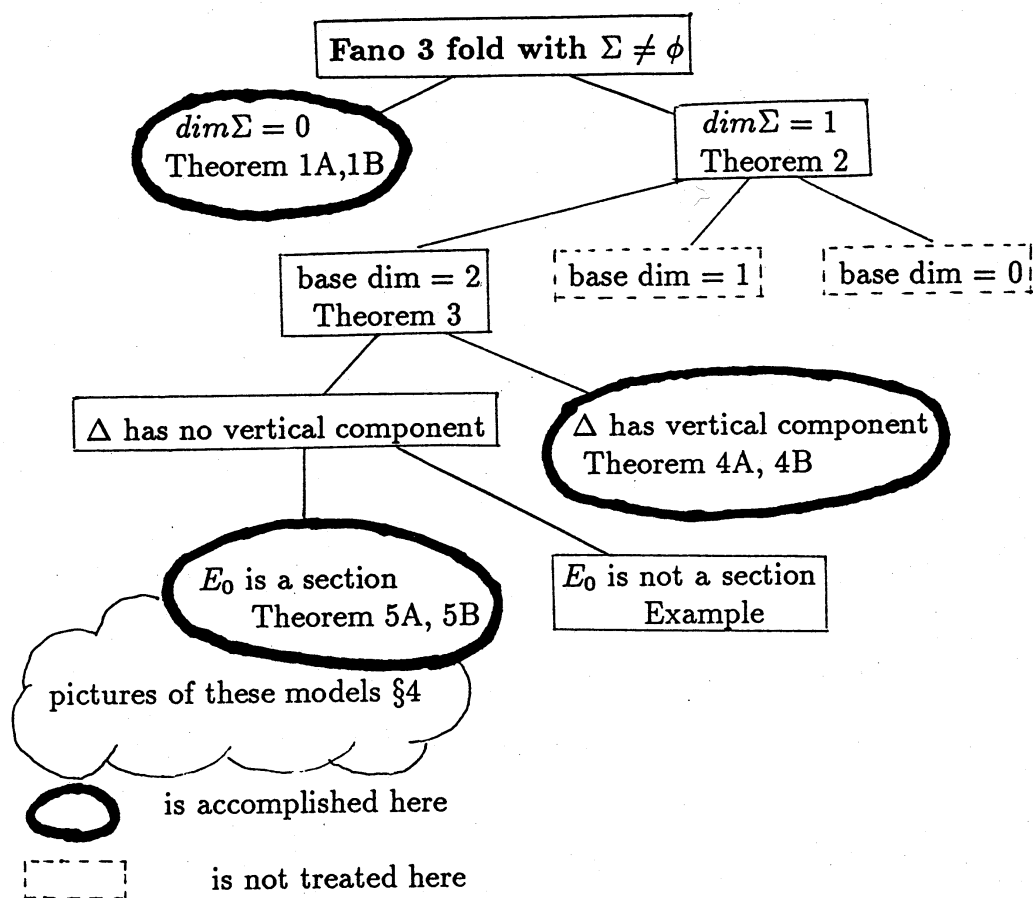


A Fano 3-fold with the 1-dimensional locus of non-rational singularities

東工大理 石井志保子 (Shihoko ISHII)

The structure of this paper



Introduction

In this paper a Fano 3-fold means a normal projective variety of dimension three over $\mathbb{C}$ whose anticanonical sheaf is ample and invertible. During the past fifteen years, there has been big progress in the investigation of a non-singular Fano 3-fold owing to Iskovskih, Mori, Mukai and Shokulov. And it is still developing. On the other hand, in singular Fano 3-folds, progress seems to have started recently. Here we study the structure of a Fano 3-fold with non-rational singularities.

Let Σ be the locus of non-rational singular points of a Fano 3-fold X . As X is normal, $\dim \Sigma \leq 1$. If $\dim \Sigma = 0$, then X is isomorphic to a projective cone over a normal K3-surface or an Abelian surface (Theorem 1A, 1B). The proof of this theorem also works in the case that Σ contains an isolated point. So what we should study next is the case that Σ has pure dimension one. Such a Fano 3-fold is classified in three families according to the maximal basis-dimension of its $\mathbb{Q}$ -factorial terminal modification (Theorem 2, Definition 1). We obtain the fact that a Fano 3-fold with the maximal basis-dimension 2 admits a projective bundle over a non-singular surface as a $\mathbb{Q}$ -factorial terminal modification (Theorem 3). We try to make clear the structure of a Fano 3-fold in this family: what kind of surface occurs as a basis, what kind of projective bundle appears as a $\mathbb{Q}$ -factorial terminal modification and which parts on the projective bundle are contracted in a Fano 3-fold.

