

Contractions on Hilbert space

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Let T be a contraction, that is $\|T\| \leq 1$, on a separable Hilbert space $\mathcal{H}$. Then $D_T = (I - T^*T)^{1/2}$ is well defined, which is called defect operator of T . In this case we have $\sigma(T) \subset \tilde{D}$, where D and $\tilde{D}$ denote the open unit disc and its closure respectively. Contractions which have defect operators of finite ranks have been studied by many mathematicians. For investigations of contraction T with $D_T \in (\sigma, c)$, that is $I - T^*T \in (\tau, c)$, where (σ, c) and (τ, c) denote the Hilbert Schmidt class and the trace class respectively, some mathematicians added a condition $\sigma(T) \neq \tilde{D}$. Such a contraction T was called weak contraction by M.G.Krein. The spectral decomposition for weak contraction T or accretive operator

$$(I+T)(I-T)^{-1}$$

were obtained by Sz.-Nagy and Foias, Brodskii and Ginzburg (cf [7]).

Since T is a contraction, $\|T^n x\|$ is decreasing for each x . Sz.-Nagy and Foias defined contractions' classes as following:

$$C_1 = \{T: \lim_{n \rightarrow \infty} \|T^n x\| > 0 \text{ for each } x \neq 0\},$$

$$C_0 = \{T: \lim_{n \rightarrow \infty} \|T^n x\| = 0 \text{ for each } x\},$$

$$C_{.1} = \{T: T^* \in C_1\}, \quad C_{.0} = \{T: T^* \in C_0\},$$

$$C_{ij} = C_i \cap C_{.j} \quad (0 \leq i, j \leq 1).$$

These formal notations are playing important roles in the studies of contraction. In particular they showed that every weak contraction in C_{00} belongs to C_0 (about this notation see [7]),

and every weak contraction is decomposed to direct sum of the contraction in C_0 and the contraction in $C_{1,1}$. The Jordan models for weak contractions were constructed by P.Y.Wu [10].

In [9] the author applied the results of Bercovici and Voiculescu's paper [1] to investigate a contraction T satisfying $\sigma(T) = \widetilde{D}$ and $D_T \in (\sigma, c)$, in particular, showed that T belongs to $C_{1,0}$ iff there is a quasi-affinity X such that

$$X T = S_E X ,$$

where E is a Hilbert space with $\dim E = - \text{index } T$ (this "index" is Fredholm index) and S_E is the unilateral shift on $\ell_+^2(E)$. From the results of [9], he conjectured that contraction in $C_{0,0}$ with (σ, c) -defect operator belongs to C_0 . In [8] Takahashi and Uchiyama showed that this was true.

In this note we will clear the structure of a contraction T with D_T in (σ, c) . In particular, setting

$$\alpha = \min \{ \dim N(T - \lambda) : \lambda \in D \}, \quad \beta = \min \{ \dim N(T^* - \lambda) : \lambda \in D \},$$

where $N(T) = \{x : Tx = 0\}$, we will show that there are vector valued holomorphic functions $h_i(\lambda), f_j(\lambda)$ ($1 \leq i \leq \alpha, 1 \leq j \leq \beta$) defined on D satisfying

$$(T - \lambda)h_i(\lambda) \equiv 0, \quad (T^* - \lambda)f_j(\lambda) \equiv 0$$

, and that if $\alpha = \beta = 0$, then T is a weak contraction.

In section 4, we will study the weighted shifts with finite matrices' weights.

From now on, we use the symbol $D(T)$ instead of D_T for convenience

1. Upper triangulation

Let T be a contraction on $\mathcal{H}$ with $D(T) \in (\sigma, c)$. Then, since $\sum_i (1 - \|Te_i\|^2) < \infty$ for a C.O.N.B. $\{e_i\}$ of $\mathcal{H}$, we have $\dim N(T) < \infty$. Let $T = V|T|$ be the polar decomposition of T . Then there is a isometric (or co-isometric) extension V_1 of V such that $V_1 - V$ is of finite rank. In this case $\dim N(V_1 - \lambda)$ is constant on D and finite, also $\dim N(V_1^* - \lambda)$ is constant on D . Since $\text{range}(V_1 - \lambda)$ is closed $(V_1 - \lambda)$ is a semi-Fredholm operator, and $\text{index}(V_1 - \lambda)$ is constant on D . Since $T - \lambda = V_1 - \lambda + (V - V_1) - V(I - |T|)$, $T - \lambda$ is a semi-Fredholm operator, and $\text{index}(T - \lambda)$ is constant on D and less than ∞ . Thus we have

$$(1.1) \quad \sigma(T) \cap D = \{\sigma_p(T) \cup \overline{\sigma_p(T^*)}\} \cap D.$$

Now we notice that if $\dim N(T^*)$ is finite, then $(T - \lambda)$ is a Fredholm operator for each $\lambda \in D$.

