

ON THE CROSSED BURNSIDE RINGS

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1. BOUC FUNCTORS

1.1. Notation and Definition. Let G and H be finite groups. An (H, G) -biset, or a biset shortly, is a set with a left $(H \times G^{\text{op}})$ -action, i.e., a set U with a left H -action and a right G -action which commute.

If K is another group, and V is a (K, H) -biset, then the product $V \times U$ by the right action of H given by $(v, u)h = (vh, h^{-1}u)$ for $v \in V$, $u \in U$, and $h \in H$. The class of (v, u) in $V \times_H U$ is denoted by $(v, {}_H u)$. The set $V \times_H U$ is a (K, G) -biset for the action given by

$$k(v, {}_H u)g = (kv, {}_H ug)$$

for $k \in K$, $g \in G$, $u \in U$, and $v \in V$.

Denote by $\mathcal{C}_p$ the following category:

- The objects of $\mathcal{C}_p$ are the finite p -groups.
- If P and Q are finite p -groups, then $\text{Hom}_{\mathcal{C}_p}(P, Q) = B(Q \times P^{\text{op}})$ is the Burnside group of finite (Q, P) -bisets. An element of this group is called a **virtual (Q, P) -biset**.
- The composition of morphisms is $\mathbb{Z}$ -bilinear, and if P, Q, R are finite p -groups, if U is a finite (Q, P) -biset, and V is a finite (R, Q) -biset, then the composition of (the isomorphism classes of) V and U is the (isomorphism class) of $V \times_Q U$. The identity morphism Id_P of the p -group P is the class of the set P , with left and right action by multiplication.

Let $\mathcal{F}_p$ denote the category of additive functors from $\mathcal{C}_p$ to the category $\mathbb{Z}\text{-Mod}$ of abelian groups. An object of $\mathcal{F}_p$ is called a **Bouc functor** (defined over p -groups, with values in $\mathbb{Z}\text{-Mod}$) (see [Th06], [Bo06]).

1.2. Notation. The Bouc functor of Burnside group will denote by B . The Bouc functor of rational representations will denote by R_Q . The G -poset of the family of all subgroups of a finite group G will denote by $\mathcal{S}(G)$. If X is a G -set, denote by $G \backslash X$ a family of G -orbits, and by $[G \backslash X]$ a set of representatives of $G \backslash X$.

2. THE DADE GROUP

2.1. Some known Dade groups: The structure of $D(P)$ is known for any 2-group P of normal 2-rank 1: when P is generalized quaternion, the result is due to Dade, and the other cases have been solved by Carlson and Thévenaz:

Theorem 2.2. (Dade [Da78a], Carlson-Thévenaz [CT00])

- (1) $D(C_{2^n}) \cong (\mathbb{Z}/2\mathbb{Z})^{n-1}$, and $D(C_{p^n}) \cong (\mathbb{Z}/2\mathbb{Z})^n$, if $p \geq 3$.
- (2) $D(D_{2^n}) \cong \mathbb{Z}^{2n-3}$.
- (3) $D(SD_{2^n}) \cong \mathbb{Z}^{2n-4} \oplus \mathbb{Z}/2\mathbb{Z}$.
- (4) $D(Q_{2^n}) \cong \mathbb{Z}^{2n-5} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, for $n \geq 4$.
- (5) $D(Q_8) \cong \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, if the ground field contains all cubic roots of unity, and $D(Q_8) \cong \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ otherwise.

The ingredient of the present note is

Theorem 2.3. (Bouc-Thévenaz [BT00] Theorem 10.4) *There is an exact sequence of functors*

$$0 \longrightarrow \mathbb{Q}D \xrightarrow{\alpha} \mathbb{Q}B \xrightarrow{\varepsilon} \mathbb{Q}R_Q \longrightarrow 0$$

where $\varepsilon(P) : \mathbb{Q}B(P) \rightarrow \mathbb{Q}R_Q(P)$ is the morphism mapping a P -set to the corresponding permutation module over $\mathbb{Q}$.

We could determine the difference $\text{rank} B^c(P) - \text{rank} R_Q(\mathcal{D}(P))$ by using a result as follows:

Theorem 2.4. (Bouc-Thévenaz [BT00] Theorem A) *The torsion-free rank of the Dade group $D(P)$ is equal to the number of conjugacy classes of non-cyclic subgroups of P .*

3. THE CROSSED BURNSIDE RING AND THE RATIONAL REPRESENTATIONS OF DRINFEL'D DOUBLE

3.1. Definition. Let M be one of the Bouc functors $\mathbb{Q}D$, $\mathbb{Q}B$ and $\mathbb{Q}R_Q$. We use a construction of Dress for Mackey functors for obtaining a module from M . Let P be a p -group. Now we set

$$\begin{aligned} M(X) &= \left(\bigoplus_{x \in X} M(P_x) \right)^P \\ &= \left\{ (m(x)) \in \bigoplus_{x \in X} M(P_x) \mid g(m(x)) = m(gx) \forall g \in P \right\} \end{aligned}$$

where P_x is the stabilizer of x in P .

