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**Parametric Study
of the Rossby Wave Instability
in a Two-Dimensional Barotropic Disk**

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Abstract

The recent discovery of lopsided structures in protoplanetary disks was so striking that it has attracted much attention among researchers. The lopsided structures are very likely to be produced by large-scale vortices of gas, and one of the promising mechanisms of producing vortices is the Rossby Wave Instability (RWI). Previous works about the RWI have been done in order to understand the formation mechanisms of not only the lopsided structures but also transitional disks. It is also expected that the studies on the RWI will lead to full understanding the disk evolution and the planet formation. However, our knowledge on the RWI itself and its outcomes were very limited. We did not know the *necessary and sufficient* condition for the onset of the RWI nor how the instability evolves in a real space during the linear growth of the RWI. It was also difficult to estimate the properties of vortices formed by the RWI from arbitrary initial conditions.

In this thesis, I study the RWI in two-dimensional (2D), barotropic, and hydrodynamic disks. I first perform linear stability analyses in order to understand its physical nature in the linear regime. I newly establish a methodology to accurately find the configuration that is marginally stable against the RWI and derive the *necessary and sufficient* condition for the onset of the RWI in a semi-analytic form. I find that the physical nature of the RWI can be well explained by the interaction between two Rossby waves in the same manner as that of the shear instability. I then perform the numerical simulations in order to investigate its evolution in the non-linear regime and its final outcomes. The important consequences of its non-linear evolution are that one quasi-stationary vortex (RWI vortex) is finally formed and that the RWI vortices have strong similarity with the steady solution of vortices in the Kepler shear. Further, I obtain empirical relations between the properties of the vortices and the initial conditions. The results of this thesis provide a solid theoretical ground for quantitative interpretation of the observed lopsided structures in protoplanetary disks.

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Chapter 1

Introduction

We are no longer suprised at the discovery of extrasolar plantes. Since Mayor & Queloz (1995) discovered 51 Peg b, more than 3700 extrasolar planets have been found in our neighborhood (The Extrasolar Planets Encyclopedia, 2018). The extrasolar planetary systems are various, and only a few of them are similar to our solar system. Figure 1.1 shows the correlation diagram between planetary mass and semi-major axis of planets. A part of our goals is to explain what causes the diversity of planets and to know whether our solar system is typical or not in the universe.

It is considered that the diversity of planets is partially caused from protoplanetary disks which are birthplaces of planets and constituted with gas and dust particles surrounding young stars. There are two scenarios of the planet formation in protoplanetary disks. One is core-accretion model (Safronov, 1972; Hayashi et al., 1985), and the other is disk instability model (Cameron, 1978). In the core-accretion model, small dust particles ($\sim 0.1 \mu\text{m}$ – 1 cm) coagulate each other and grow up into protoplanets ($\sim 10^3 \text{ km}$) via planetesimals ($\sim 1 \text{ km}$) in protoplanetary disks. Protoplanets grow up into planets through gas accretion or coagulation. In the disk instability model, planets are formed through the gravitational instability of the massive protoplanetary disk. In either model, the structure and evolution of protoplanetary disks have significant effects on the planet formation. Therefore, it is essential to study the structure and evolution of protoplanetary disks for understanding the diversity

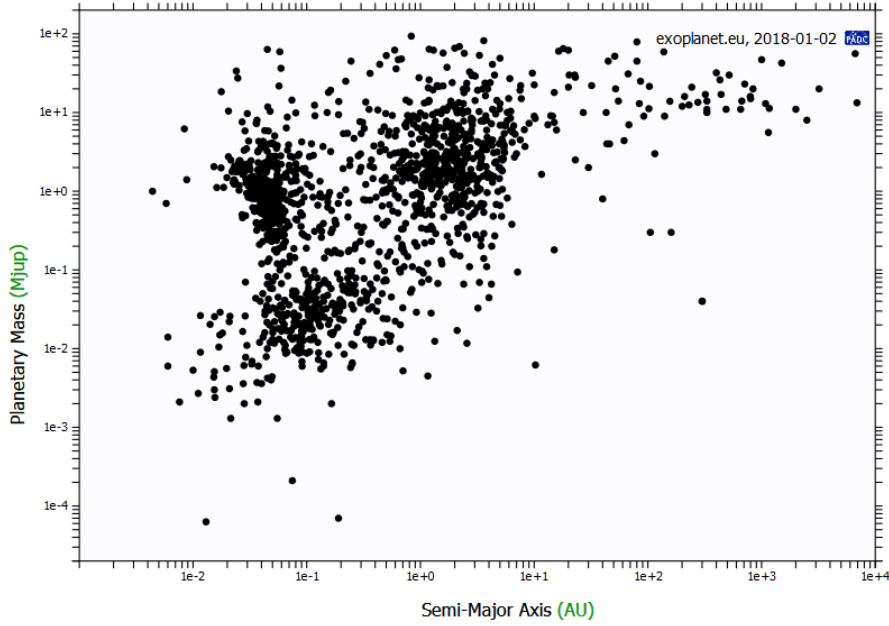


Figure 1.1: The correlation diagram between planetary mass and semi-major axis of extrasolar planets (The Extrasolar Planets Encyclopedia, 2018).

of planets.

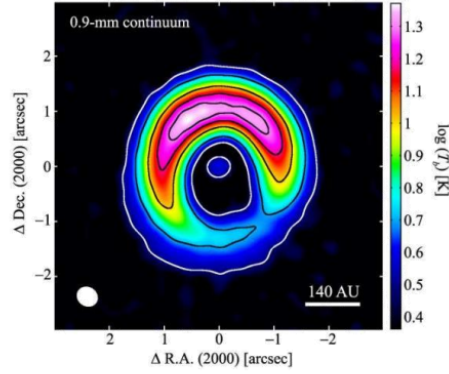
In theoretical works, it is often assumed that a protoplanetary disk has a simple power-law profile for simplicity. It, however, has been revealed that not all protoplanetary disks have such a simple profile since the discovery of the transitional disks of which spectral energy distributions (SEDs) have small near infrared (near-IR) excesses but significant middle and far infrared (mid- and far-IR) excesses (Strom et al., 1989). Since the near-IR traces hot inner part of the disks and the mid- and far-IRs trace cold outer part of the disks, it is considered that transitional disks have inner holes. Recently, the inner holes of transitional disks have been observed directly with various observational instruments (e.g., Subaru, GPI, VLA, ALMA Mayama et al., 2012; Follette et al., 2017; Osorio et al., 2014; van der Marel et al., 2016a). Observations in sub-mm dust continuum emission with Atacama Large Millimeter/sub-millimeter Array (ALMA) have shown various features of protoplanetary disks, such as lopsided structure (e.g., Fukagawa et al., 2013), multiple rings (e.g., ALMA Partnership et al., 2015), and spiral arm (Pérez et al., 2016), as well as the

inner hole (see Figure 1.2). It is important to investigate the formation mechanisms of these features and to understand how these features affect on the disk evolution and the planet formation.

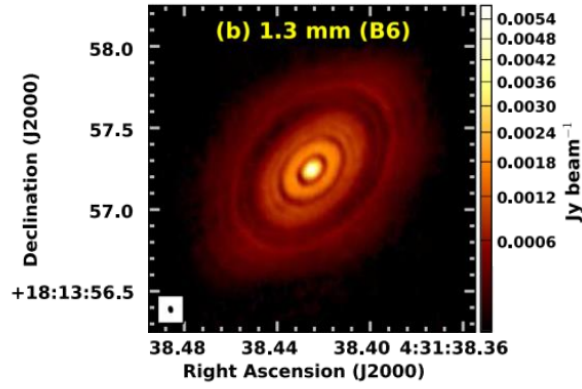
In this thesis, I focus on the lopsided structure in protoplanetary disks. One possible mechanism of producing such lopsided structure is capturing of dust particles ($\sim$ mm in size) by large-scale gas vortices. Theoretically, it has been known that a large-scale vortex of gas can survive for a long time (Godon & Livio, 2000) and can efficiently trap dust particles due to gas drag (e.g., Barge & Sommeria, 1995). Several vortex formation mechanisms are proposed; the Rossby wave instability (RWI) (e.g., Lovelace & Hohlfield, 1978; Lovelace et al., 1999), the baroclinic instability (e.g., Klahr & Bodenheimer, 2003; Lesur & Papaloizou, 2010), the vertical shear instability (e.g., Goldreich & Schubert, 1967; Nelson et al., 2013), the zombie vortex instability (e.g., Marcus et al., 2013) and the pebble accretion onto planets (Owen & Kollmeier, 2017). As far as the observations show, all the protoplanetary disks with lopsided structures are transitional disks. In the case of a transitional disk, the existence of a rapid radial variation of physical quantities (e.g., surface density) is naturally expected. When protoplanetary disks have the rapid radial variations, large-scale vortices of gas are expected to be formed by the RWI. The RWI is one of the most promising mechanisms of explaining the observed lopsided structures.

The linear regime of the RWI has been studied with linear stability analyses (Lovelace et al., 1999; Li et al., 2000; Umurhan, 2010; Lin, 2013; Meheut et al., 2012a). It has been known that the existence of a vortensity local extremum is necessary for the RWI to take place (Lovelace et al., 1999). Tsang & Lai (2008) pointed out that the physical nature of the RWI is explained using the over reflection of waves at co-rotation resonance in the same manner as that of the shear instability or the Drury-Papaloizou-Pringle instability (DPPI: Drury, 1980, 1985; Papaloizou & Pringle, 1984, 1985, 1987; Goldreich et al., 1986; Narayan et al., 1987). However, the necessary and sufficient conditions for the onset of the RWI have not been known. We are ignorant of what happens in a real space during the linear growth of the RWI. The non-linear regime of the RWI has been also studied with numerical simulations (Li et al., 2001; Varnière & Tagger, 2006; Lyra et al., 2008, 2009a; Meheut et al., 2010, 2012b; Richard

lopsided structure (HD142527)



multiple rings (HL Tau)



spiral arm (Elias 2-27)

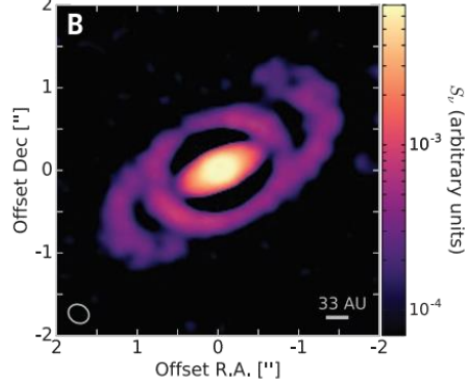


Figure 1.2: Top: 0.89 mm ALMA continuum image of HL142527 (Fukagawa et al., 2013). Middle: 1.3 mm ALMA continuum image of HL Tau (ALMA Partnership et al., 2015). Bottom: 1.3 mm ALMA continuum image of Elias 2-27 of which contrast is increased using an unsharp masking filter (Pérez et al., 2016).

et al., 2013). Nevertheless, our knowledge on the final outcomes of the RWI for various initial profiles is still limited, partially due to lack of systematic parameter survey in 2D cylindrical coordinates.

In this thesis, I study the RWI in 2D, barotropic and hydrodynamic disks with linear stability analyses and numerical simulations. I explore a wide parameter space of initial surface density profiles and the disk temperature. From the linear stability analyses, I newly establish a methodology to accurately find the configuration that is marginally stable against the RWI and derive the *necessary and sufficient* condition for the onset of the RWI in a semi-analytic form. I also find that the physical nature of the RWI can be explained by the interaction between two Rossby waves in the same manner as that of the shear instability. From the numerical simulations, I investigate the properties of the vortices formed by the RWI and obtain some empirical relations between the vortices and the initial conditions.

I first review the previous works about transitional disks and lopsided disks in Chapter 2. I investigate the linear regime of the RWI using linear stability analyses in Chapter 3. I show the physical nature of the RWI in a comparison with that of the shear instability in Chapter 4 and results of my numerical simulations in Chapter 5. I summarize my work and present my future works in Chapter 6.

Chapter 2

Review of Transitional Disks and Lopsided Disks

Recent observations of protoplanetary disks have shown a rich variety of features, such as inner hole, lopsided structure, multiple rings, and spiral arms. Since all protoplanetary disks with the lopsided structures are transitional disks, it is expected that there is a relation between transitional disks and lopsided disks. In this chapter, I review the observations and the possible formation mechanisms of transitional disks and lopsided structures.