From the definition of C_1 , it follows that

$$(1.2) \quad \sigma_p(T) \cap D = \emptyset \quad \text{for } T \in C_1.$$

In this section we obtain an upper triangulation of T whose diagonal elements were already studied.

The next lemma is trivial, but for the sake of the completeness we prove it.

Lemma 1.1. Let Y be a bounded operator and F a Fredholm

operator such that $FY \in (\tau, c)$. Then we have $Y \in (\tau, c)$.

Proof. There are bounded operators F' and P such that

$$F'F = I - P, \quad \text{range } P = N(F).$$

Thus $(I - P)Y = F'FY \in (\tau, c)$ implies $Y = (I - P)Y + PY \in (\tau, c)$. Q.E.D.

Lemma 1.2. Let T be a contraction with $D(T) \in (\sigma, c)$ and

let

$$(1.3) \quad T = \begin{bmatrix} T_0 & B \\ 0 & T_1 \end{bmatrix}$$

be the decomposition of T such that $T_0 \in C_0$, $T_1 \in C_1$. (see [7]).

Then $D(T_0)$ and $D(T_1)$ are in (σ, c) and B in (τ, c) .

Proof. Since $I - T^*T \in (\tau, c)$,

$$I - T_0^* T_0, \quad B^* T_0 \quad \text{and} \quad I - (B^* B + T_1^* T_1)$$

belong to (τ, c) , where I of " $I - T_0^* T_0$ " is the identity on the

space where T_0 is defined. From next lemma, it follows

that T_0 is a Fredholm operator. Thus, by Lemma 1.1, we have

$B \in (\tau, c)$ and hence $I - T_1^* T_1 \in (\tau, c)$. Q.E.D.

Lemma 1.3. Suppose $T_0 \in C_0$ and $D(T_0) \in (\sigma, c)$, then

T_0 is a Fredholm operator.

Proof. Let

$$(1.4) \quad T_0 = \begin{bmatrix} T_{01} & A \\ 0 & T_0 \end{bmatrix}$$

be the decomposition of T_0 . satisfying $T_{01} \in C_{01}$ and $T_0 \in C_{00}$ ([7]). Since $I - T_0^* T_0 \in (\tau, c)$, $I - T_{01}^* T_{01}$, $A^* T_{01}$ and $I - (A^* A + T_0^* T_0)$ are in (τ, c) too. From (1.2) we have $\sigma_p(T_{01}^*) \cap D = \emptyset$, hence T_{01} is a Fredholm operator. Consequently, from Lemma 1.1, $A \in (\tau, c)$ and hence $I - T_0^* T_0 \in (\tau, c)$. Since $T_0 \in C_{00}$, we have $T_0 \in C_0$ [8], which implies $\dim N(T_0) = \dim N(T_0^*) < \infty$ [7]. Therefore T_0 is a Fredholm operator. Thus

$$T_0 = \begin{bmatrix} T_{01} & 0 \\ 0 & T_0 \end{bmatrix} + \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}$$

is a Fredholm operator.

Q.E.D.

Lemma 1.4. Suppose $T_1 \in C_1$ and $D(T_1) \in (\sigma, c)$ and let

$$T_1 = \begin{bmatrix} T_{11} & F \\ 0 & T_0 \end{bmatrix}$$

be a decomposition of T_1 such that $T_{11} \in C_{11}$, $T_0 \in C_{00}$ ([7]).

Then $D(T_{11})$ and $D(T_0)$ are in (σ, c) and F in (τ, c) , and $T_0 \in C_{10}$.

