $\{R_m\}$  of positive numbers which satisfy

$$r_{m+1} + R_m < |p_m| < r_m - R_m \quad (m = 1, 2, \dots).$$

Here we set

$$\Sigma_m = \{x \in H^n \mid |x - p_m| = R_m\}.$$

Then  $\{\Sigma_m\}$  is a sequence of hemispheres in  $H^n$  which are mutually disjoint to each other. Let denote by  $\psi_m$  the reflection in  $\Sigma_m$  and set  $\psi_m(\sigma) = S_m, \psi_m(\sigma_m) = S'_m$  ( $m = 1, 2, \dots$ ). We can easily see that  $S_m, S'_m \subset \text{Int}(\Sigma_m)$  and  $\text{Int}(S_m) \cap \text{Int}(S'_m) = \emptyset$  ( $m = 1, 2, \dots$ ).

We put  $\gamma_m = \psi_m \circ g_m \circ \psi_m^{-1}$ . Then we have that  $\gamma_m$  is loxodromic and the translation length of  $\gamma_m$  is equal to  $\log r_m$ . Let  $\Gamma$  be the group generated by  $\gamma_1, \gamma_2, \dots$ . We show that  $\Gamma$  is a non-elementary, free, discrete subgroup of  $M(H^n)$ . Since  $\Gamma$  contains two loxodromic transformations which do not have common fixed points,  $\Gamma$  is a non-elementary group. Let  $\gamma$  be an element of  $\Gamma$  which is represented as a reduced word  $\gamma = \gamma_{m_k} \circ \dots \circ \gamma_{m_1}, \gamma_{m_i} \in \{\gamma_1^{\pm 1}, \gamma_2^{\pm 1}, \dots\}$  ( $i = 1, \dots, k$ ). Note that hemispheres  $S_1, S'_1, S_2, S'_2, \dots$  are mutually disjoint to each other. Take a point  $x_0 = (x_1, \dots, x_n) \in H^n$  with  $x_n$  sufficiently large. We may suppose that  $B(x_0, \delta) = \{x \in H^n \mid \rho(x, x_0) < \delta\} \subset \bigcap_{i=1}^{\infty} (\text{Ext}(S_i) \cup \text{Ext}(S'_i))$ . We can easily see  $\gamma_{m_1}(B(x_0, \delta)) \subset \text{Int}(S_l)$  or  $\text{Int}(S'_l)$  for some  $l = 1, 2, \dots$  and  $\gamma_{m_1}(B(x_0, \delta)) \cap B(x_0, \delta) = \emptyset$ . Repeat this procedure. Then we obtain  $\gamma(B(x_0, \delta)) \subset \text{Int}(S_j)$  or  $\text{Int}(S'_j)$  for some  $j = 1, 2, \dots$ . It follows that  $\gamma(B(x_0, \delta)) \cap B(x_0, \delta) = \emptyset$  and  $\gamma \neq \text{id}$ . Hence we have that  $\Gamma$  is free and discrete. Furthermore  $\{\gamma_m\}$  is the sequence of loxodromic elements and  $l_m = \log r_m \searrow \alpha$  ( $m \rightarrow \infty$ ). It completes the proof of this lemma.

q.e.d.

By using Lemma 6, we have the following result immediately.

**THEOREM 7.** *For any positive integer  $n \geq 2$  there exists a non-elementary, discrete subgroup  $\Gamma$  of  $M(H^n)$  such that  $N_{\epsilon, 2} \neq \emptyset$  for any  $\epsilon > 0$ .*

Next we apply Lemma 6 to geometrically finite groups. Let  $\epsilon \in (0, \epsilon(n)]$  be sufficiently small. Then, by using Lemma 6, we can take loxodromic transformations  $\gamma_1, \dots, \gamma_r$ , such that  $l_k < \epsilon$  ( $k = 1, 2, \dots, r$ ) and  $\Gamma = \langle \gamma_1, \dots, \gamma_r \rangle$  is a non-elementary, geometrically finite subgroup of  $M(H^n)$ . Hence we have the following:

THEOREM 8. *For any positive integer  $n \geq 2$  and any  $\epsilon > 0$ , there exists a geometrically finite subgroup  $\Gamma$  of  $M(H^n)$  such that  $N_{\epsilon,2} \neq \emptyset$ .*

## REFERENCES

- [1] APANASOV, B. N., Discrete groups in space and uniformization problems. MIA (Soviet Ser. ) 40, Kluwer Academic Publishers, 1991.
- [2] BEARDON, A. F. AND B. MASKIT, *Limit points of Kleinian groups and finite-sided fundamental polyhedra*. Acta Math. 132 (1974), 1-12.
- [3] BOWDITCH, B. H., *Geometric finiteness for hyperbolic groups*. Mathematical Institute, University of Warwick, Coventry, 1988.
- [4] SUSSKIND, P. AND G. A. SWARUP, *Limit sets for geometrically finite hyperbolic groups*. Amer. J. Math. 114 (1992), 233-250.

DEPARTMENT OF LIBERAL ARTS  
 SCHOOL OF ALLIED MEDICAL PROFESSION  
 KANAZAWA UNIVERSITY  
 KANAZAWA, 920  
 JAPAN