The author would like to thank Professors Nakayama and Kei-ichi Watanabe and also other members of Waseda Seminar for their stimulating discussion during the preparation of this article. In particular Nakayama's proof of Proposition 2 helped her very much and also K-i. Watanabe's comment "a Weil divisor on a $\mathbb{Q}$ -factorial terminal singularity is Cohen-Macaulay" was very helpful in the proof of Theorem 2.

§1. The case $\dim \Sigma = 0$

Theorem 1A([I]). Let X be a Fano 3-fold with $\dim \Sigma = 0$. Then there exist a normal surface S which is either an Abelian surface or a normal K3-surface and an ample invertible sheaf $\mathcal{L}$ on S such that X is the contraction of the negative section of a projective bundle $\mathbf{P}(\mathcal{O}_S \oplus \mathcal{L})$. Here a normal K3-surface implies a normal projective surface with the trivial canonical sheaf and has only rational singularities.

Theorem 1B([I]). Let X be a projective cone over a surface S which is either an Abelian or a normal K3-surface. Then X is a Fano 3-fold with $\Sigma = \{\text{the vertex}\}$.

§2. Basic structure theorem of $\mathbb{Q}$ -factorial terminal modifications for the case $\dim \Sigma = 1$

Theorem 2. Let X be a Fano 3-fold with Σ of pure dimension one. Let $g : Y \rightarrow X$ be a $\mathbb{Q}$ -factorial terminal modification whose existence is proved by Mori ([M]). Denote $K_Y = g^*K_X - \Delta$. Then we have a sequence of projective morphisms:

$Y = Y_0 \xrightarrow{\varphi_0} Y_1 \xrightarrow{\varphi_1} Y_2 \dots \rightarrow Y_r \xrightarrow{\varphi_r} Z$, where for each i , φ_i is the contraction of an extremal ray R_i on Y_i such that $R_i \Delta_i > 0$ (here, $\Delta_0 = \Delta$, and $\Delta_i = (\varphi_{i-1})_* \Delta_{i-1}$). For $i \leq r-1$, φ_i is a birational contraction of a divisor isomorphic to $F_{a,0}$ ($a \geq 1$) to a non-singular point and φ_r is a fibration to a lower dimensional variety Z .

Definition 1. The variety Z above is called a basis of X . And each φ_i is called a Δ -extremal contraction. Of course a basis of X is not unique for X . It depends on the choice of a $\mathbb{Q}$ -factorial terminal modification Y and also on the choice of extremal rays R_i 's.

From now on, we devote to study X which has a two dimensional basis Z . In this case, the last contraction $\varphi_r : Y_r \rightarrow Z$ satisfies the assumption of the following proposition. So we can see that it is a $\mathbb{P}^1$ -bundle over a non-singular surface Z .

Proposition 1 (Nakayama). Let $\varphi : Y \rightarrow Z$ be a contraction of an extremal ray on a 3-fold Y with at worst $\mathbb{Q}$ -factorial terminal singularities on it to a surface Z . Assume there exists an invertible sheaf on Y whose degree on a general fiber is 1. Then Z is non-singular and Y is a $\mathbb{P}^1$ -bundle over Z .

Theorem 3. Let X be a Fano 3-fold with one dimensional Σ and a two dimensional basis. Then there exists a $\mathbb{Q}$ -factorial terminal modification $g : Y \rightarrow X$ such that a Δ -extremal contraction $\varphi_0 : Y \rightarrow Z$ gives a $\mathbb{P}^1$ -bundle over a non-singular surface Z .

This theorem is proved by applying the following lemma successively.

Lemma. Let X be as above and $Y = Y_0 \xrightarrow{\varphi_0} Y_1 \xrightarrow{\varphi_1} Y_2 \dots \rightarrow Y_r \xrightarrow{\varphi_r} Z$ be a sequence of Δ -extremal contractions of $\mathbb{Q}$ -factorial terminal modification Y of X with 2-dimensional basis Z . If $r > 0$, then there is a flop Y'_i of Y_i for each i ($i \leq r-1$) such that $g' : Y' = Y'_0 \rightarrow X$ is a $\mathbb{Q}$ -factorial terminal modification of X and $Y' = Y'_0 \xrightarrow{\varphi'_0} Y'_1 \xrightarrow{\varphi'_1} Y'_2 \dots \rightarrow Y'_{r-1} \xrightarrow{\varphi'_{r-1}} Z'$ is a sequence of Δ' -extremal contractions with 2-dimensional basis Z' , where Δ' is a $\mathbb{Q}$ -divisor such that $K_{Y'} = g'^* K_X - \Delta'$.