Proof. $I - T_{11}^* T_{11}$, $F^* T_{11}$ and $I - (F^* F + T_0^* T_0)$ belong to (τ, c) . From (1.2) we have

$$\sigma_p(T_{11}) \cap D = \emptyset \quad \text{and} \quad \sigma_p(T_{11}^*) \cap D = \emptyset,$$

and hence, by (1.1) we have

$$(1.5) \quad \sigma(T_{11}) \cap D = \emptyset.$$

Thus $F \in (\tau, c)$ and hence $I - T_0^* T_0 \in (\tau, c)$. To show $T_0 \in C_{10}$, decompose T_0 as

$$(1.6) \quad T_{\cdot 0} = \begin{bmatrix} T_{00} & F_3 \\ 0 & T_{10} \end{bmatrix},$$

where $T_{00} \in C_{00}$, $T_{10} \in C_{10}$. Then we have $I - T_{00}^* T_{00} \in (\tau, c)$ and hence $T_{00} \in C_0$, from which we get

$$(1.7) \quad \sigma(T_{00}) \cap D \neq D.$$

Denote the space on which $T_{1\cdot}$ is defined by $\mathcal{L}$, and let $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3$ be a decomposition of $\mathcal{L}$ corresponding to

$$T_{1\cdot} = \begin{bmatrix} T_{11} & F_1 & F_2 \\ 0 & T_{00} & F_3 \\ 0 & 0 & T_{10} \end{bmatrix},$$

where $[F_1, F_2] = F$. Set

$$(1.8) \quad T_2 = \begin{bmatrix} T_{11} & F_1 \\ 0 & T_{00} \end{bmatrix}.$$

Then, since $T_2 = T_{1\cdot}|_{\mathcal{L}_1 \oplus \mathcal{L}_2}$, we have $T_2 \in C_1$ and $D(T_2) \in (\sigma, c)$.

Above triangulation of T_2 implies that

$$\sigma(T_2) \subset \sigma(T_{11}) \cup \sigma(T_{00}).$$

From this relation and (1.5), (1.7), it follows that

$$\sigma(T_2) \cap D \neq D.$$

Therefore T_2 is a weak contraction. The $C_0 - C_{11}$ decomposition of T_2 ([7]) implies T_2 has no C_0 -part, because $T_2 \in C_1$, and so $T_2 \in C_{11}$. From (1.8) we have $T_{00}^* = T_2^*|_{\mathcal{L}_2}$, which belongs to C_0 and C_1 ; this is impossible. Thus $\mathcal{L}_2$ reduces to 0, so that from (1.6) we have $T_{\cdot 0} = T_{10} \in C_{10}$. Q.E.D.

Theorem 1.5. Let T be a contraction with $D(T) \in (\sigma, c)$.

Then we have an upper triangulation :

$$T = \begin{bmatrix} T_{01} & & & \\ 0 & T_0 & * & \\ 0 & 0 & T_{11} & \\ 0 & 0 & 0 & T_{10} \end{bmatrix},$$

where $D(T_{01})$, $D(T_0)$, $D(T_{11})$ and $D(T_{10})$ belong to (σ, c) , and $T_{01} \in C_{01}$, $T_0 \in C_0$, $T_{11} \in C_{11}$, $T_{10} \in C_{10}$, and $*$ belongs to (τ, c) .

Proof. At first, decompose T as Lemma 1.2, next decompose T_0 as (1.4). In the proof of Lemma 1.3 we showed that T_{01} and T_0 satisfy the conditions in theorem. At last decompose T_1 as Lemma 1.4 and set $T_{10} = T_{10}$. Q.E.D.

Definition. Above upper triangulation is called the canonical triangulation for T with $D(T) \in (\sigma, c)$.

Remark. We showed that T_{01} and T_0 are Fredholm operators and T_{11} is invertible. But $\dim N(T_{10}^*)$ may be infinite.

2. Eigenvectors

Let T be a contraction on $\mathcal{H}$ with $D(T) \in (\sigma, c)$. Set

$$\alpha = \min \{ \dim N(T-\lambda) : \lambda \in D \} , \quad \beta = \min \{ \dim N(T^*-\lambda) : \lambda \in D \} ,$$

$$i(\lambda) = \dim N(T-\lambda) - \alpha \quad (< \infty) , \quad \Lambda = \{ \lambda \in D : i(\lambda) > 0 \} .$$

Now we note that if a bounded operator A is decomposed as

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} , \quad \text{where } A_1 \text{ is a surjection,}$$

then $\dim N(A) = \dim N(A_1) + \dim N(A_3)$. In fact, we have

$$N(A) = N(A_1) + \{ (-B^{-1}A_2x, x) : x \in N(A_3) \} ,$$

where B is the restriction of A_1 to $N(A_1)^\perp$.