2.1 Transitional Disks

Transitional disks are defined with IR excesses in spectral energy distribution (SED). The SED is composed with a stellar component and a disk component. The stellar component is approximated by that of the point source at the stellar surface temperature. As shown in Figure 2.1, the disk component is the superposition of the contributions of all annuli in the multi-color region. Typically, the near- and mid-IR emission comes from inner part of the disk, while the far-IR comes from the outer part of the disk. SEDs of transitional disks have small near-IR excesses but significant mid- and far-IR excesses (Strom et al., 1989). Therefore, transitional disks are considered to have inner holes.

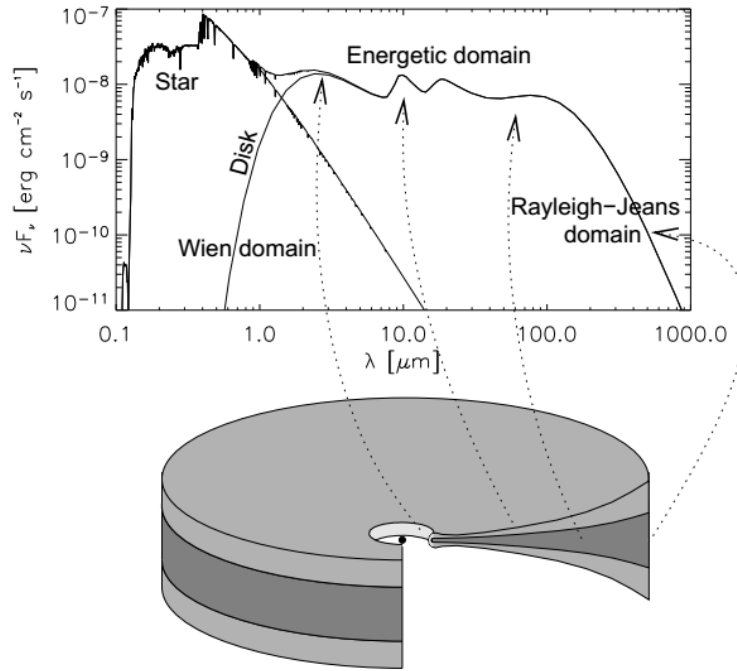


Figure 2.1: SED of a protoplanetary disk and the origin of various components (Dullemond et al., 2007). Typically, the near- and mid-IR emission comes from inner part of the disk, while the far-IR comes from the outer part of the disk. The (sub-)millimeter emission mostly comes from the mid-plane of the outer disk. Scattering is not included here.

Recently, the inner holes have been observed in radio dust continuum using various observational instruments (e.g., SMA, ALMA) as shown in Figures 2.2 and 2.3. The inner holes of some transitional disks have been also observed in radio gas emissions (see Figure 2.3).

Protoplanetary disks have dissipation mechanisms. The lifetime of the disks is estimated at about 1–10 Myr from statistical observations (e.g., Haisch et al., 2001). It is considered that the main dissipation process of protoplanetary disks is viscous accretion due to the MRI turbulence (e.g., Balbus & Hawley, 1998). However, it is difficult to produce various features by only the viscous accretion. The existence of transitional disks suggest that protoplanetary disks drain from the inside out. Since number of the transitional disks is about 10% of that of protoplanetary disks (Espanlat et al., 2014), the period of the

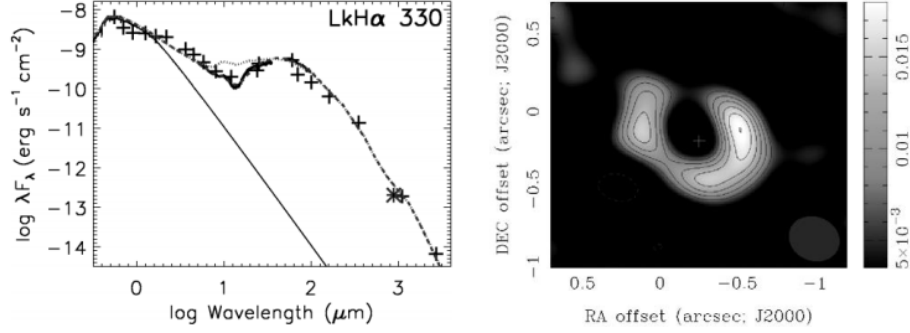


Figure 2.2: SED and SMA 340 GHz dust continuum image of LkHα 330 (Brown et al., 2008).

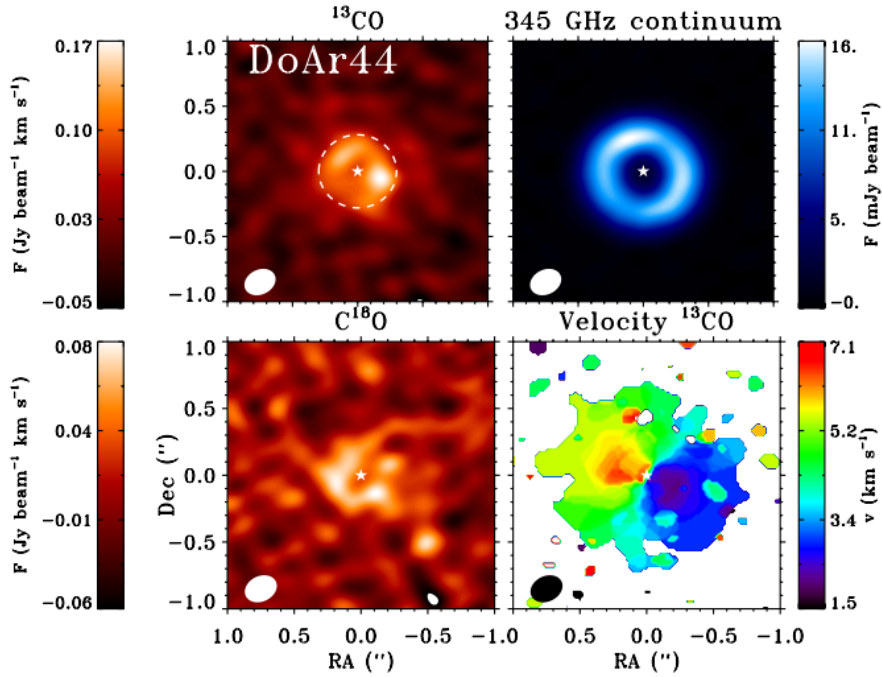


Figure 2.3: ALMA observations of the continuum emission, ^{13}CO and C^{18}O 3–2 line emission of DoAr44 (van der Marel et al., 2016a). Top left: zero-moment ^{13}CO map. Top right: continuum map. Bottom left: zero-moment C^{18}O map. Bottom right: first moment ^{13}CO map (velocity map). The beam is indicated in each map by a white ellipse in the lower left corner. The dotted white ellipse indicates the dust cavity radius.

inside-out clearing is expected to be about 0.1–1 Myr. In order to explain the rapid inside-out clearing of protoplanetary disks, some additional dissipation mechanisms have been proposed. I briefly review the three most possible mechanisms; photoevaporation (e.g., Hollenbach et al., 2000), magnetically driven disk wind (Suzuki & Inutsuka, 2009; Suzuki et al., 2016), and planet-disk interaction (e.g., Zhu et al., 2012).

2.1.1 Photoevaporation

A protoplanetary disks are irradiated by a central star. The central star's energetic radiation (the extreme-ultraviolet, far-ultraviolet, and/or the X-ray) heats columns of gas to of order the escape temperature. Then, the irradiated protoplanetary disk has a photoevaporative mass loss from the surface. The photoevaporative mass loss depends on the distance from the central star and gets most efficient at the critical radius ~ 1 AU (Liffman, 2003). When the photoevaporative mass loss is comparable to the viscous accretion at the critical radius, the inner hole is rapidly formed. Clarke et al. (2001) reported that it is possible to explain the rapid inside-out clearing using the combination between the photoevaporation and the viscous accretion, as shown in Figure 2.4.

Figure 2.4 shows the formation of a inner disk after the photoevaporative mass loss gets significant. The inner disk, however, dissipates by the viscous accretion immediately. Therefore, the photoevaporation model predicts that most transitional disks should have little mass accretion rate on the central star. Nevertheless, most transitional disks observed have larger mass accretion rate than expected (Owen et al., 2011). The inner disks are also discovered in some transitional disks. In these points, the photoevaporation model is inconsistent with the observations of transitional disks. Some ways to overcome these problems have been proposed (Owen et al., 2012, 2013), but those are under discussion.

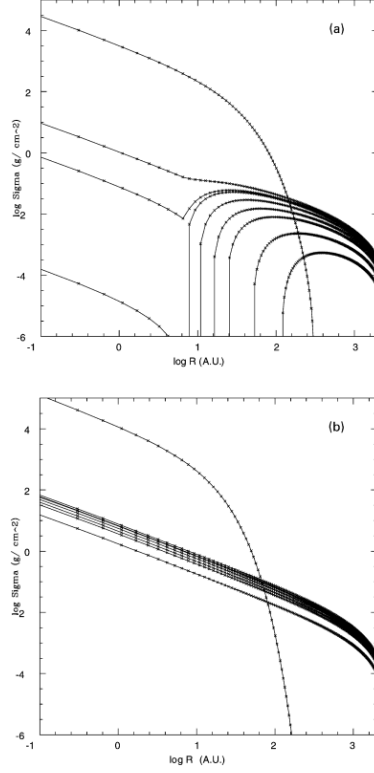


Figure 2.4: The viscous evolution of the surface density distribution in the disk (a) with the photoevaporation and (b) without photoevaporation (Clarke et al., 2001). The curves represent (from the top down), $t = 0, 1.3 \times 10^7, 1.41 \times 10^7, 1.6 \times 10^7, 1.8 \times 10^7, 2.0 \times 10^7, 2.4 \times 10^7, 2.8 \times 10^7$ yr.

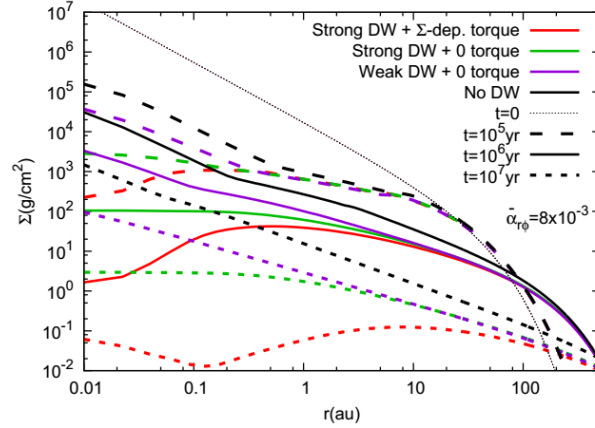


Figure 2.5: Comparison of time evolutions of four MRI-active protoplanetary disks (Suzuki et al., 2016). The four cases are (i) strong disk wind with the torque dependent on density (red), (ii) strong disk wind without the torque dependent on density (green), (iii) weak disk wind without the torque dependent on density (purple), and (iv) no disk wind (black). Radial profiles of surface density at $t = 0$ (dotted lines), 10^5 (long dashed lines), 10^6 (solid lines), and 10^7 (short dashed lines) years.

2.1.2 Magnetically Driven Disk Wind

In protoplanetary disks, vertical magnetic fields are effectively amplified by MRI and winding due to the shear. Large-scale channel flows can develop around the disk surface. By the reconnection of those channel flows, disk wind is produced (Suzuki & Inutsuka, 2009). Suzuki et al. (2016) reported that the magnetically driven disk wind possibly has significant effects on the disk evolution and that the inner hole like that of transitional disks can be formed in the case with the strong disk wind (see Figure 2.5). Due to the uncertainty of vertical magnetic flux, however, it has been not clear whether the observed transitional disks can be explained by the magnetically driven disk wind or not.

2.1.3 Planet-Disk Interaction

A giant planet embedded in a protoplanetary disk creates a gap in surface density distribution of gas (e.g., Lin & Papaloizou, 1979; Goldreich & Tremaine, 1979). The gravity torque of the planet repel gas fluid around the planet. The larger the planetary mass is, the deeper and the wider the planetary gap is due to the gravitational planet-disk interaction. At the edges of the planetary gap, pressure bumps are formed. Since dust particles in a protoplanetary disk move to the local maxima of pressure (Adachi et al., 1976), a gap in surface density distribution of dust is also formed by the dust filtration (Rice et al., 2006).