§3. Fano 3-folds which have $\mathbb{P}^1$ -bundles as $\mathbb{Q}$ -factorial terminal modi-

fications.

Let X be a Fano 3-fold with a 2-dimensional basis. Then, by Theorem 3, we can take a $\mathbf{Q}$ -factorial terminal modification $g : Y \rightarrow X$ such that a Δ -extremal contraction $\varphi : Y \rightarrow Z$ gives a $\mathbf{P}^1$ -bundle over a non-singular surface Z . Then we have the following facts:

- (i) $-g^*K_X\ell = \Delta\ell = 1$, where ℓ is a fiber of $\varphi : Y \rightarrow Z$.
- (ii) Δ is denoted by $E_0 + \varphi^*(\Delta')$, where E_0 is an irreducible component with $E_0\ell = 1$ and $\Delta' \in \text{Pic}(Z)$.

The case $\text{Supp}\Delta$ contains a vertical component

We call an irreducible divisor D in Y a vertical divisor for g , if D is mapped to a point of X by g .

Theorem 4A. Let $X, g : Y \rightarrow X, \Delta$ and $\varphi : Y \rightarrow Z$ be as in the beginning of this section. Assume $\text{Supp}\Delta$ contains a vertical component.

Then, (i) a vertical component is unique and coincides with E_0 and it is a section of the projection φ ,

(ii) there exists a normal surface Z_0 with at least one non-rational singular point on it whose canonical sheaf is trivial and whose minimal resolution is $h : Z \rightarrow Z_0$ and

(iii) the $\mathbf{P}^1$ -bundle $\varphi : Y \rightarrow Z$ is a pull back of a $\mathbf{P}^1$ -bundle $\varphi_0 : Y_0 \rightarrow Z_0$ by h and $g : Y \rightarrow X$ factors as $Y \xrightarrow{h} Y_0 \xrightarrow{g_0} X$, where g_0 is a contraction of the negative section $h(E_0)$.

Theorem 4B. Let S be a normal surface with trivial canonical sheaf and at least one non-rational singular point on it. Then an arbitrary projective cone X over S is a Fano 3-fold and Σ is generating lines over a non-rational singular points of S .

Remark. Normal surfaces with the trivial canonical sheaf and at least one non-rational singular point are studied in [U] among others. The number of non-rational singular points is less than or equal to 2. It is 2, if and only if both of them are simple elliptic singularities [U, Theorem 1].

The case $\text{Supp}\Delta$ contains no vertical component

In the previous case, E_0 is a section of φ . But in this case, it is not necessarily true. First we consider the case that E_0 is a section. Since E_0 is not a vertical component, $g|_{E_0} : E_0 \rightarrow C$ is a fibration to a curve C .

Proposition 2. The possible triples $(E_0, g|_{E_0}, \Delta')$ are the following:

- (i) $(\mathbf{P}^1 \times \text{elliptic curve}, \text{the first projection } p_1, \phi)$,
- (ii) (a rational elliptic surface, the elliptic fibration, ϕ),
- (iii) E_0 is the composite of r -blowing ups $E_0 \xrightarrow{\sigma_r} \dots \xrightarrow{\sigma_1} \mathbf{P}^1 \times \text{elliptic curve}$, where σ_1 is the blow up at a point on the fiber $C = p_1^{-1}(z)$ of a point $z \in \mathbf{P}^1$ and σ_i ($i > 1$) is the blow up at the intersection of the proper transform of C and the exceptional curve of σ_{i-1} . The morphism $g|_{E_0}$ is $p_1 \sigma_1 \sigma_2 \dots \sigma_r$ and Δ' = the proper transform of C .
- (iv) E_0 is a ruled surface $p : E_0 \rightarrow S$ such that there exist a covering $\pi : S \rightarrow \mathbf{P}^1$ and a member D in $|-K_{E_0}|$ of type $D = (\pi p)^*(z) + \Delta'$, where $z \in \mathbf{P}^1$ and Δ' is an effective divisor with $K_{E_0} C \geq 0$ for every component $C \subset \Delta'$. The morphism $g|_{E_0}$ is πp .