Theorem 2.1. Let T be a contraction with $D(T) \in (\sigma, c)$. And consider the canonical triangulation of T . Then

$$\alpha = \dim N(T_{01}) \quad \text{and} \quad \beta = \dim N(T_{10}^*) .$$

Proof. At first, we notice (1.3). Since $\sigma_p(T_1) \cap D = \emptyset$, it is not difficult to show $N(T-\lambda) = N(T_0-\lambda)$ for $\lambda \in D$. Next we notice (1.4). Since $D(T_{01}) \in (\sigma, c)$ and $\sigma_p(T_{01}^*) \cap D = \emptyset$, $(T_{01}-\lambda)$ is a surjection for each $\lambda \in D$. Thus we have

$$\begin{aligned} (2.1) \quad \dim N(T-\lambda) &= \dim N(T_0-\lambda) = \dim N(T_{01}-\lambda) + \dim N(T_0-\lambda) \\ &= \text{index } (T_{01}-\lambda) + \dim N(T_0-\lambda) = \text{index } T_{01} + \dim N(T_0-\lambda). \end{aligned}$$

$T_0 \in C_0$ implies that $\sigma(T_0) \cap D$ is countable. Hence we have

$$\alpha = \text{index } T_{01} = \dim N(T_{01}) .$$

To show $\beta = \dim N(T_{10}^*)$, take the adjoint of (1.3), that

is

$$T^* = \begin{bmatrix} T_1 . * & B^* \\ 0 & T_0 . * \end{bmatrix} .$$

Since $\sigma_p(T_1 .) \cap D = \emptyset$ and $D(T_1 .) \in (\sigma, c)$, $(T_1 . * - \lambda)$ is a surjection for each $\lambda \in D$. Thus we have

$$\dim N(T^* - \lambda) = \dim N(T_1 . * - \lambda) + \dim N(T_0 . * - \lambda) .$$

From (1.4), it follows that $N(T_0 . * - \lambda) = N(T_0 * - \lambda)$ for $\lambda \in D$, because $\sigma_p(T_{0,1} *) \cap D = \emptyset$. Now we notice the decomposition of $T_1 .$ in Lemma 1.4 and remark that we set $T_{1,0}$ instead of T_0 in the canonical triangulation of T . Since $\sigma_p(T_{1,1} *) \cap D = \emptyset$, it is clear that $N(T_1 . * - \lambda) = N(T_{1,0} * - \lambda)$ for $\lambda \in D$, so that

$$\dim N(T^* - \lambda) = \dim N(T_{1,0} * - \lambda) + \dim N(T_0 * - \lambda) .$$

Consequently we have $\beta = \dim N(T_{1,0} *)$.

Q.E.D.

Corollary 2.2. Let T be a contraction with $D(T) \in (\sigma, c)$.

Then $\sum_{\lambda \in \Lambda} (1 - |\lambda|) \cdot i(\lambda) < \infty$.

Proof. From (2.1), we have $i(\lambda) = \dim N(T_0 - \lambda)$. Thus, by [7]

we can conclude the proof.

Q.E.D.

Theorem 2.3. Let T be a contraction with $D(T) \in (\sigma, c)$.

Then there are holomorphic vector valued functions $h_i(\lambda)$, $f_j(\lambda)$, $(1 \leq i \leq \alpha, 1 \leq j \leq \beta)$ defined on D such that

$$(T - \lambda) h_i(\lambda) \equiv 0 \quad (T^* - \lambda) f_j(\lambda) \equiv 0 ,$$

and for each $\lambda \in D$ $\{h_1(\lambda), \dots, h_\alpha(\lambda)\}$ are linearly independent, also $\{f_1(\lambda), \dots, f_\beta(\lambda)\}$ are. In this case, setting

$\mathcal{L}^\perp = \bigvee \{h_i(\lambda), f_j(\lambda) : i, j, \lambda\}, P_{\mathcal{L}} T|_{\mathcal{L}}$ is a weak contraction.