Zhu et al. (2012) performed 2D two-fluid simulations and compare their numerical results with observations of transitional disks. In their calculations, enough wide gaps are formed when they consider multiple planets or the dust filtration. However, they found a difficulty to explain transitional disks using the planet-disk interaction. Since relative motion of small dust particles ($\lesssim 10 \mu\text{m}$) to gas fluid is slow, the small dust particles remain in the planetary gap. Then, the NIR excess in SED is stronger than that in the observations (see Figure 2.6). Additional physical processes to make a gap of the micron sized dust particles deeper, such as the dust growth, are required. It should

be noted that the planet migration is neglected in their calculations. A giant planet embedded in a protoplanetary disk migrates toward the central star with a planetary gap due to the planet-disk interaction (Type II planet migration Lin & Papaloizou, 1986a,b). In order to explain transitional disks using a giant planet, it is also required to overcome the planet migration problem.

Regardless of some theoretical difficulties, the planet-disk interaction is one of the promising mechanisms. Planetary exploration in the inner holes have been done actively. Sallum et al. (2015) detected $H\alpha$ emission from the inner hole of the LkCa transitional disk. This detection evinces hot ($\sim 10^4$ K) gas falling deep into the potential well, and protoplanets in the inner hole.

2.2 Lopsided Structures of Protoplanetary Disks

Recent ALMA observations in dust continuum emission have revealed protoplanetary disks with the lopsided structure especially in transitional disks (e.g., van der Marel et al., 2013; Fukagawa et al., 2013; Casassus et al., 2015; van der Marel et al., 2016b; Soon et al., 2017). On the other hand, the lopsided structure has not been detected apparently in line emission due to low resolution and sensitivity as shown in Figure 2.7. Muto et al. (2015) modeled the gas distribution of HD 142527, which is a lopsided disk, based on ALMA observations of gas and dust emissions. They find that the dust distribution is strongly non-axisymmetric and the surface density contrast of dust is about 70 at about 170-200 AU. In addition, they concluded that the gas distribution is weakly non-axisymmetric with the surface density contrast of 3-10 and the disk aspect ratio is about 0.1 around 170 AU. Boehler et al. (2017) also estimated the surface density contrast of gas at about 3.75 independently.

Two mechanisms have been proposed in order to explain the observations of the lopsided structures. One is an eccentric disk (Kley & Dirksen, 2006), and the other is a large-scale vortex of gas (e.g., Lyra & Mac Low, 2012). A giant planet embedded in a protoplanetary disk can produce an eccentric gap if the planetary mass is larger than $3 M_J$ and disk viscosity is moderate ($\alpha_\nu \approx 4 \times 10^{-3}$), where α_ν is the kinematic viscosity normalized by the sound

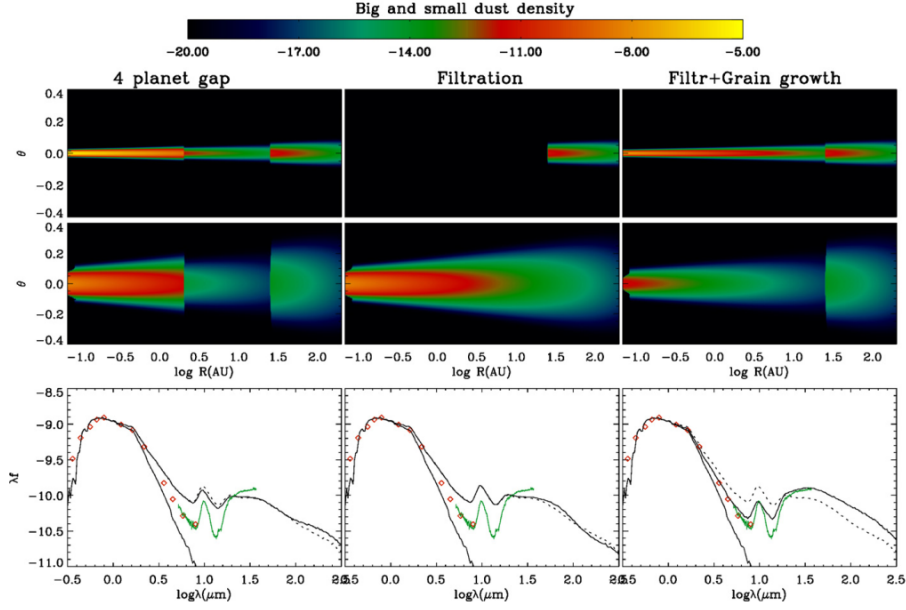


Figure 2.6: Various scenarios to explain transitional disks GM Aur (Zhu et al., 2012). Various scenarios to explain transitional disks GM Aur. (1) A wide gap opened by multiple planets (left panels), (2) a deep gap opened by one planet which can filter large dust particles (middle panels), and (3) after big particles are filtered by the gap, small particles can grow (right panels). The upper panels show the dust density distribution for large particles ($\gtrsim 10 \mu\text{m}$) in the disk (big particles are totally filtered in the second scenario and some new big particles are generated in the third scenario), while the middle panels show the dust density distribution for small particles ($\lesssim 10 \mu\text{m}$) in the disk. The lower panels show the SED from these models. The dotted curve is the SED from a full disk. Photometric (red, open symbols) and IRS data (green) for GM Aur are from (Espaillat et al., 2011).

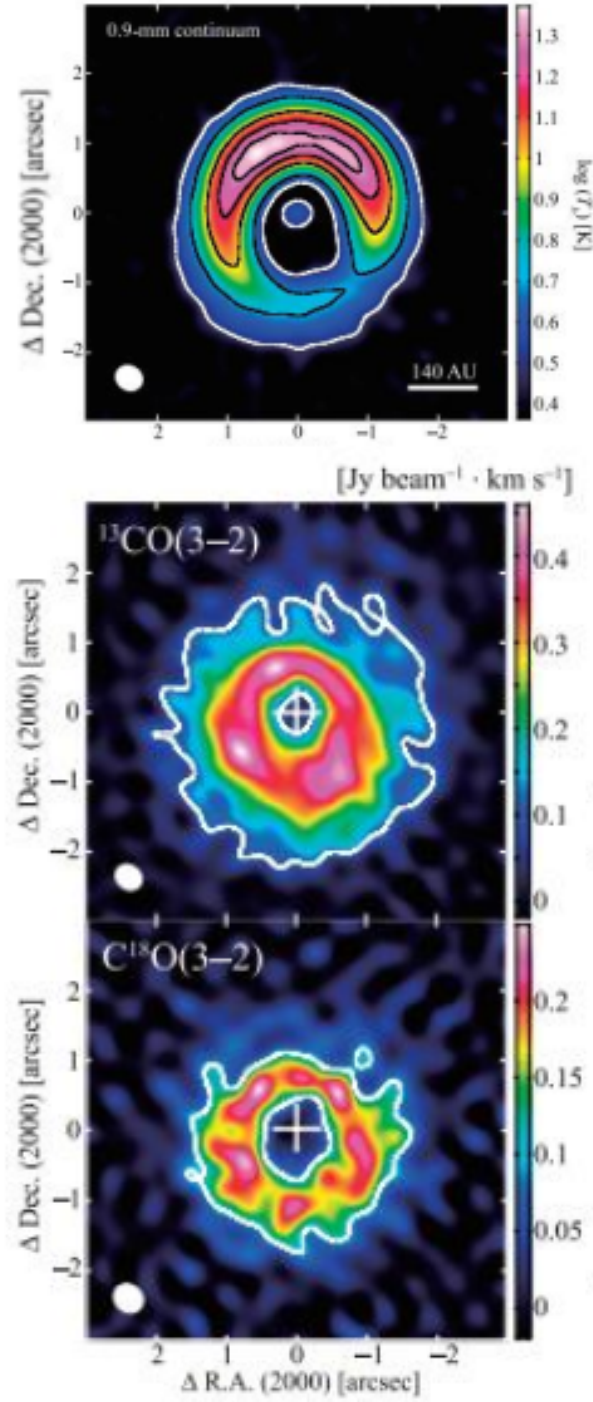


Figure 2.7: ALMA image of HD142527 (Fukagawa et al., 2013). Top: Dust continuum map at 336 GHz (890 μm). Middle: Moment 0 maps in ^{13}CO . Bottom: Moment 0 maps in ^{18}CO .

speed and Kepler time (the α -parameter, Shakura & Sunyaev, 1973). If the disk is eccentric, i.e., fluid particles of gas are on elliptic orbits, rotation velocity of the particles slower at apocenter than at pericenter due to Kepler's laws. Around apocenter, the traffic jam of the fluid particles occur and the surface density is increased. As a result of the traffic jam, it is expected that such a eccentric protoplanetary disk looks lopsided. However, the surface density contrast of dust is almost same as that of gas in a eccentric disk (Ataiee et al., 2013). In this point, it is difficult that the eccentric disk model explains the observed lopsided disks.

Theoretically, it has been known that a anticyclic large-scale vortex of gas can survive for a long time (Godon & Livio, 2000). On the other hand, a cyclic vortex is destroyed by shear only for a few orbits. Assuming the geostrophic balance, the anticyclic vortex can be regarded as a pressure bump. A large-scale vortex of gas can capture large dust particles ($\sim$ mm in size) due to gas drag (e.g., Barge & Sommeria, 1995). Therefore, a large-scale vortex of gas is a promising mechanism to explain the lopsided structures.

The vortex model has also some difficulties. It is known that the viscosity can destroy the vortices (Lin, 2014). For the vortices to survive for a long term, very low viscosity ($\alpha_\nu \lesssim 10^{-4}$) is required, such as the MRI-dead zone (Gammie, 1996). It is reported that the effect of 3D can also destroy vortices by the ellipsoidal instability (Lesur & Papaloizou, 2009). Lesur & Papaloizou (2009) showed the 2D steady vortex is strongly unstable and destroyed for $\chi \leq 4$ and weakly unstable $\chi \gtrsim 6$ with 3D linear stability analyses and local numerical simulations of incompressible flows with the Keplerian shear, where χ is a vortex aspect ratio (see Section 5.2 for details). For $\chi \lesssim 4$, the velocity fields of vortices violate the Rayleigh's condition (Chandrasekhar, 1960, see also Section 3.2.1), and the rotational instability destroys vortices. On the other hand, vortices with $\chi \gtrsim 6$ are destroyed due to the resonance between the turnover motion of vortices and the epicyclic motion of the disk. Richard et al. (2013) performed 3D simulations of vortices in compressible flow. In those simulations, the destruction of vortices with $\chi \lesssim 4$ is verified. However, vortices with $\chi \approx 7$ are not destroyed and survive for enough long time. It is considered that the effects of the dust particles on the gas flow are

negligible in typical protoplanetary disks due to the low dust-to-gas mass ratio ($\sim 10^{-3}$). The vortex captures the dust particles and concentrate them at the vortex center (Barge & Sommeria, 1995). If the high dust-to-gas mass ratio is realized at the vortex center, the vortex is destroyed due to the effects of dusts (Fu et al., 2014b; Crnkovic-Rubsamen et al., 2015). Regardless of these difficulties, the vortex model is still promising.

With a vortex induced by an embedded planet, Zhu & Stone (2014) showed that some ALMA observations can be reproduced with three-dimensional (3D) MHD simulations including dust particles. Several vortex formation mechanisms are proposed; the Rossby wave instability (RWI) (e.g., Lovelace & Hohlfeld, 1978; Lovelace et al., 1999), the baroclinic instability (e.g., Klahr & Bodenheimer, 2003; Lesur & Papaloizou, 2010), the vertical shear instability (e.g., Goldreich & Schubert, 1967; Nelson et al., 2013), the zombie vortex instability (e.g., Marcus et al., 2013) and the pebble accretion onto planets (Owen & Kollmeier, 2017). It is considered that the RWI, the baroclinic instability, and the pebble accretion are promising mechanisms producing large-scale vortices. I briefly review the RWI, the baroclinic instability, and the pebble accretion.

2.2.1 Rossby Wave Instability

The Rossby wave instability (RWI) is one of hydrodynamic instabilities forming vortices in differential rotation disks. When protoplanetary disks have the rapid radial variations of physical quantities (i.e., surface density, pressure, temperature), vortices are formed by the RWI. When we attempt to explain the observed lopsided structures using the vortices formed by the RWI (RWI vortices), the formation mechanism of the rapid radial variation is discussing matter. Note that the details of the RWI are shown in Chapters 3–5.