Corollary 3.2. *Let P be a p -group and X a P -set. Then there is an exact sequence of $\mathbb{Q}$ -vector spaces*

$$0 \longrightarrow \mathbb{Q}D(X) \xrightarrow{\alpha} \mathbb{Q}B(X) \xrightarrow{\epsilon} \mathbb{Q}R_Q(X) \longrightarrow 0.$$

3.3. Notation. We denote by $B^c(P)$ the *crossed Burnside ring* of P , i.e. the Grothendieck ring of the category of finite crossed P -sets over P^c , for relations given by decomposition into disjoint union of crossed P -sets, the ring structure being induced by the product of crossed P -sets. Also we denote by $R_Q(\mathcal{D}(P))$ the *rational representation ring* of the Drinfel'd double $\mathcal{D}(P) = (\mathbb{Q}P)^* \otimes \mathbb{Q}P$ for the group algebra $\mathbb{Q}P$.

Corollary 3.4. *Let P be a p -group. Then there is an exact sequence of $\mathbb{Q}$ -vector spaces*

$$0 \longrightarrow \mathbb{Q}D(P^c) \xrightarrow{\alpha} \mathbb{Q}B^c(P) \xrightarrow{\epsilon} \mathbb{Q}R_Q(\mathcal{D}(P)) \longrightarrow 0.$$

In particular, we have

$$\text{rank} B^c(P) = \text{rank} R_Q(\mathcal{D}(P)) + \dim_{\mathbb{Q}} \mathbb{Q}D(P^c).$$

Corollary 3.5. *Let P be a p -group. Then the following numbers are equal:*

- (1) $\text{rank} B^c(P)$.
- (2) $\text{rank} R_Q(\mathcal{D}(P)) + \sum_{g \in [P \setminus P^c]} \dim_{\mathbb{Q}} \mathbb{Q}D(C_P(g))$.
- (3) $\sum_{Q \in [P \setminus \mathcal{S}(P)]} \frac{|C_P(Q)|}{|N_P(Q)|} \cdot |Q| \left(\sum_{x \in Q/Q'} \frac{1}{|x|} \right)$.
- (4) $\sum_{Q \in [P \setminus \mathcal{S}(P)]} |N_P(Q) \setminus C_P(Q)|$.
- (5) $\sum_{g \in [P \setminus P^c]} |C_P(g) \setminus \mathcal{S}(C_P(g))|$.

Corollary 3.6. *Let P be a p -group. Then*

$$\text{rank} B^c(P) = \text{rank} R_Q(\mathcal{D}(P)) + \sum_{g \in [P \setminus P^c]} |C_P(g) \setminus \mathcal{S}_{\text{non}}(C_P(g))|,$$

where $\mathcal{S}_{\text{non}}(C_P(g))$ is the $C_P(g)$ -poset of non-cyclic subgroups of $C_P(g)$ with $C_P(g)$ -action defined by conjugation.

Corollary 3.7. *Let P be a cyclic p -group. Then*

$$\text{rank} B^c(P) = \text{rank} R_Q(\mathcal{D}(P)).$$

3.8. Some small 2-groups. We summarize basic facts on the structure of the centralizers of the representative of a conjugacy class of dihedral, semi-dihedral and generalized quaternion 2-groups (see, for instance, III.17 of [Er90]). In the rest of the paper, we always denote by z the central elements of order 2 of the group considered. Suppose that

$$D_{2^n} = \langle x, y | x^{2^{n-1}} = y^2 = 1, y^{-1}xy = x^{-1} \rangle$$

is a dihedral group of order 2^n ($n \geq 2$). Then the centralizers of 1 and z are D_{2^n} . The centralizers of y and xy are Klein four groups. The centralizers of the representative of the other $2^{n-2} - 1$ conjugacy classes are cyclic subgroups ($n \geq 3$). Suppose that

$$SD_{2^n} = \langle x, y | x^{2^{n-1}} = y^2 = 1, y^{-1}xy = x^{-1+2^{n-2}} \rangle$$

is a semi-dihedral group of order 2^n ($n \geq 4$). Then the centralizers of 1 and z are SD_{2^n} . The centralizer of y is a Klein four group. The centralizers of the representative of the other 2^{n-2} conjugacy classes are cyclic subgroups. Suppose that

$$Q_{2^n} = \langle x, y | x^{2^{n-2}} = y^2, y^4 = 1, y^{-1}xy = x^{-1+2^{n-2}} \rangle$$

is a generalized quaternion group of order 2^n ($n \geq 3$). Then the centralizers of 1 and z are Q_{2^n} . The centralizers of the representative of the other $2^{n-2} + 1$ conjugacy classes are cyclic subgroups.

Corollary 3.9. *Let P be a dihedral group D_{2^n} of order 2^n ($n \geq 2$). Then*

$$\text{rank} B^c(P) - \text{rank} R_Q(\mathcal{D}(P)) = 4n - 4.$$

Corollary 3.10. *Let P be a semi-dihedral group SD_{2^n} of order 2^n ($n \geq 4$). Then*

$$\text{rank} B^c(P) - \text{rank} R_Q(\mathcal{D}(P)) = 4n - 7.$$

Corollary 3.11. *Let P be a generalized quaternion group Q_{2^n} of order 2^n ($n \geq 3$). Then*

$$\text{rank} B^c(P) - \text{rank} R_Q(\mathcal{D}(P)) = 4n - 10.$$

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