Proof. We showed that T_{01} in the canonical triangulation of T is a Fredholm operator. Hence

$$T_{01}^*(I - T_{01}T_{01}^*) = (I - T_{01}^*T_{01})T_{01}^* \in (\tau, c)$$

implies, by Lemma 1.1, $D(T_{01}^*) \in (\sigma, c)$. Therefore there is a quasi-affinity X such that $X T_{01}^* = S_E X$, where

$$\dim E = -\text{index } T_{01}^* = \dim N(T_{01}) = \alpha < \infty \quad [9]. \text{ Let}$$

$\{e_1, \dots, e_\alpha\}$ be a C.O.N.B. of E . Then $g_i(\lambda) = \{e_i, \lambda e_i, \lambda^2 e_i, \dots\}$ ($1 \leq i \leq \alpha$) is holomorphic function defined on D with value in $\ell_+^2(E)$. And for each $\lambda \in D$ $\{g_1(\lambda), \dots, g_\alpha(\lambda)\}$ are orthogonal each other. It is trivial to show that

$$(S_E^* - \lambda)g_i(\lambda) \equiv 0, \quad \bigvee_{i, \lambda} g_i(\lambda) = \ell_+^2(E).$$

Since $T_{01}X^* = X^*S_E^*$,

$$h_i(\lambda) = \begin{bmatrix} X^*g_i(\lambda) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (1 \leq i \leq \alpha)$$

satisfy the conditions given in the theorem. Since $T_{10} \in C_{10}$ and $D(T_{10}) \in (\sigma, c)$, there is a quasi-affinity Y such that

$$Y T_{10} = S_F Y, \quad \text{where } \dim F = \beta < \infty.$$

We can show the existence of $f_j(\lambda)$ with the same way as above; hence we omit it. We must show the last assertion. To this end, we notice that $\{h_i(\lambda) : 1 \leq i \leq \alpha, \lambda \in D\}$ and $\{f_j(\lambda) : 1 \leq j \leq \beta, \lambda \in D\}$ span the spaces on which T_{01} and T_{10} , respectively, are defined. Thus, by Theorem 1.5 we have

$$(2.2) \quad P_{\mathcal{L}} T|_{\mathcal{L}} = \begin{bmatrix} T_0 & * \\ 0 & T_{11} \end{bmatrix}.$$

In this case $*$ clearly belongs to (τ, c) . Now we set $T_{\mathcal{L}} = P_{\mathcal{L}} T|_{\mathcal{L}}$. From (2.2), $D(T_0) \in (\sigma, c)$ and $D(T_{11}) \in (\sigma, c)$ imply that $D(T_{\mathcal{L}}) \in (\sigma, c)$. Since T_{11} is invertible, we have

$$\sigma_p(T_{\mathcal{L}}) = \sigma_p(T_0) \quad \sigma_p(T_{\mathcal{L}}^*) = \sigma_p(T_0^*) .$$

$T_0 \in C_0$ implies that $\sigma_p(T_0^*) = \overline{\sigma_p(T_0)} \neq D$ [7]. Thus by (1.1) we have $\sigma(T_{\mathcal{L}}) \cap D = \sigma_p(T_0) = \Lambda \neq D$. Thus $T_{\mathcal{L}}$ is a weak contraction. Q.E.D.

Theorem 2.4. Let T be a contraction with $D(T) \in (\sigma, c)$; then the following are equivalents:

- (a) $\alpha = \beta = 0$;
- (b) T is a weak contraction ;
- (c) T is decomposable (about definition see [2]).

Proof. (a) $\Rightarrow$ (b): From Theorem 2.1. $N(T_{01}) = 0$, which implies T_{01} is a weak contraction. Therefore there is a C_0 - C_{11} decomposition of T_{01} , but it is impossible, because $T_{01} \in C_{01}$. Thus the space on which T_{01} is defined reduces to 0. Similarly the space on which T_{10} is defined reduces to 0. Thus $\mathcal{L}$ in Theorem 2.3 is $\mathcal{H}$. Therefore T is a weak contraction.

(b) $\Rightarrow$ (c): This was shown by Jafarian [5].

(c) $\Rightarrow$ (a): Since decomposable T has the single valued extension

property, $\alpha=0$ follows. Thus for $\lambda \notin \Lambda$, $(T-\lambda)$ is injective semi-Fredholm operator. Hence $\sigma_p(T) \cap D \subset \Lambda$. Thus we have $\sigma(T) \cap D \subset \Lambda$ (see p.30 of [2]). Consequently $\beta=0$. Q.E.D.