Theorem 5A. Let $X, g : Y \rightarrow X, \Delta, \Delta'$ and $\varphi : Y \rightarrow Z$ be as in the beginning of this section. Assume $\text{Supp} \Delta$ contains no vertical component and E_0 is a section of φ . Denote $-g^*K_X = E_0 + \varphi^*L$ for $L \in \text{Pic} Z$. Then the triple $(E_0, g|_{E_0}, \Delta')$ is as one of (i)~(iv) in Proposition 2 and the $\mathbf{P}^1$ -bundle $\varphi : Y \rightarrow Z$ is obtained by an sheaf $\mathcal{E}$ which satisfies the following properties:

(I) $\mathcal{E}$ is an extension of $\mathcal{N}$ by $\mathcal{O}_Z$, where $\mathcal{N} = \mathcal{O}_Z(-K_Z - \Delta' - L)$ such that $\mathcal{E}|_{\Delta'} = \mathcal{O}_{\Delta'}(-L) \oplus \mathcal{O}_{\Delta'}(-L)$ and $(L \otimes \mathcal{E})_y$ is generated by its global sections for each $y \in \Delta'$.

(II) $L - \Delta'$ is semi-ample and $(L - \Delta')(L - \Delta' - K_Z) > 0$.

Theorem 5B. Let a triple $(Z, \tilde{g}, \Delta')$ be as one of (i) ~ (iv) in Proposition 2 and $L \in \text{Pic} Z$ and $\mathcal{E}$ be as in (I) and (II) in Theorem 5A.

Let $Y = \mathbf{P}(\mathcal{E}) \xrightarrow{\varphi} Z$ be the projective bundle defined by $\mathcal{E}$ and E_0 be a section of φ defined by the surjection $\mathcal{E} \rightarrow \mathcal{N}$. Denote $E_0 + \varphi^*L$ by H .

Then $|mH|$ is base point free for $m \gg 0$ and the image X of the morphism $g = \Phi_{|mH|} : Y \rightarrow \mathbf{P}^M$ becomes a Fano 3-fold with one dimensional Σ and $g|_{E_0} = \tilde{g}$ under the identification of E_0 with Z .

Now we give an example of a Fano 3-fold with E_0 not a section.

Example. Let Z be the projective plane $\mathbf{P}^2$, C and C' be two general curves of degree 3 on Z . Let $\sigma : \tilde{Z} \rightarrow Z$ be the blowing up at 9-distinct points $\{p_1, p_2, \dots, p_9\} = C \cap C'$, then $\tilde{Z}$ becomes an elliptic surface with elliptic fibers $[C]$, $[C']$, where $[C]$ is the proper transform of C on $\tilde{Z}$. Denote the fiber $\sigma^{-1}(p_i)$ by ℓ_i . Let L be $\sigma^*L_0 + \sum_{i=1}^9 \ell_i$ where L_0 is an ample divisor on Z .

Since $H^1(\tilde{Z}, L - [C]) \simeq \oplus H^1(\ell_i, L - [C]|_{\ell_i}) \simeq \mathbf{C}^{\oplus 9}$, we can take an extension sheaf $\tilde{\mathcal{E}}$ of $\mathcal{O}([C] - L)$ by $\mathcal{O}_{\tilde{Z}}$ such that the restriction $[\tilde{\mathcal{E}}|_{\ell_i}] \in H^1(\ell_i, L - [C]|_{\ell_i})$ is not zero for every i ($i = 1, 2, \dots, 9$). Now $0 \rightarrow \mathcal{O}_{\tilde{Z}}|_{\ell_i} \rightarrow \tilde{\mathcal{E}}|_{\ell_i} \rightarrow \mathcal{O}([C] - L)|_{\ell_i} = \mathcal{O}_{\mathbf{P}^1}(2) \rightarrow 0$ does not split, so $\tilde{\mathcal{E}}|_{\ell_i} \simeq \mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(1)$. Put $\tilde{\mathcal{E}}' = \tilde{\mathcal{E}}(\Sigma \ell_i)$, then $\tilde{\mathcal{E}}'|_{\ell_i}$ is trivial for each i . By Schwarzenberger's Theorem, $\tilde{\mathcal{E}}' = \sigma^*\mathcal{E}$ for some locally free sheaf $\mathcal{E}$ on Z . Let Y be the projective bundle $\mathbf{P}(\mathcal{E})$ and $\tilde{Y}$ be $\mathbf{P}(\tilde{\mathcal{E}}')$. Then we have the diagram of a fiber product