It is interesting that all the protoplanetary disks with lopsided structures are transitional disks as far as the observations show. As shown in 2.1, transitional disks have inner holes. Transitional disks should have a rapid radial variation of surface density at the edges of the inner holes. It is expected that environment for the onset of the RWI is regulated well in some transitional disks. Some

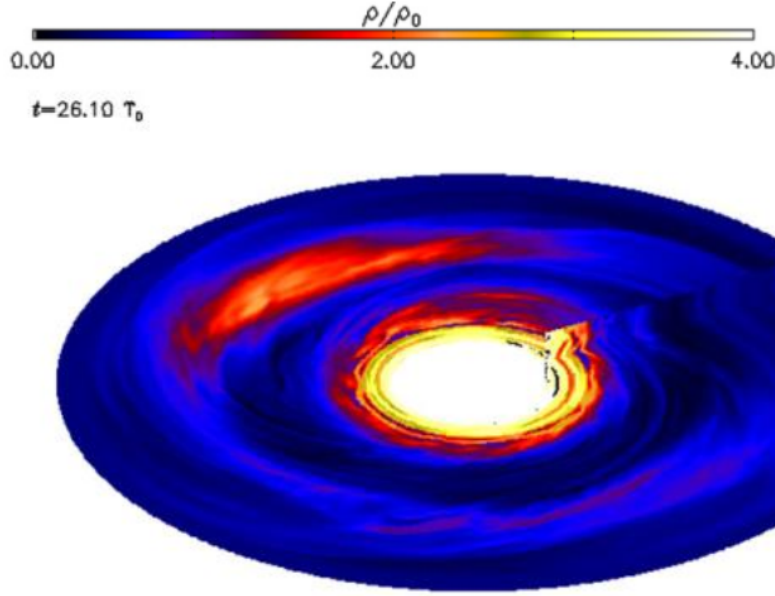


Figure 2.8: Snapshot of the density distribution in the 3D calculations of the RWI induced at the boundary of the dead zones (Lyra & Mac Low, 2012).

numerical simulations have confirmed that the RWI vortices are formed at the edges of a planetary gap (Lyra et al., 2009b; Zhu & Stone, 2014, see Figure 2.6). Other numerical simulations (e.g., Lyra & Mac Low, 2012) also show that the RWI vortices are formed at the boundaries between magnetically active and non-active regions ("dead zones"; Gammie, 1996). Therefore, the RWI is one of the most promising mechanisms to explain the lopsided structures.

2.2.2 Baroclinic Instability

The (global) baroclinic instability was reported by Klahr & Bodenheimer (2003) using a purely numerical approach and considering a disk with a radial entropy gradient. They reported the formation of large-scale vortices, but the actual baroclinic origin of these vortices is unclear since thermal transfer is absent from their calculations. The non-linear nature of this instability was underlined by Lesur & Papaloizou (2010) using incompressible shearing box

simulations. They clarify that the (subcritical) baroclinic instability is basically non-linear and differs from the linear instability that forms cyclones in geophysical flows. Klahr & Hubbard (2014) proposed that a convective overstability could develop following a linear amplification of the epicyclic oscillations. The baroclinic instability can produce a large-scale vortex in a protoplanetary disk with a radial entropy gradient and moderate thermal relaxation times on the order of the orbital frequency. By some numerical simulations (e.g., Lyra, 2014) have confirmed the vortex formation by the baroclinic instability.

2.2.3 Pebble Accretion onto Planets

Planet formation is likely to proceed through pebble accretion (e.g., Ormel & Klahr, 2010). A pebble is a small body (~ 10 cm) and rapidly migrates toward the region with larger pressure (Adachi et al., 1976). If a protoplanetary disk has a planet and a planetary gap, it is expected that the rapid pebble accretion onto the planet makes a large accretion luminosity. This accretion luminosity is sufficient to heat the disk outside the gravitational influence of the planet and makes the disk locally baroclinic, and a source of vorticity (Owen & Kollmeier, 2017). In their numerical simulations (Figure 2.10), the vortex formation is confirmed.

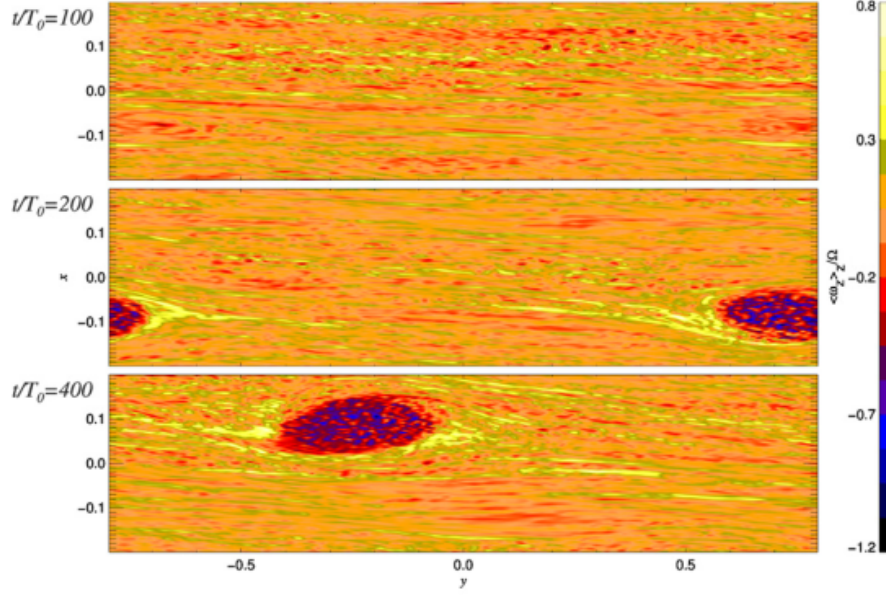


Figure 2.9: Nonlinear evolution of the baroclinic instability in 3D disks (Lyra, 2014). Each panel shows the averaged vertical vorticity distribution.

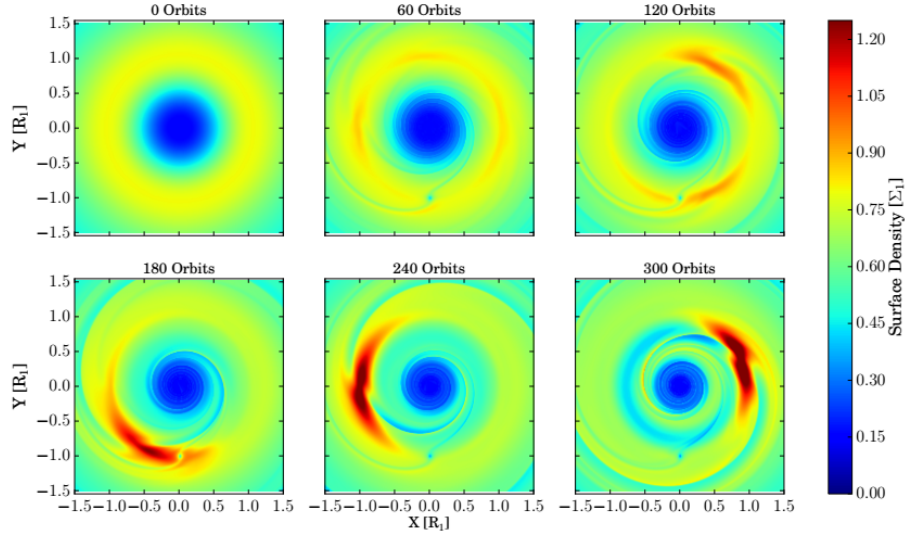


Figure 2.10: Snapshots of the surface density distribution every 60 planetary orbits very low viscosity of $\alpha_\nu = 10^{-6}$ (Owen & Kollmeier, 2017). The planet is located at a position $[0; -1]$.

Chapter 3

The Disk Stability against the RWI

Conditions for the RWI have been studied. Lovelace et al. (1999) derived a necessary condition (Lovelace's condition) for the onset of the RWI. However, the sufficiency of the Lovelace's condition has not been investigated well, and a *necessary and sufficient* condition has not been derived. In this chapter, I investigate the stability against the RWI with linear stability analyses considering various types of background flow on a wider parameter space. I establish a method to accurately find the background flow which is marginally stable against the RWI. As a result, I newly derive the *necessary and sufficient* condition for the onset of the RWI in a semi-analytic form.

I show formulation in Section 3.1, and review the known conditions associated with the RWI in Section 3.2. The results of the linear stability analyses are given in Section 3.3. I investigate marginally stable states against the RWI and derive a new condition for the RWI in Section 3.4. Section 3.5 is the summary of this chapter.

3.1 Formulation

3.1.1 Basic Equations

I consider barotropic and purely hydrodynamic disks orbiting a central star of mass M in global 2D cylindrical coordinates. I assume that the disks are geometrically thin and neglect the effects of magnetic fields, viscosity, and self-gravity. Previous works showed that viscosity, self-gravity, vertical stratification and baroclinicity have some effects on the RWI or the vortices formed by the RWI (Lin, 2012a,b, 2013; Lovelace & Hohlfield, 2013; Lin, 2014; Zhu & Baruteau, 2016; Miranda et al., 2016, 2017). Several numerical calculations of the RWI with dust particles, planets and magnetic fields are also performed (Li et al., 2005; Inaba & Barge, 2006; Lin & Papaloizou, 2011a,b; Lyra & Mac Low, 2012; Regály et al., 2013; Fu et al., 2014a,b; Hammer et al., 2017). However, it is considered that the essential physics of the RWI is captured within the 2D, barotropic and purely hydrodynamic framework.

I employ a non-rotating frame centered on the star and 2D cylindrical coordinates with (r, φ) . The gravitational potential of the central star is denoted by $\Phi(r) = -GM/r$, where G is the gravitational constant. I denote the disk surface density profile by $\Sigma(r, \varphi, t)$ and (vertically integrated) pressure by $P(r, \varphi, t)$. The disk is assumed to be barotropic, i.e., $P = P(\Sigma) \propto \Sigma^\Gamma$, where Γ is the effective adiabatic index of the gas. The adiabatic sound speed $c_s(r, \varphi, t)$ is given by $c_s = \sqrt{\Gamma P / \Sigma}$.

I first show basic equations used in this thesis. The continuity equation is

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r \Sigma v_r) + \frac{1}{r} \frac{\partial}{\partial \varphi} (\Sigma v_\varphi) = 0, \quad (3.1)$$

where t is time, $\mathbf{v}(r, \varphi, t) \equiv v_r(r, \varphi, t) \hat{\mathbf{r}} + v_\varphi(r, \varphi, t) \hat{\boldsymbol{\varphi}}$ is the velocity field, $\hat{\mathbf{r}}$ is the unit vector in the r direction and $\hat{\boldsymbol{\varphi}}$ is the unit vector in the φ direction.

direction. The equations of motion are

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\varphi}{r} \frac{\partial v_r}{\partial \varphi} - \frac{v_\varphi^2}{r} = -\frac{GM}{r^2} - \frac{\partial \Pi}{\partial r}, \quad (3.2)$$

$$\frac{\partial v_\varphi}{\partial t} + v_r \frac{\partial v_\varphi}{\partial r} + \frac{v_\varphi}{r} \frac{\partial v_\varphi}{\partial \varphi} + \frac{v_r v_\varphi}{r} = -\frac{1}{r} \frac{\partial \Pi}{\partial \varphi}, \quad (3.3)$$

where Π is the pressure function. For barotropic flow and $\Gamma \neq 1$, Π is written as

$$\Pi \equiv \frac{\Gamma}{\Gamma - 1} \frac{P}{\Sigma}. \quad (3.4)$$

From equations (3.1)-(3.3), the equation of the vortensity conservation is obtained as

$$\frac{\partial q}{\partial t} + v_r \frac{\partial q}{\partial r} + \frac{v_\varphi}{r} \frac{\partial q}{\partial \varphi} = 0, \quad (3.5)$$

where $q(r, \varphi, t) \equiv (\text{rot } \mathbf{v})_z / \Sigma$ is the vortensity.

It should be noted that I have assumed, for simplicity, that the barycenter of the system always coincide with the central star, and is located at the origin. This means that the indirect term of the gravitational potential, which arises from the deviation of the barycenter of the system from the position of the central star, is negligible. The indirect term can have a significant effect on the disk morphology if the disk is massive (e.g., Mittal & Chiang, 2015; Zhu & Baruteau, 2016). In this thesis, I restrict myself to consider low-mass disks.