Proposition 2.5. Let T be a contraction on $\mathcal{H}$ with $D(T) \in (\sigma, c)$. Then $T \in C_{10}$ if and only if there are vector valued holomorphic functions $h_i(\lambda)$ such that

$$(T^* - \lambda)h_i(\lambda) \equiv 0, \quad \bigvee_{i, \lambda} h_i(\lambda) = \mathcal{H}.$$

Proof. "Only if" part follows from Theorem 2.3 and its proof. We must show "if" part. Since

$$T^{*n}h_i(\lambda) = \lambda^n h_i(\lambda) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

T^{*n} strongly converges to 0 on linear span of $\{h_i(\lambda): i, \lambda\}$.

Suppose

$$T^{*n}x_i \rightarrow 0 \text{ (} n \rightarrow \infty \text{)} \text{ and } x_i \rightarrow x \text{ (} i \rightarrow \infty \text{)}.$$

Since $\|T^{*n}x\| \leq \|T^{*n}x_i\| + \|T^{*n}(x-x_i)\| \leq \|T^{*n}x_i\| + \|x-x_i\|$,

we have $\lim_{n \rightarrow \infty} \|T^{*n}x\| \leq \|x-x_i\|$. Since we can make the right side arbitrary small, $T^{*n}x \rightarrow 0$ ($n \rightarrow \infty$). Thus T belongs to C_0 ,

therefore the canonical triangulation of T becomes

$$T = \begin{bmatrix} T_0 & * \\ 0 & T_{10} \end{bmatrix}.$$

Let P be the orthogonal projection to the space which T_0 is defined on. Then we have

$$0 = P(T^* - \lambda)h_i(\lambda) = P(T^* - \lambda)Ph_i(\lambda) = (T_0^* - \lambda)Ph_i(\lambda).$$

Since $\sigma_p(T_0^*)$ are countable, $\text{Ph}_1(\lambda) \equiv 0$. Consequently $P\mathcal{H} = 0$ and hence $T = T_{10}$. Q.E.D.

Alternately we have

Proposition 2.6. Let T be a contraction on $\mathcal{H}$ with $D(T) \in (\sigma, c)$. Then $T \in C_{01}$ iff there are vector valued holomorphic functions $f_j(\lambda)$ defined on D such that

$$(T - \lambda) f_j(\lambda) \equiv 0 \quad \bigvee_{j \in \mathbb{N}} f_j(\lambda) = \mathcal{H}.$$

3. m -accretive operators

Let A be an m -accretive operator densely defined in $\mathcal{H}$ (about the definition see [6]). Then

$$(3.1) \quad T = (A - I)(A + I)^{-1}$$

is a contraction defined on $\mathcal{H}$ and

$$\sigma_p(T) \not\supset 1 \quad \text{and} \quad T^* = (A^* - I)(A^* + I)^{-1}$$

(see Chap IV of [7]). It is trivial to show that

$$((I - T^*T)h, h) = 4 \operatorname{Re} (A(A + I)^{-1}h, (A + I)^{-1}h) \quad \text{for } h \in \mathcal{H}.$$

Since $A(A + I)^{-1}$ and $(A + I)^{-1}$ are bounded, we have a relation:

$$I - T^*T \in (\tau, c) \iff u(A) \in (\tau, c),$$

where $u(A) = \operatorname{Re}((A^* + I)^{-1}A(A + I)^{-1})$. In this section we denote the

open right half plane by Ω . The mapping

$$\psi: \mu \longrightarrow \frac{\mu - 1}{\mu + 1}$$

transforms Ω onto D . It is clear that

$$(3.2) \quad (A - \mu)x = 0 \iff (T - \psi(\mu))(A + I)x = 0.$$

Set

$$\alpha = \min\{\dim N(A - \mu) : \mu \in \Omega\}, \quad \beta = \min\{\dim N(A^* - \mu) : \mu \in \Omega\},$$

$$i(\mu) = \dim N(A - \mu) - \alpha, \quad \Gamma = \{\mu : i(\mu) > 0\}.$$

Proposition 3.1. Let A be an m -accretive operator densely defined in $\mathcal{H}$. If $u(A) \in (\tau, c)$, then it follows that

$$\sum_{\mu \in \Gamma} \left(\frac{\operatorname{Re} \mu}{1 + |\mu|^2} \right) \cdot i(\mu) < \infty.$$