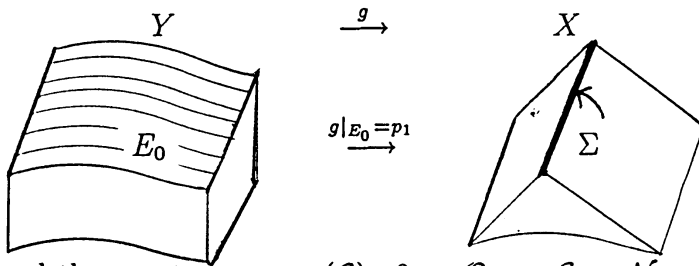
$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\sigma} & Y \\ \downarrow \tilde{\varphi} & \square & \varphi \downarrow \\ \tilde{Z} & \xrightarrow{\sigma} & Z \end{array}$$

Let $\tilde{E}_0$ be the section of $\tilde{\varphi}$ defined by the surjection $\tilde{\mathcal{E}} \rightarrow \mathcal{O}([C] - L)$ and E_0 be the image $\sigma(\tilde{E}_0)$. Then $H = E_0 + \varphi^*L_0$ is a semipositive divisor on Y . The image X of the morphism $\Phi_{|mH|} : Y \rightarrow \mathbf{P}^M$ becomes a Fano 3-fold with $\Sigma \simeq \mathbf{P}^1$ and Y is a $\mathbf{Q}$ -factorial terminal modification of X . It is easy to see that $\Delta = E_0$ and E_0 contains the fibers of φ over $p_1, p_2, \dots, p_9 \in Z$.

§4. Pictures of Y and X of Theorem 5

(i) In the case the triple is $(\mathbf{P}^1 \times C, \text{the first projection } p_1, \phi)$, where C is an elliptic curve. Then $\Delta = E_0$. If we denote $L = p_1^*\mathcal{O}_{\mathbf{P}^1}(a) \otimes p_2^*B$, then $a \geq 0$ and B is ample.

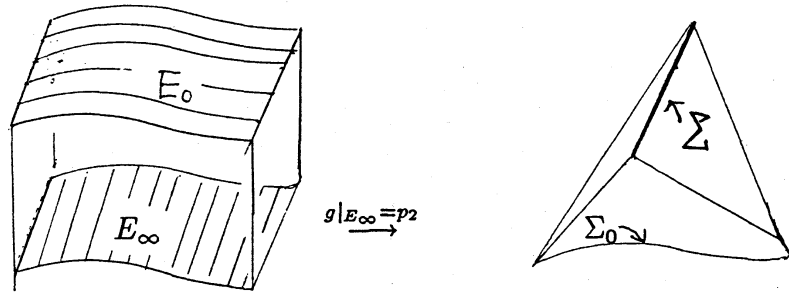
(i-1) $a > 0$. $g|_{Y-E_0} : Y - E_0 \simeq X - \Sigma$.



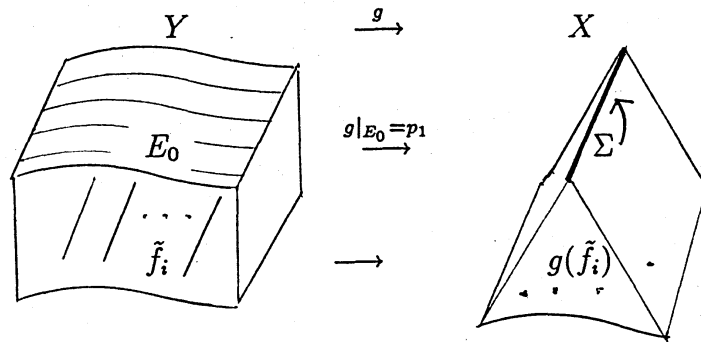
(i-2) $a = 0$ and the exact sequence $(\mathcal{E}) : 0 \rightarrow \mathcal{O}_Z \rightarrow \mathcal{E} \rightarrow \mathcal{N} \rightarrow 0$ splits.

$g|_{Y-E_0-E_\infty} : Y - E_0 - E_\infty \simeq X - \Sigma - \Sigma_0$, and $g|_{E_\infty} = p_2$, where Σ_0 is the locus of canonical singularities.

$$\begin{array}{ccc} Y & \xrightarrow{g} & X \\ \vdots & & \vdots \\ E_0 & \xrightarrow{g|_{E_0=p_1}} & \Sigma \end{array}$$