3.1.2 Background Profiles

Here, I define profiles of the background disks in order to perform linear stability analyses of the RWI. I adopt stationary ($\partial/\partial t = 0$), axisymmetric ($\partial/\partial \varphi = 0$) and circular ($v_r = 0$) flow as the background profiles, which are denoted by subscript ‘0’, e.g., $\Sigma_0(r)$, $P_0(r)$ and $\mathbf{v}_0(r) = v_{\varphi 0}(r) \hat{\varphi}$.

I assume that the background profiles have radially smooth “base” profiles and rapid radial variations. As a “base” profile, the surface density is given by a simple power-law profile with the index $-f$ as $\Sigma_b(r) = \Sigma_n (r/r_n)^{-f}$, where r_n is a representative radius and Σ_n is the surface density of this base disk model at $r = r_n$. As a form of Σ_0/Σ_b , I consider four representative cases,

namely, step jump up (SJU), step jump down (SJD), Gaussian bump (GB), and Gaussian gap (GG). All of them are represented by two parameters: radial contrast $\mathcal{A}_0$ and radial half width Δw_0 . The specific forms of these four profiles are as follows.

- Step Jump Up (SJU)

Σ_0/Σ_b is given by

$$\frac{\Sigma_0}{\Sigma_b} = 1 + \frac{\mathcal{A}_0}{2} \left[\tanh \left(\frac{r - r_n}{\Delta w_0} \right) + 1 \right]. \quad (3.6)$$

The variation of Σ_0/Σ_b is a smooth jump up from 1 to $(\mathcal{A}_0 + 1)$ over $2\Delta w_0$ around $r = r_n$. I call this the step jump up (SJU) type. The SJU type is same as the "HSJ" type in Li et al. (2000).

- Step Jump Down (SJD)

Σ_0/Σ_b is given by

$$\frac{\Sigma_0}{\Sigma_b} = 1 + \frac{\mathcal{A}_0}{2} \left[1 - \tanh \left(\frac{r - r_n}{\Delta w_0} \right) \right]. \quad (3.7)$$

The variation of Σ_0/Σ_b is a smooth jump down from $(\mathcal{A}_0 + 1)$ to 1 over $2\Delta w_0$ around $r = r_n$. I call this the step jump down (SJD) type. It is noted that the SJD type is not equivalent to the SJU type due to curvature effects even if Σ_b is constant.

- Gaussian Bump (GB)

Σ_0/Σ_b is given by

$$\frac{\Sigma_0}{\Sigma_b} = 1 + \mathcal{A}_0 \exp \left[-\frac{1}{2} \left(\frac{r - r_n}{\Delta w_0} \right)^2 \right]. \quad (3.8)$$

The variation of Σ_0/Σ_b is a Gaussian bump whose height is $\mathcal{A}_0$ and radial half width is Δw_0 , and the peak of the bump is located at $r = r_n$. I call this the Gaussian bump (GB) type. The GB type is intrinsically same as the "HGB" type in Li et al. (2000). However, equation (3.8) is slightly different from the equation (4) in Li et al. (2000) about the definition of

$\mathcal{A}_0$.

- Gaussian Gap (GG)

Σ_0/Σ_b is given by

$$\frac{\Sigma_0}{\Sigma_b} = (\mathcal{A}_0 + 1) - \mathcal{A}_0 \exp \left[-\frac{1}{2} \left(\frac{r - r_n}{\Delta w_0} \right)^2 \right]. \quad (3.9)$$

The variation of Σ_0/Σ_b is a Gaussian gap whose depth is $\mathcal{A}_0$ and radial half width is Δw_0 , and the minimum of the gap is located at $r = r_n$. I call this the Gaussian gap (GG) type.

Since I consider barotropic flow, the relation between P_0 and Σ_0 is written as

$$\frac{P_0}{\Sigma_0^\Gamma} = \frac{P_n}{\Sigma_n^\Gamma} = S_0 = \text{constant.}, \quad (3.10)$$

where P_n is the (vertically integrated) pressure of the base disk model at $r = r_n$, and S_0 is the entropy and constant. Since I consider the disk to be in vertically hydrostatic equilibrium in terms of the structure in the vertical (z) direction, the scale height $H_0(r)$ is given by $H_0(r) = c_{s0}/\Omega_0$, where $c_{s0}(r) \equiv \sqrt{\Gamma P_0/\Sigma_0}$ is background adiabatic sound speed and $\Omega_0(r) \equiv v_{\varphi 0}/r$ is background rotation velocity. I define the dimensionless parameter h by

$$h \equiv \frac{\sqrt{\Gamma P_n/\Sigma_n}}{r_n \Omega_n}, \quad (3.11)$$

where $\Omega_K(r) \equiv \sqrt{GM/r^3}$ is the Keplerian rotation velocity, and $\Omega_n \equiv \Omega_K(r_n)$ is the rotation velocity of the base disk model with $\mathfrak{f} = 0$ at $r = r_n$. The parameter h equals the disk aspect ratio (scale height divided by radius) of the base model at $r = r_n$. It is noted that h can be regarded as the dimensionless scale height, or, equivalently, the dimensionless sound speed. For the background disks, equation (3.2) is rewritten as a equation of the radial force balance;

$$v_{\varphi 0}(r) = \sqrt{v_K^2 + r \frac{d\Pi_0}{dr}}, \quad (3.12)$$

or

$$\left(\frac{\Omega_0}{\Omega_K}\right)^2 = 1 + h^2 \left(\frac{r^2}{r_n}\right) \left(\frac{\Sigma_0}{\Sigma_n}\right)^{\Gamma-2} \left(\frac{\Sigma_0}{\Sigma_n}\right)', \quad (3.13)$$

where $\Pi_0(r) \equiv \Gamma S_0 \Sigma_0^{\Gamma-1} / (\Gamma - 1)$ is the pressure function of the background flow, and the prime denotes d/dr . When h , Γ and Σ_0 are given, it is possible to obtain P_0 , c_{s0} , S_0 , $v_{\varphi 0}$ and Ω_0 using equations (3.10)–(3.13). The square of the background epicyclic frequency $\kappa_0^2(r)$;

$$\kappa_0^2(r) \equiv \frac{(r^4 \Omega_0^2)'}{r^3}, \quad (3.14)$$

and the background vortensity $q_0(r)$;

$$q_0(r) = \frac{(\text{rot } \mathbf{v}_0)_z}{\Sigma_0} = \frac{\kappa_0^2}{2\Sigma_0\Omega_0} \quad (3.15)$$

can be also calculated.

The background profiles are characterized by five parameters, $(h, \Gamma, \mathfrak{f}, \mathcal{A}_0, \Delta w_0)$. I fix $\mathfrak{f} = 0$ and explore the parameter space for the four other quantities. First, I fix $h = 0.1$ and $\Gamma = 5/3$, and vary $\mathcal{A}_0$ and Δw_0 for the four types (SJU, SJD, GB, and GG) of the background flow in cases (i)–(iii) and (vi). To see the effect of h and Γ , I calculate $h = 0.2$ case and $\Gamma = 1$ case for the GB type in cases (iv) and (v), respectively. Note that $\Gamma = 1$ indicates the case of isothermal disks. I investigate the stability of the back ground profiles against the RWI in each case and for each azimuthal mode ($1 \leq m \leq 10$). Table 3.1 shows the parameter space explored in my linear stability analyses.

Figure 3.1 shows the radial distribution of Σ_0/Σ_b , Ω_0/Ω_K , κ_0^2/Ω_K^2 , and q_0 and q'_0/q_0 for the SJU type with $h = 0.1$, $\Gamma = 5/3$, $\mathcal{A}_0 = 0.5$, and $\Delta w_0 = 0.05r_n$. In like manner, Figure 3.2 is for the SJD type with $\mathcal{A}_0 = 0.5$, Figure 3.3 is for the GB type with $\mathcal{A}_0 = 0.2$, and Figure 3.4 is for the GG type with $\mathcal{A}_0 = 0.2$.

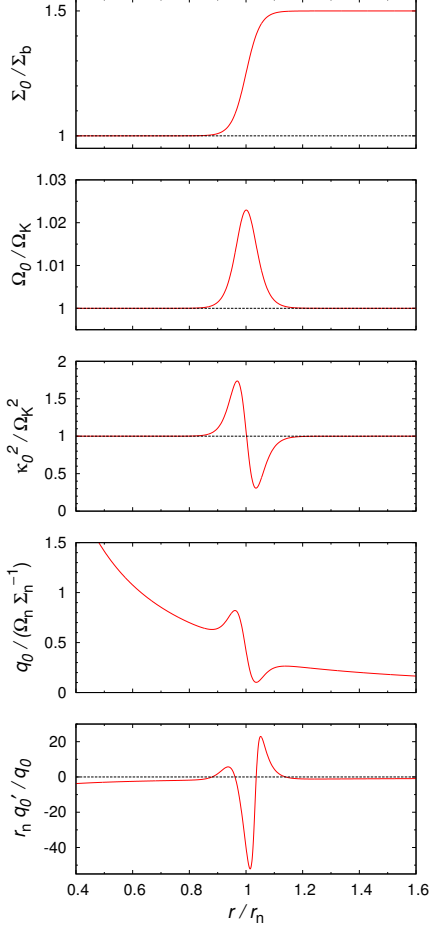


Figure 3.1: The radial distribution of Σ_0/Σ_b , Ω_0/Ω_K , κ_0^2/Ω_K^2 , $q_0/(\Omega_n \Sigma_n^{-1})$ and $r_n q'_0/q_0$ are shown for the step jump up (SJU) type with $h = 0.1$, $\Gamma = 5/3$, $\mathcal{A}_0 = 0.5$, and $\Delta w_0 = 0.05r_n$. The variation of Σ_0/Σ_b is a smooth jump up from 1 to $(\mathcal{A}_0 + 1)$ over $2\Delta w_0$ around $r = r_n$.

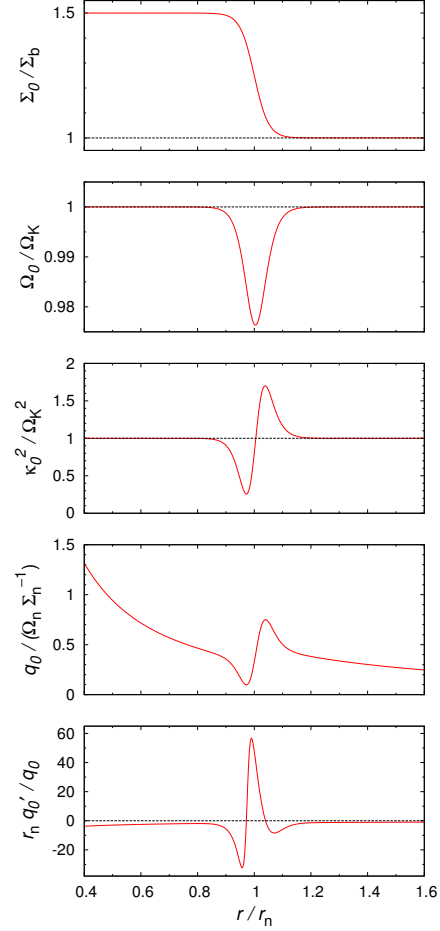


Figure 3.2: Similar to Figure 3.1, but for the step jump down (SJD) type with $h = 0.1$, $\Gamma = 5/3$, $\mathcal{A}_0 = 0.5$, and $\Delta w_0 = 0.05r_n$. The variation of Σ_0/Σ_b is a smooth jump down from $(\mathcal{A}_0 + 1)$ to 1 over $2\Delta w_0$ around $r = r_n$.