Proof. Since range of $(A + I)$ is $\mathcal{H}$, by (3.2), we have

$$\dim N(A - \mu) = \dim N(T - \psi(\mu)), \quad \alpha = \min\{\dim N(T - \lambda) : \lambda \in D\},$$

$$\dim N(T - \lambda) - \alpha = \dim N(A - \psi^{-1}(\lambda)) - \alpha = i(\psi^{-1}(\lambda)),$$

$$\{\lambda : i(\psi^{-1}(\lambda)) > 0\} = \psi(\Gamma).$$

Thus from Corollary 2.2, it follows that

$$\sum_{\lambda \in \psi(\Gamma)} (1 - |\lambda|) \cdot i(\psi^{-1}(\lambda)) < \infty$$

so that
$$\sum_{\mu \in \Gamma} (1 - |\psi(\mu)|) \cdot i(\mu) < \infty.$$

Therefore we have

$$\sum_{\mu \in \Gamma} \frac{\operatorname{Re} \mu}{1 + |\mu|^2} \cdot i(\mu) < \infty \quad (\text{cf. p.132 of [4]}).$$

Theorem 3.2. Let A be an m -accretive operator densely defined in $\mathcal{H}$. If $u(A) \in (\tau, c)$, then there are vector valued

holomorphic functions $x_i(\mu), y_j(\mu), (1 \leq i \leq \alpha, 1 \leq j \leq \beta)$ defined on Ω such that

$$(A - \mu) x_i(\mu) \equiv 0 \quad \text{and} \quad (A^* - \mu) y_j(\mu) \equiv 0.$$

Proof. From Theorem 2.3, for T defined by (3.1) there are holomorphic functions $h_i(\lambda)$ ($1 \leq i \leq \alpha$) such that

$$(T - \lambda) h_i(\lambda) \equiv 0.$$

Then $x_i(\mu) = (A + I)^{-1} h_i(\psi(\mu))$

is a holomorphic function defined on Ω , and for each $\mu \in \Omega$ $x_i(\mu)$ belongs to the domain of A . From (3.2), we have

$$(A - \mu) x_i(\mu) \equiv 0.$$

We can similarly show the existence of $y_j(\mu)$ from the alternate relation of (3.2), that is

$$(A^* - \mu)x = 0 \iff (T^* - \psi(\mu))(A^* + I)x = 0. \quad \text{Q.E.D.}$$

4. Weighted unilateral shifts

In this section we study weighted unilateral shifts with (σ, c) -defect operators. Let E be an N -dimensional finite Hilbert space, and A_n ($n=0,1,2,\dots$) invertible contraction on E . Let T be a weighted unilateral shift on $\mathcal{Q}_+^2(E)$ defined by

$$T \{x_0, x_1, \dots\} = \{0, A_0 x_0, A_1 x_1, \dots\}$$

Lemma 4.1. Let B be an invertible operator on E . Then we have

$$\|B^{-1}\| \leq \frac{\|B\|^{N-1}}{|\det B|}, \quad \frac{1}{|\det B|} \leq \|B^{-1}\|^N.$$

Proof. Let $\lambda_1 \geq \dots \geq \lambda_N > 0$ be eigen values of B^*B .

Then we have

$$\|B^{-1}\|^2 = \|(B^*B)^{-1}\| = \frac{1}{\lambda_N} \leq \frac{\lambda_1^{N-1}}{\lambda_1 \dots \lambda_N} = \frac{\|B^*B\|^{N-1}}{\det(B^*B)}.$$

Thus we have

$$\|B^{-1}\| \leq \frac{\|B\|^{N-1}}{|\det B|}.$$

The second inequality similarly follows (cf. p.200 of [3]). Q.E.D.

Now we remember next fact:

for scalar a_n such that $0 < |a_n| < 1$, $\prod_{n=0}^{\infty} |a_n|$ converges
iff $\sum_{n=0}^{\infty} (1 - |a_n|) < \infty$.