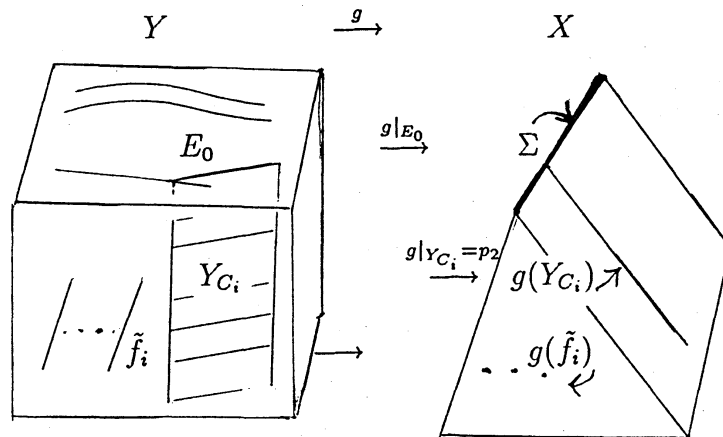


(i-3) $a = 0$ and the exact sequence $(\mathcal{E}) : 0 \rightarrow \mathcal{O}_Z \rightarrow \mathcal{E} \rightarrow \mathcal{N} \rightarrow 0$ does not split. There exists a divisor $\sum_{i=1}^s m_i q_i \in |B|$ such that the restriction $(\mathcal{E})|_{f_i}$ splits for each $i, (i = 1, \dots, s)$, where $f_i = p_2^{-1}(q_i)$. For a general fiber $f = p_2^{-1}(q), q \in C$, Y_f is $\mathbf{P}^1 \times \mathbf{P}^1$ and for f_i , $Y_{f_i} \simeq \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(2))$. $E_0|_{Y_f}$ is an ample section for general f and is the disjoint section from the negative section for $f = f_i$. Denote the negative section of Y_{f_i} by $\tilde{f}_i$. Then the restriction $g|_{Y-E_0-\cup \tilde{f}_i}$ is an isomorphism, $g|_{E_0} = p_1$, and each $\tilde{f}_i$ is contracted to a canonical singularity in X .

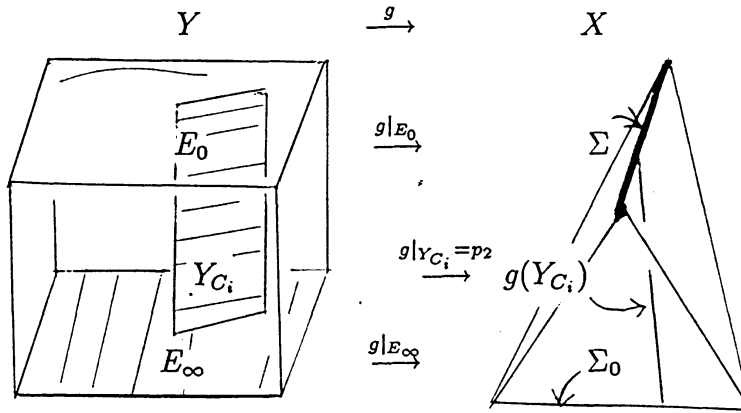


(ii) The case that the triple $(E_0, g|_{E_0}, \Delta')$ is (rational elliptic surface, the elliptic fibration, ϕ). Then $E_0 = \Delta$ in this case too. If L is big then the exact sequence $(\mathcal{E})$ splits and if L is not big $|L|$ gives a fibration $\Phi = \Phi|_L : Z \rightarrow \mathbf{P}^1$ with a general fiber $\mathbf{P}^1$. Let C_i ($i = 1, 2, \dots, r$) be (-2) -curves on Z with $LC_i = 0$ and f_j ($j = 1, \dots, s$) be (-1) -curves on Z with $Lf_j = 0$. Then $E_0|_{Y_{f_i}}$ is the section disjoint from the negative section. Denote the negative section of Y_{f_i} by $\tilde{f}_j$. Then the normal bundle of $\tilde{f}_j$ in Y is $\mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(1)$.

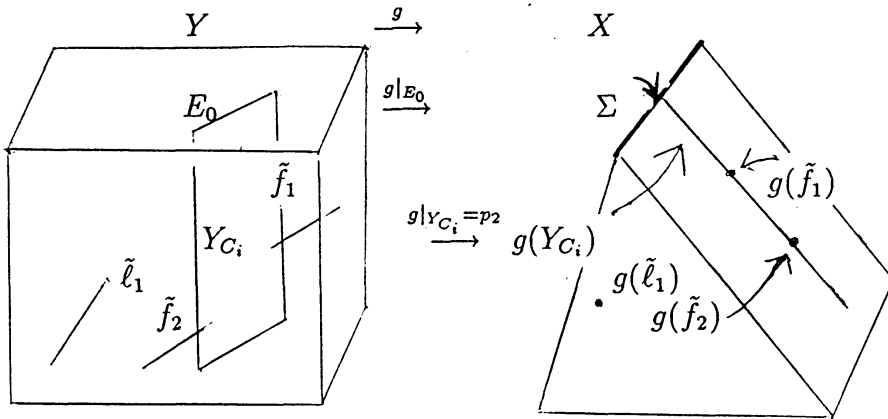
(ii-1) L is big. Then the restriction $g|_{Y-E_0-\cup Y_{C_i}-\cup \tilde{f}_i}$ is an isomorphism, $g|_{Y_{C_i}} : Y_{C_i} \simeq C_i \times \mathbf{P}^1 \rightarrow \mathbf{P}^1$ is the projection to the second factor and $g(\tilde{f}_j)$ is an isolated canonical singular point for each j . A point of $g(Y_{C_i})$ away from $g(E_0)$ is non-isolated canonical singularities.