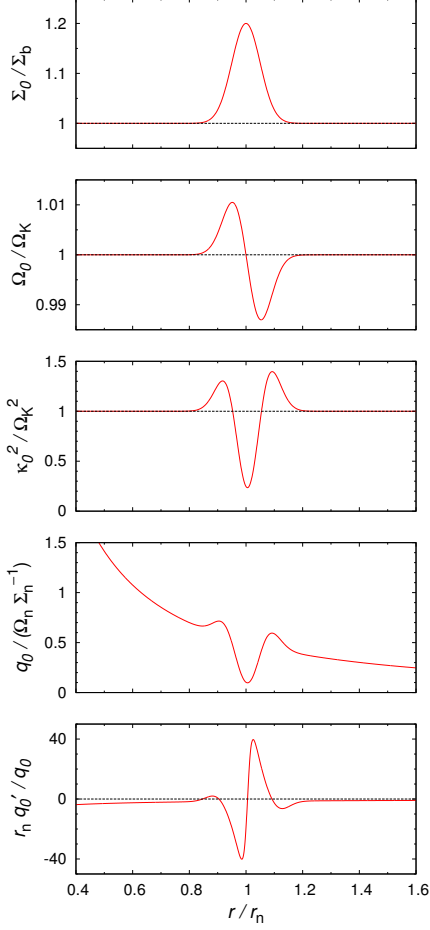


Figure 3.3: Similar to Figure 3.1, but for the Gaussian bump (GB) type with $h = 0.1$, $\Gamma = 5/3$, $\mathcal{A}_0 = 0.2$, and $\Delta w_0 = 0.05r_n$. The variation of Σ_0/Σ_b is a Gaussian bump whose height is $\mathcal{A}_0$ and radial half width is Δw_0 , and the peak of the bump is at $r = r_n$.

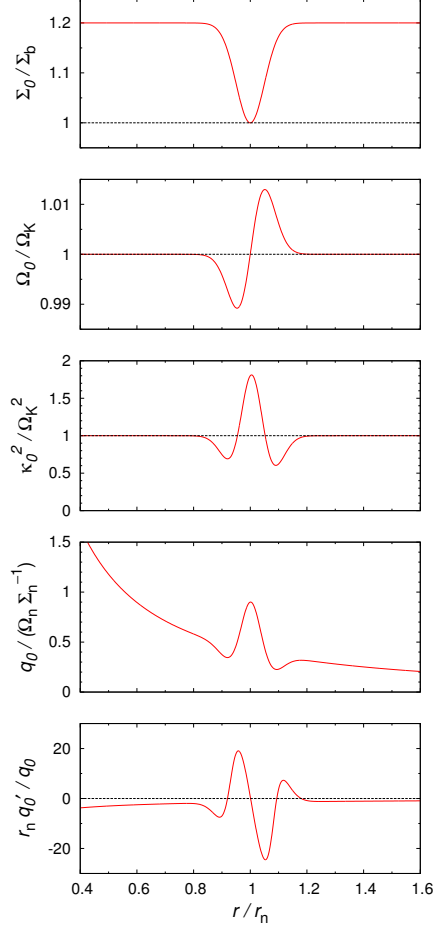


Figure 3.4: Similar to Figure 3.1, but for the Gaussian gap (GG) type with $h = 0.1$, $\Gamma = 5/3$, $\mathcal{A}_0 = 0.2$, and $\Delta w_0 = 0.05r_n$. The variation of Σ_0/Σ_b is a Gaussian gap whose depth is $\mathcal{A}_0$ and width is Δw_0 , and the minimum of the gap is at $r = r_n$.

Table 3.1: The parameter space for each case.

Case	Type	h	Γ	$\mathcal{A}_0$	$\Delta w_0/r_n$	m
i	SJU	0.1	5/3	0.001–10.0	0.02–0.25	1–10
ii	SJD	0.1	5/3	0.001–10.0	0.02–0.25	1–10
iii	GB	0.1	5/3	0.001–10.0	0.02–0.25	1–10
iv	GB	0.1	1.0	0.001–10.0	0.02–0.25	1–10
v	GB	0.2	5/3	0.001–2.0	0.02–0.25	1–10
vi	GG	0.1	5/3	0.001–10.0	0.02–0.25	1–10

3.1.3 Perturbation Equations

I perform linear stability analyses and use subscript "1" to denote perturbations, e.g., $\Sigma_1(r, \varphi, t)$, $P_1(r, \varphi, t)$ and $\mathbf{v}_1(r, \varphi, t)$, where $\mathbf{v}_1(r, \varphi, t) = v_{r1}(r, \varphi, t)\hat{\mathbf{r}} + v_{\varphi1}(r, \varphi, t)\hat{\boldsymbol{\varphi}}$. The first-order forms of equations (3.1)–(3.3) are

$$\begin{aligned}
\left(\frac{\partial}{\partial t} + \Omega_0 \frac{\partial}{\partial \varphi}\right) \Sigma_1 + \frac{1}{r} \frac{\partial}{\partial r} (r \Sigma_0 v_{r1}) + \frac{\Sigma_0}{r} \frac{\partial v_{\varphi1}}{\partial \varphi} &= 0, \\
\left(\frac{\partial}{\partial t} + \Omega_0 \frac{\partial}{\partial \varphi}\right) v_{r1} - 2\Omega_0 v_{\varphi1} + \frac{1}{\Sigma_0} \frac{\partial P_1}{\partial r} - \frac{1}{\Sigma_0^2} P'_0 \Sigma_1 &= 0, \\
\left(\frac{\partial}{\partial t} + \Omega_0 \frac{\partial}{\partial \varphi}\right) v_{\varphi1} + \frac{\kappa_0^2}{2\Omega_0} v_{r1} + \frac{1}{r \Sigma_0} \frac{\partial P_1}{\partial \varphi} &= 0.
\end{aligned} \tag{3.16}$$

Since I assume the barotropic approximation,

$$P_1 = c_{s0}^2 \Sigma_1 \tag{3.17}$$

is satisfied. I consider perturbations as

$$\begin{aligned}
\Sigma_1(r, \varphi, t) &= \Sigma_1(r) \exp(im\varphi - i\omega t), \\
P_1(r, \varphi, t) &= P_1(r) \exp(im\varphi - i\omega t), \\
v_{r1}(r, \varphi, t) &= v_{r1}(r) \exp(im\varphi - i\omega t), \\
v_{\varphi1}(r, \varphi, t) &= v_{\varphi1}(r) \exp(im\varphi - i\omega t),
\end{aligned} \tag{3.18}$$

where $m(= 1, 2, 3, \dots)$ is the azimuthal mode and ω is the mode frequency. In general, ω is a complex number and written as $\omega \equiv \omega_r + i\gamma$. The linearized

forms of equations (3.16) are

$$i\Delta\omega\Sigma_1 = \left(\frac{\Sigma_0}{r} + \Sigma'_0\right)v_{r1} + \Sigma_0v'_{r1} + ik_\varphi\Sigma_0v_{\varphi1}, \quad (3.19)$$

$$i\Delta\omega v_{r1} + 2\Omega_0v_{\varphi1} = \frac{P'_1}{\Sigma_0} - \frac{P'_0}{\Sigma_0^2}\Sigma_1, \quad (3.20)$$

$$i\Delta\omega v_{\varphi1} - \frac{\kappa_0^2}{2\Omega_0}v_{r1} = ik_\varphi\frac{P_1}{\Sigma_0}, \quad (3.21)$$

where $\Delta\omega(r, \omega, m) \equiv \omega - m\Omega_0$ and $k_\varphi(r, m) \equiv m/r$.

I define enthalpy perturbation by $\mathfrak{E}(r) \equiv P_1/\Sigma_0$. It is noted that I omit the subscript "1" from $\mathfrak{E}$ even though $\mathfrak{E}$ is perturbation. Using $\mathfrak{E}' = P'_1/\Sigma_0 - \mathfrak{E}\Sigma'_0/\Sigma_0$, I obtain

$$i\Delta\omega v_{r1} + 2\Omega_0v_{\varphi1} = \mathfrak{E}', \quad (3.22)$$

$$i\Delta\omega v_{\varphi1} - \frac{\kappa_0^2}{2\Omega_0}v_{r1} = ik_\varphi\mathfrak{E}, \quad (3.23)$$

from equations (3.19)–(3.21). Solving equations (3.22)–(3.23) for v_{r1} and $v_{\varphi1}$, I have

$$\Sigma_0v_{r1} = i\mathcal{F}\left(\frac{\Delta\omega}{\Omega_0}\mathfrak{E}' - 2k_\varphi\mathfrak{E}\right), \quad (3.24)$$

$$\Sigma_0v_{\varphi1} = \mathcal{F}\left(\frac{\kappa_0^2}{2\Omega_0^2}\mathfrak{E}' - k_\varphi\frac{\Delta\omega}{\Omega_0}\mathfrak{E}\right), \quad (3.25)$$

where

$$\mathcal{F}(r, \omega, m) \equiv \frac{\Sigma_0\Omega_0}{\kappa_0^2 - \Delta\omega^2}. \quad (3.26)$$

The perturbation equation for $\mathfrak{E}$ is derived from equations (3.17), (3.19), (3.24)–(3.25) as

$$\frac{1}{r}\left(\frac{r\mathcal{F}}{\Omega_0}\mathfrak{E}'\right)' - \frac{k_\varphi^2\mathcal{F}}{\Omega_0}\mathfrak{E} = \left(\frac{\Sigma_0}{c_{s0}^2} + \frac{2k_\varphi\mathcal{F}'}{\Delta\omega}\right)\mathfrak{E}. \quad (3.27)$$

This perturbation equation corresponds to the equation (14) in Li et al. (2000). Equation (3.27) can be written as

$$\mathfrak{E}'' + B(r, \omega, m)\mathfrak{E}' + C(r, \omega, m)\mathfrak{E} = 0, \quad (3.28)$$

where

$$B(r, \omega, m) \equiv \frac{1}{r} + \frac{\mathcal{F}'}{\mathcal{F}} - \frac{\Omega'_0}{\Omega_0}, \quad (3.29)$$

$$C(r, \omega, m) \equiv -k_\varphi^2 - \frac{\kappa_0^2 - \Delta\omega^2}{c_{s0}^2} - 2k_\varphi \frac{\Omega_0}{\Delta\omega} \frac{\mathcal{F}'}{\mathcal{F}}. \quad (3.30)$$

When I define

$$\Xi(r) \equiv \sqrt{r \frac{\mathcal{F}}{\Omega_0}} \mathfrak{E}, \quad (3.31)$$

equation (3.28) can be rewritten as

$$\Xi'' - D(r, \omega, m)\Xi = 0, \quad (3.32)$$

where

$$D(r, \omega, m) \equiv \frac{B'}{2} + \frac{B^2}{4} - C. \quad (3.33)$$

Equation (3.32) is similar to the stationary Schrödinger equation with zero energy, and $D(r, \omega, m)$ can be regarded as the effective potential. The region where $\text{Re}[D] > 0$ is an evanescent region and the region where $\text{Re}[D] < 0$ is a wave propagation region (Goldreich & Tremaine, 1979). I also omit the subscript "1" from Ξ even though Ξ is perturbation.

I solve equation (3.28) or (3.32) under appropriate boundary conditions (see Section 3.1.4) at the outer and inner radii of the disk. Now, the mode frequency $\omega \equiv \omega_r + i\gamma$ is regarded as a complex eigenvalue. From the real part of the mode frequency, ω_r , I can calculate the co-rotation radius r_c by solving $\Omega_0(r_c) = \omega_r/m$. The imaginary part of the mode frequency, γ , gives the linear growth rate for the m mode against the RWI if γ is positive. For $\gamma = 0$, the mode is marginally stable against the RWI. I obtain the eigenfunction, $\mathfrak{E}(r)$ or $\Xi(r)$, as well as the eigenvalue ω . From $\mathfrak{E}(r)$ or $\Xi(r)$, other perturbations (e.g., Σ_1 , P_1 , v_{r1} , and $v_{\varphi1}$) are evaluated. Discretizing equation in the radial direction with N grids and using matrix inversion (see Section 3.1.5), I solve equation (3.28) or (3.32). The inner and outer boundaries are taken at $r_{\text{in}} = 0.3r_n$ and $r_{\text{out}} = 3.0r_n$. I adopt $N = 3000$ unless otherwise stated.

Figure 3.5 shows the real part of the effective potential $\text{Re}[D]$ for the base disk model with $\mathfrak{f} = 0$ and $r_c = r_n$ in the cases of $m = 1$ and $m = 2$. Turning

points of $\text{Re}[D]$, where $\text{Re}[D] = 0$ is satisfied, except in the vicinity of the co-rotation radius are the effective Lindblad resonances (ELRs; Artymowicz, 1993). Outer one is the outer effective Lindblad resonance (OELR), and inner one is the inner effective Lindblad resonance (IELR). It is noted that the IELR does not exist for $m = 1$. The case of higher azimuthal mode ($m = 3, 4, \dots$) is similar to the case of $m = 2$, but the ELRs are closer to the co-rotation radius. For the base disk models where the radial surface density distribution is given by a simple power-law profile, the region between the ELRs is an evanescent region except around the singularity at the co-rotation radius. When the background profiles have a rapid radial variation, the region between the ELRs is not always an evanescent region (see Section 3.4.1). On the other hand, the regions inside the IELR and outside the OELR are always wave propagation regions.