Theorem 4.2. Let T be a contractive weighted shift defined above. Then the following are equivalents :

- (a) $T \in C_{10}$;
- (b) $D(T) \in (\sigma, c)$;
- (c) T is similar with simple shift S_E ;
- (d) there is a $\delta > 0$ such that

$$\|A_n \cdots A_0 x\| \geq \delta \|x\| \text{ for every } x \in E \text{ and every } n.$$

Proof. (d) $\Rightarrow$ (c): For each m we have

$$\begin{aligned} \|A_{m+n} \cdots A_m x\| &= \|A_{m+n} \cdots A_m A_{m-1} \cdots A_0 (A_{m-1} \cdots A_0)^{-1} x\| \\ &\geq \delta \| (A_{m-1} \cdots A_0)^{-1} x \| \geq \delta \frac{1}{\|A_m \cdots A_0\|} \|x\| \geq \delta \|x\|, \end{aligned}$$

because each A_i is a contraction. Thus for each $f \in \mathcal{L}_+^2(E)$, we have

$$\|T^n f\| \geq \delta \|f\| \quad \text{for every } n.$$

By the well known Sz.-Nagy's theorem, T is similar with an isometry V . Since T belongs to C_0 , so do V , hence V is a unilateral shift. Since

$$\dim N(V^*) = \dim N(T^*) = \dim E = N$$

dimension of the wandering space for V is N . Thus V is unitarily equivalent with S_E .

(c) $\Rightarrow$ (a): This is obvious.

(a) $\Rightarrow$ (d): Set $\ell(x) = \lim_{n \rightarrow \infty} \|T^n \{x, 0, 0, \dots\}\|$ for $x \in E$.

Since ℓ is continuous and $\ell(x) \neq 0$ for $x \neq 0$, there is a $\delta > 0$ such that

$$\ell(x) \geq \delta \quad \text{for } x \text{ in the unit surface of } E.$$

Since $\ell(\alpha x) = |\alpha| \ell(x)$, we have

$$\lim_{n \rightarrow \infty} \|A_n \cdots A_0 x\| = \ell(x) \geq \delta \|x\| \quad \text{for } x \in E.$$

(b) $\Rightarrow$ (d): From

$$\infty > \|I - T^*T\|_1 = \sum_{n=0}^{\infty} \|I - A_n^* A_n\|_1 \geq \sum_{n=0}^{\infty} \|I - A_n^* A_n\|,$$

it follows that

$$\prod_{n=0}^{\infty} (1 - \|I - A_n^* A_n\|)$$

converges and we denote its limit by δ^2 . In view of

$$\|A_i^{-1}\|^2 = \|(A_i^* A_i)^{-1}\| = \|(I - (I - A_i^* A_i))^{-1}\| \leq \frac{1}{1 - \|I - A_i^* A_i\|},$$

we have

$$\begin{aligned} \|A_n \cdots A_0 x\|^2 &\geq \frac{\|x\|^2}{\|(A_n \cdots A_0)^{-1}\|^2} \geq \frac{\|x\|^2}{\|A_n^{-1}\|^2 \cdots \|A_0^{-1}\|^2} \\ &\geq \prod_{i=0}^n (1 - \|I - A_i^* A_i\|) \|x\|^2 \geq \delta^2 \|x\|^2 \text{ for every } n. \end{aligned}$$

(d) $\Rightarrow$ (b): Since each A_n is an invertible contractive matrix,

$$\begin{aligned} \text{we have } \|I - A_n^* A_n\| &= 1 - \min \{ \lambda : \lambda \in \sigma_p(A_n^* A_n) \} \\ &= 1 - \frac{1}{\|(A_n^* A_n)^{-1}\|} = 1 - \frac{1}{\|A_n^{-1}\|^2} \leq 2 \left(1 - \frac{1}{\|A_n^{-1}\|} \right) \end{aligned}$$

from Lemma 4.1,

$$\leq 2 \left(1 - \frac{|\det A_n|}{\|A_n\|^{N-1}} \right) \leq 2(1 - |\det A_n|).$$

From (d) and Lemma 4.1, we have

$$|\det A_n| \cdots |\det A_0| = |\det (A_n \cdots A_0)|$$

$$\geq \|(A_n \cdots A_0)^{-1}\|^{-N} \geq \delta^N,$$

which implies that $\prod_{n=0}^{\infty} |\det A_n|$ converges, and hence

$$\sum_{n=0}^{\infty} \|I - A_n^* A_n\| \leq 2 \sum_{n=0}^{\infty} (1 - |\det A_n|) < \infty \quad . \text{ Q.E.D. }$$

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