(ii-2) L is not big and $(\mathcal{E})$ splits. Let E_∞ be the section of φ disjoint from E_0 . Then the restriction $g|_{E_0-E_\infty-\cup Y_{C_i}}$ is an isomorphism, $g|_{Y_{C_i}}$ is as above and $g|_{E_\infty} = \Phi$.

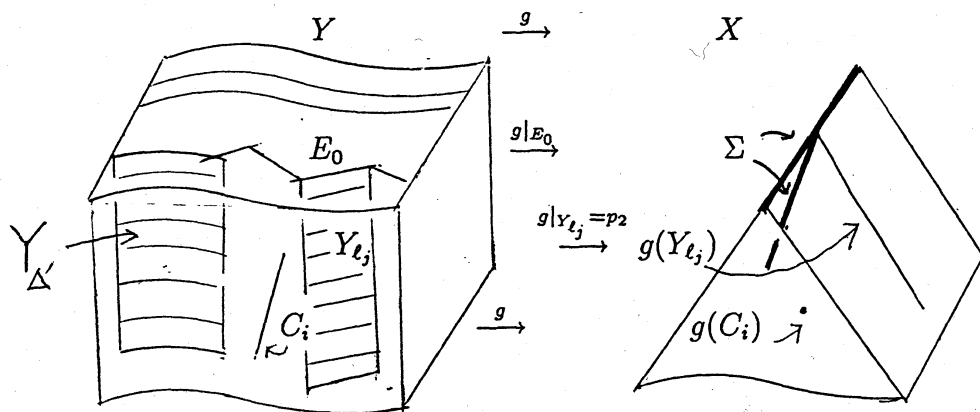


(ii-3) L is not big and $(\mathcal{E})$ does not split. Denote $L = \Phi^* L_0$ for a Cartier divisor L_0 on $\mathbf{P}^1$. Then the extension $\mathcal{E}$ of $\mathcal{N}$ corresponds to a non-zero section $\phi_{\mathcal{E}}$ of $\Gamma(\mathbf{P}^1, L_0 + K_{\mathbf{P}^1})$. Let $\phi_{\mathcal{E}}$ define a divisor $\sum_{k=1}^d m_k q_k$ ($d \geq 0, m_k > 0$) and ℓ_k $k = 1, \dots, b$ ($0 \leq b \leq d$) be smooth fibers among $\{\Phi^{-1}(q_k)\}$. A component of a singular fiber of Φ is either one of C_i 's or f_i 's defined above. For a general fiber $\ell = \Phi^{-1}(q)$ $q \in \mathbf{P}^1$, Y_ℓ is $\mathbf{P}^1 \times \mathbf{P}^1$ and for ℓ_k , $Y_{\ell_k} \simeq \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(2))$. $E_0|_{Y_\ell}$ is an ample section for general ℓ , while it is the section disjoint from the negative section for $\ell = \ell_k$ ($0 \leq k \leq b$). Denote the negative section of Y_{ℓ_k} by $\tilde{\ell}_k$. Then the restriction $g|_{Y-E_0-\cup_{i=1}^r Y_{C_i}-\cup_{j=1}^s \tilde{f}_j-\cup_{k=1}^b \tilde{\ell}_k}$ is isomorphic, $g|_{Y_{C_i}}$ is the second projection, $\tilde{f}_j$'s and $\tilde{\ell}_k$'s are contracted to canonical singularities in X .