3.1.4 Boundary Conditions

To solve equation (3.28) or (3.32), appropriate boundary conditions should be set. I adopt the outgoing boundary conditions. The region that is sufficiently far away from the transition of the background surface density Σ_0 , $r = r_n$, is a wave propagation region except for the inner region of $m = 1$. I first consider boundary conditions except in the case of the inner boundary for $m = 1$. When I choose the inner boundary inside the IELR and the outer boundary outside the OELR, I can use the Wentzel–Kramers–Brillouin–Jeffreys (WKBJ) solution of the form $\mathfrak{E}(r) \propto \exp[i\mathfrak{g}(r)]$. When I substitute this into equation (3.28) and neglect the term which includes $\mathfrak{g}''$,

$$\mathfrak{g}'^2 - iB\mathfrak{g}' - C = 0 \quad (3.34)$$

is obtained. From equation (3.34), $\mathfrak{g}'$ is solved as

$$\mathfrak{g}'(r) = \frac{iB \pm \sqrt{4C - B^2}}{2}. \quad (3.35)$$

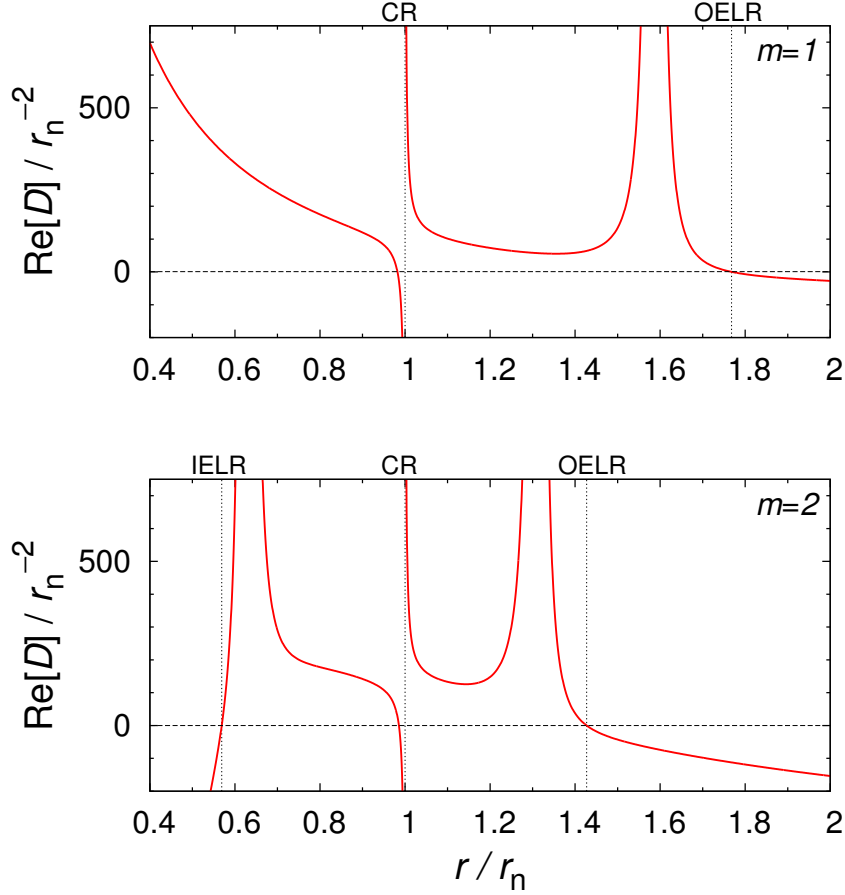


Figure 3.5: The real part of the effective potential $\text{Re}[D]$ in the base disk model with $h = 0.1$, $\Gamma = 5/3$, $\mathfrak{f} = 0$ and $\omega/\Omega_n = (m, 10^{-5})$. Top panel shows the $m = 1$ case and bottom panel shows the $m = 2$ case. Since this base disk model has the Keplerian profile, the co-rotation radius is located at $r = r_n$. CR denotes the co-rotation radius. $\text{Re}[D]$ has both the IELR and the OELR for $m = 2$, while only the OELR for $m = 1$. Note that I set γ to the infinitesimal value (10^{-5}) in order to avoid the divergence of the numerical operation.

The positive sign in equation (3.35) indicates the trailing, or outgoing, wave. Therefore, I apply

$$\mathfrak{E}' = \frac{-B(r) + i\sqrt{4C - B^2}}{2} \mathfrak{E} \quad (3.36)$$

as the boundary conditions on $\mathfrak{E}(r)$ except in the case of the inner boundary for $m = 1$.

In the case for $m = 1$, the inner boundary falls in a evanescent region. The perturbation $\mathfrak{E}$ is required to be regular at $r = 0$. I assume that the solution is of the form of $\mathfrak{E}(r) \propto r^{\mathfrak{h}}$. From equations (3.29) and (3.30), I have $B \propto 1/r$ and $C \propto 1/r^2$ for $r \sim 0$. From equation (3.28), $\mathfrak{h}$ is solved as

$$\mathfrak{h} = \frac{(1 - rB) \pm \sqrt{(rB - 1)^2 - 4r^2C}}{2}. \quad (3.37)$$

Since $\mathfrak{E}$ is regular at $r = 0$, I have to choose positive sign in equation (3.37). Therefore, I apply

$$\mathfrak{E}' = \frac{1}{2} \left\{ - \left[B - \frac{1}{r} \right] + \sqrt{\left[B - \frac{1}{r} \right]^2 - 4C} \right\} \mathfrak{E} \quad (3.38)$$

as the boundary condition on $\mathfrak{E}(r)$ in the case of the inner boundary for $m = 1$.

In the same way, boundary conditions on $\Xi(r)$ are also obtained from equation (3.32) as

$$\Xi' = i\sqrt{-D}\Xi \quad (3.39)$$

except in the case of the inner boundary for $m = 1$, and

$$\Xi' = \left[\frac{1}{2r} + \sqrt{\frac{1}{4r^2} + D} \right] \Xi \quad (3.40)$$

in the case of the inner boundary for $m = 1$.

3.1.5 Method for Solving the Perturbation Equations

In this section, I show the method to solve equation (3.28). Note that it is able to solve equation (3.32) with the same method. I discretize equation (3.28) on a grid $i = 0, \dots, N$ over equal intervals $\mathfrak{d}r$, and define r_i as the radius at the i -th grid. Equation (3.28) is rewritten as

$$\sum_{j=0}^N W_{i,j}(\omega) \mathfrak{E}(r_j) = 0, \quad (3.41)$$

where $\mathbf{W}$ is the $(N+1) \times (N+1)$ matrix and $W_{i,j}$ denotes the element of $\mathbf{W}$ in the i -th row and the j -th column. When the discretization of equation (3.28) is done in the manner of

$$\mathfrak{E}''|_{r=r_i} \rightarrow \frac{\mathfrak{E}(r_{i+1}) - 2\mathfrak{E}(r_i) + \mathfrak{E}(r_{i-1}))}{(\mathfrak{d}r)^2},$$

and

$$\begin{aligned} \mathfrak{E}'|_{r=r_i} &\rightarrow \frac{\mathfrak{E}(r_{i+1}) - \mathfrak{E}(r_{i-1}))}{2\mathfrak{d}r} \quad (\text{without boundaries}), \\ \mathfrak{E}'|_{r=r_1} &\rightarrow \frac{\mathfrak{E}(r_1) - \mathfrak{E}(r_0)}{\mathfrak{d}r} \quad (\text{at the inner boundary}), \\ \mathfrak{E}'|_{r=r_{N-1}} &\rightarrow \frac{\mathfrak{E}(r_N) - \mathfrak{E}(r_{N-1}))}{\mathfrak{d}r} \quad (\text{at the outer boundary}), \end{aligned} \quad (3.42)$$

$\mathbf{W}$ is given as a tridiagonal matrix; $W_{i,j} = 0$ for $|i - j| > 1$.

The elements of $\mathbf{W}$ in the 0-th row are obtained from the inner boundary conditions as

$$W_{0,0} = 1, \quad W_{0,1} = \mathfrak{d}r \frac{-B(r_1) + i\sqrt{4C(r_1) - B(r_1)^2}}{2} - 1 \quad (3.43)$$

for $m \neq 1$, and

$$W_{0,0} = 1, \quad W_{0,1} = \frac{\mathfrak{d}r}{2} \left\{ - \left[B(r_1) - \frac{1}{r_1} \right] + \sqrt{\left[B(r_1) - \frac{1}{r_1} \right]^2 - 4C(r_1)} \right\} \quad (3.44)$$

for $m = 1$. The elements of W in the N -th row are

$$W_{N,N-1} = \mathfrak{d}r \frac{B(r_{N-1}) - i\sqrt{4C(r_{N-1}) - B(r_{N-1})^2}}{2} - 1, \quad W_{N,N} = 1. \quad (3.45)$$

The elements of W in the other columns are obtained from equation (3.28) as

$$\begin{aligned} W_{i,i-1} &= 1 - \frac{\mathfrak{d}r}{2} B(r_i) \\ W_{i,i} &= (\mathfrak{d}r)^2 C(r_i) - 2 \\ W_{i,i+1} &= 1 + \frac{\mathfrak{d}r}{2} B(r_i) \end{aligned} \quad (3.46)$$

for $1 \leq i \leq N-1$.

The reason why I solve the perturbation equation is to obtain the eigenvalue, ω , and the eigenfunction $\mathfrak{E}$ (or Ξ) in the unstable case against the RWI, $\text{Im}[\omega] > 0$. I consider a certain unstable case with $\omega = \omega_s$ ($\text{Im}[\omega_s] > 0$) and $\mathfrak{E} = \mathfrak{E}_s$. Then, $|\det[\mathbf{W}(\omega_s)]| = 0$ should be satisfied. Since the matrix $\mathbf{W}(\omega)$ is non-linear of ω and its elements are complex numbers, the problem is a non-linear complex eigenvalue problem. Here, I call ω_s the unstable root. I have two steps to obtain ω_s and $\mathfrak{E}_s$ from the background profiles.

First, I evaluate the unstable root. I define the sequence of the numbers $\{u_i\}$, where

$$\begin{aligned} u_0 &= 1, \quad u_1 = W_{0,0} = 1, \\ u_{i+2} &= W_{i+1,i+2} u_{i+1} - W_{i,i+1} W_{i+1,i} u_i. \end{aligned} \quad (3.47)$$

Since $\mathbf{W}$ is a tridiagonal matrix, the determinant of $\mathbf{W}$ is

$$\det[W(\omega)] = u_{N+1}. \quad (3.48)$$

I estimate roughly the unstable root from the distribution of the determinant in a complex plane. Thereafter, I locate the unstable root exactly by using the Muller's method, which is the one of the root-finding algorithms and available even if I have a complex function or am ignorant of the formula of the function. Figure 3.6 shows the distribution of $|\det[\mathbf{W}(\omega)]|$ in the complex plane for the same case as Figure 3.3 and $m = 3$. I can find the region where the unstable

root resides around $\omega_s/\Omega_n = (1.0 \times 3, 0.2)$. In this case, the accurate unstable root derived by using the Muller's method is $\omega_s/\Omega_n = (0.987 \times 3, 0.173)$.

Second, I calculate the eigenfunction $\mathfrak{E}_s(r)$. The problem now becomes a linear complex eigenvalue problem;

$$\begin{cases} \sum_{j=0}^N W_{i,j}(\omega_s) \mathfrak{E}_s(r_j) = \epsilon \mathfrak{E}_s(r_i), \\ \epsilon = 0. \end{cases} \quad (3.49)$$

I solve equation (3.49) by using the LAPACK, and obtain $\mathfrak{E}_s(r)$. I have verified that my results derived from these methods agree with the results in Li et al. (2000, 2001) for some parameter sets. Note that the LAPACK is a software library for numerical linear algebra.