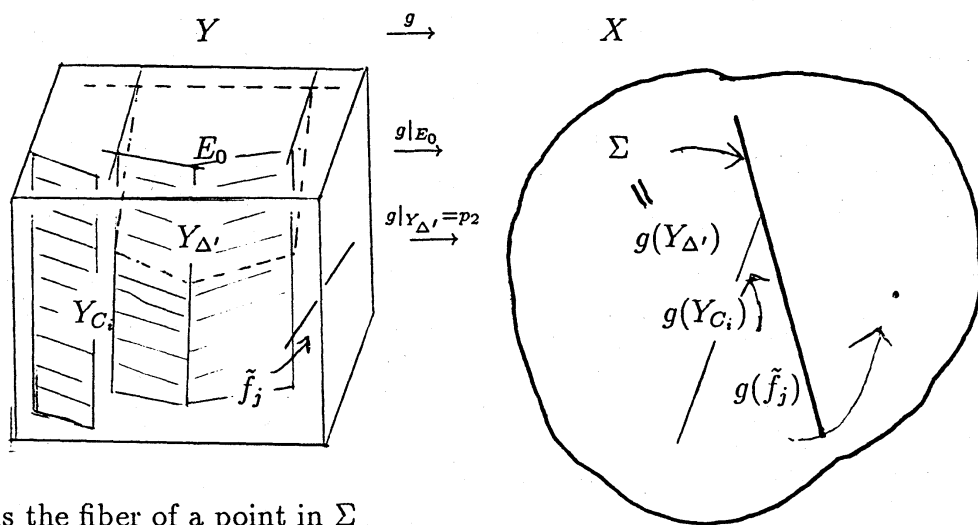
(iii) The case that the triple $(E_0, g|_{E_0}, \Delta')$ is as follows: E_0 is the composite of r -blowing ups $E_0 \xrightarrow{\sigma_r} \dots \xrightarrow{\sigma_1} \mathbf{P}^1 \times \text{elliptic curve}$, where σ_1 is the blow up at a point on the fiber $C = p_1^{-1}(z)$ of a point $z \in \mathbf{P}^1$ and σ_i ($i > 1$) is the blow up at the intersection of the proper transform of C and the exceptional curve of σ_{i-1} . The morphism $g|_{E_0}$ is $p_1 \sigma_1 \sigma_2 \dots \sigma_r$ and Δ' = the proper transform of C .

Then L is nef and big, with $L\ell_r > 0$ and the exact sequence $(\mathcal{E})$ splits, where ℓ_i ($i = 1, 2, \dots, r$) are the exceptional curves of σ_i respectively. Let E_∞ be the section of φ disjoint from E_0 , and ℓ_j $j \in J \subset \{1, 2, \dots, r-1\}$ be the exceptional curves with $L\ell_j = 0$ and C_i $i = 1, 2, \dots, s$ be the $\ell_j^{(-1)}$ curves on E_∞ with $LC_i = 0$. Then $g|_{Y-E_0-Y_{\Delta'}-\cup_{j \in J} Y_{\ell_j}-\cup_{i=1}^s C_i}$ is an isomorphism, $g|_{Y_{\ell_j}} \simeq \ell_j \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ is the projection to the second factor and $g(C_i)$ is an isolated canonical singular point on X for $i = 1, 2, \dots, s$.



(iv) The case that the triple is as (ii) of Proposition 2.

Then the exact sequence $(\mathcal{E})$ does not split. Let C_i ($i = 1, 2, \dots, r$) be (-2) -curves on Z with $eC_i = LC_i = 0$ and f_j ($j = 1, \dots, s$) be (-1) -curves on Z with $ef_j > 0$ and $Lf_j = 0$. Then we can take the negative section $\tilde{f}_j$ of $Y_{\tilde{f}_j}$ disjoint from E_0 . Then $g|_{Y-\Delta-\cup_{i=1}^r Y_{C_i}-\cup_{j=1}^s \tilde{f}_j}$ is an isomorphism, $g|_{Y_C} : Y_C \simeq C \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ is the projection to the second factor for a component $C < \Delta$ and for $C = C_i$ ($i = 1, \dots, r$) and $g(\tilde{f}_j)$ is an isolated canonical singularity for $j = 1, \dots, s$.



,where is the fiber of a point in Σ

References

- [I] S. Ishii: *Quasi-Gorenstein Fano-3-folds with isolated non-rational loci*. *Compositio Math.* **77** (1991) 335-341
- [M] S. Mori: *Flip theorem and the existence of minimal models for 3-folds*. *J. Amer. Math. Soc.* **1** (1988) 117-253
- [U] Y. Umezu: *On normal projective surfaces with trivial dualizing sheaf*. *Tokyo J. Math.* **4** (1981) 343-354

Department of Mathematics, Tokyo Institute of Technology, Oh-okayama, Meguro 152 Tokyo JAPAN

e-mail address: shihoko@math.titech.ac.jp