I have to refer that it is difficult to obtain the unstable root in the case with extremely small linear growth rate ($\text{Im}[\omega_s]/\Omega_n \ll 1$) with the above methods. Actually, there are enormous neutral roots of $|\det[\mathbf{W}(\omega)]|$ with $\gamma = 0$. For $\text{Im}[\omega_s]/\Omega_n \lesssim 10^{-5}$, the unstable root is buried in enormous neutral roots. In spite of this difficulty, I can certainly distinguish the unstable root from the neutral roots for $\text{Im}[\omega_s]/\Omega_n \geq 10^{-5}$.

3.2 Associated Known Conditions with the RWI

There are two known conditions associated with the RWI. Before presenting the results of the linear stability analyses, I briefly review these known conditions. One is the Rayleigh's condition, which is the condition for the onset of the rotational instability (e.g., Chandrasekhar, 1960). The other is the Lovelace's condition (Lovelace et al., 1999).

3.2.1 Rayleigh's Condition

When there is a radius where

$$\kappa_0^2(r) \equiv \frac{(r^4 \Omega_0^2)'}{r^3} < 0, \quad (3.50)$$

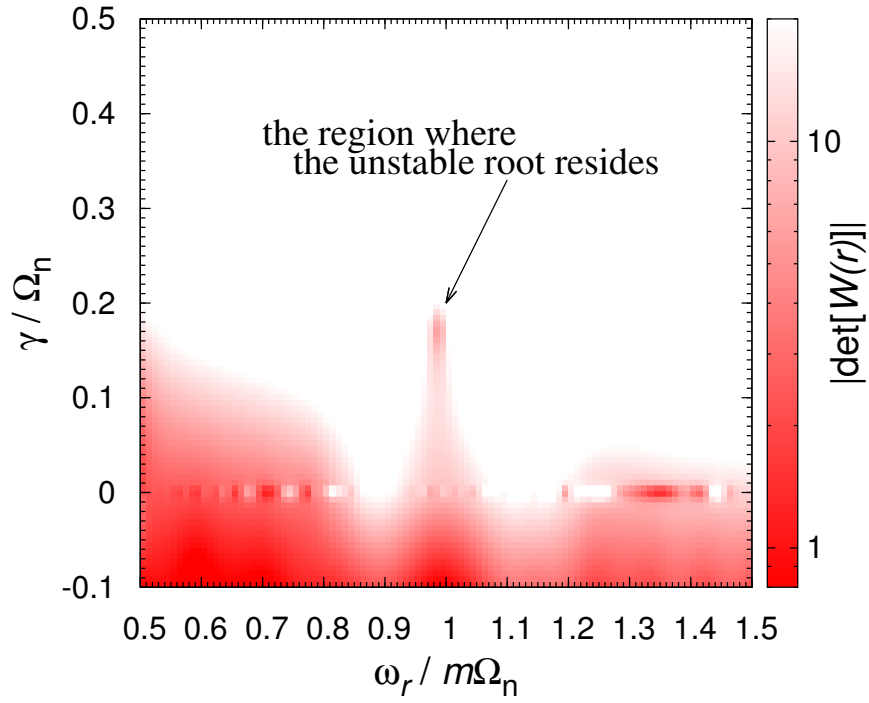


Figure 3.6: The distribution of $|\det[W(r)]|$ in a complex plane for the same case as Figure 3.3 and $m = 3$. I can find the region where the unstable root resides around $\omega_s/\Omega_n = (m, 0.2)$. The accurate unstable root is derived by using the Muller's method is located at $\omega_s/\Omega_n = (0.987 \times m, 0.173)$.

is satisfied, the background disk is unstable to the rotational instability, which is a axisymmetric hydrodynamical instability in differentially rotating disks (Chandrasekhar, 1960). Equation (3.50) is known as the Rayleigh's criterion. I use the term "Rayleigh's condition" when there is a radius where Rayleigh's criterion is violated. The rotational instability can produce axisymmetric circulations in the r - z plane known as the Taylor-Couette flow (Taylor, 1923). The circulations change the system to be marginally stable against the rotational instability on the orbital timescale. Li et al. (2000) indicated that the Rayleigh's condition is expected to be the sufficient condition for the RWI.

3.2.2 Lovelace's Condition

I multiply equation (3.27) by $r\mathfrak{E}^*$ and integrate over the disk, where $\mathfrak{E}^*$ is the complex conjugate of $\mathfrak{E}$. Assuming $r\mathcal{F}\mathfrak{E}^*\mathfrak{E}/\Omega_0 \rightarrow 0$ for $r \rightarrow 0, \infty$, I obtain

$$\int dr \left[\frac{r\mathcal{F}}{\Omega_0} (|\mathfrak{E}'|^2 + k_\varphi^2 |\mathfrak{E}|^2) + r \left(\frac{\Sigma_0}{c_0^2} + \frac{2k_\varphi \mathcal{F}'}{\Delta\omega} \right) |\mathfrak{E}|^2 \right] = 0. \quad (3.51)$$

Assuming that the dominant contribution to the integral of the left hand side of equation (3.51) comes from the region in the vicinity of the co-rotation radius, it is expected that $\Delta\omega^2 \ll \kappa_0^2$ and $\mathcal{F} \sim 1/2q_0$. Then, the imaginary part of equation (3.51) can be written as

$$\int dr \left[\frac{m|\mathfrak{E}|^2}{q_0^2} \text{Im} \left(\frac{1}{\Delta\omega} \right) q_0' \right] = 0. \quad (3.52)$$

When the system is unstable against the RWI and stable against the rotational instability, $[m|\mathfrak{E}|^2 \text{Im}(1/\Delta\omega)/q_0^2]$ is always negative due to $\text{Im}(1/\Delta\omega) = -\gamma/|\Delta\omega|^2$ and $\gamma > 0$. To satisfy equation (3.52), the existence of a turning point in the profile of q_0' is required, i.e., the background vortensity q_0 should have at least one local extremum if the disk is unstable against the RWI (Lovelace et al., 1999). However, the sufficiency of the Lovelace's condition has not been investigated. I shall show that the Lovelace's condition (Rayleigh, 1880) is almost same as the Rayleigh's inflection theorem in Section 4.2.1.

3.3 Most Unstable States to the RWI

For each set of background parameters $(\mathcal{A}_0, \Delta w_0)$ in every case, I calculate the most unstable mode frequency $\omega_* = \omega_{r*} + i\gamma_*$ and the most unstable azimuthal mode m_* against the RWI. The largest linear growth rate γ_* is the growth rate of the system against the RWI if γ_* is positive. For $\gamma_* = 0$, the system is marginally stable against the RWI. Note that I define that the "system is stable" if there is no mode that gives positive growth rate γ .

Figure 3.7 shows the largest linear growth rate γ_* with every $\mathcal{A}_0$ and Δw_0 in representative 6 cases. In each panel of Figure 3.7, I also show the parameters for "nearly" marginally stable states of the system against the RWI. Considering numerical errors, I determine that the system is "nearly" marginally stable against the RWI if γ_*/Ω_n is no more than about 10^{-3} . I note that there is no significant difference in the parameters for the marginally stable states of the system if I decrease the threshold to 10^{-4} . The methods of calculations to derive parameters exclusively for the marginally stable states will be presented in Section 3.4.3, but I obtain essentially the same results for the parameters of the marginally stable states.

In all the cases (i)-(vi), the largest linear growth rates are similar, but I see that the system can be more unstable for case (v), where h is twice as large as that in other cases. However, I have found that all the cases look very similar to each other if I use $\Delta w_0/(hr_n)$ in the horizontal axis in Figure 3.7 instead of $\Delta w_0/r_n$. Therefore, it is expected that doubling h with fixed Δw_0 has the same effect as halving Δw_0 with h fixed.

In Figure 3.7, I show the parameter space where the Rayleigh condition is not violated but Lovelace's condition is violated. It is clear that the marginally stable parameters of the system against the RWI fall in this region. Therefore, I can regard the Lovelace's condition as the necessary condition for the RWI and the Rayleigh's condition as the sufficient condition for the RWI at least within the parameter space which I have explored. The sufficiency of the Rayleigh's condition, however, has more issues, and I actually consider that this does not always hold. Further discussions will be presented in Section

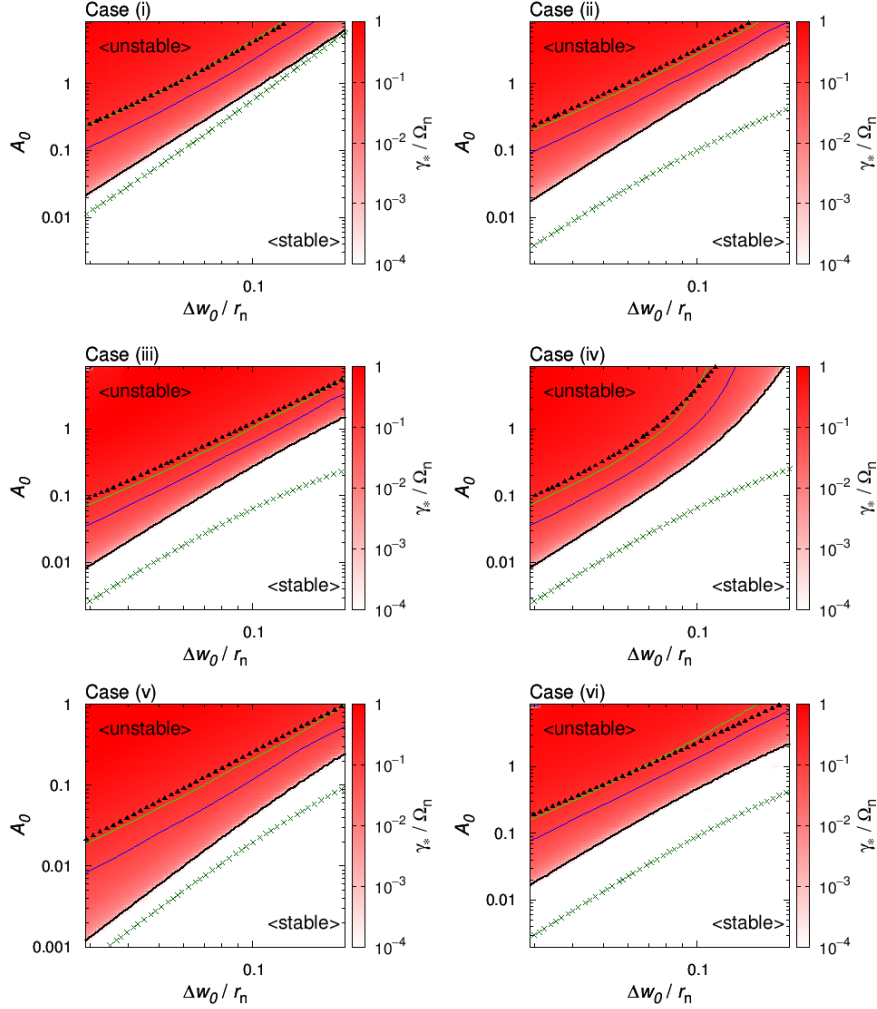


Figure 3.7: The largest growth rate γ_* is shown as color in the Δw_0 - A_0 plane for each case as seen Table 3.1. The bottom-right white region corresponds to the parameters for the stable states of the RWI. The black solid line is the contour for $\gamma_*/\Omega_n = 10^{-3}$, the blue dotted line for $\gamma_*/\Omega_n = 0.1$ and the light-green dashed line for $\gamma_*/\Omega_n = 0.2$. The line of dark-green cross represents the marginal states of the Lovelace's condition, and the region above this line is where the Lovelace's condition is satisfied. The line of black triangle represents the marginally stable states of the rotational instability, and the region above this line is where the Rayleigh's condition is violated.

3.4.5. In addition, I find that the Lovelace's condition is not the sufficient condition for the RWI.

In Figure 3.8, I show the most unstable azimuthal mode m_* for various background parameters. For the parameters where the rotational instability is stable, m_* is no larger than 7, and I describe $m_* \geq 7$ in one color (purple). From Figure 3.8, the less m_* is, the closer to the marginally stable states the system is. When the system is close to the marginally stable states against the RWI, m_* is unity. The general behavior of m_* is similar in all the six cases.

In Figure 3.9, I also show ω_{r*}/m_* for various background parameters. From the results of the linear stability analyses, I can obtain the co-rotation radius r_c for each most unstable mode by solving $\Omega_0(r_c) = \omega_{r*}/m_*$. Since my background profiles always satisfy $\Omega'_0 < 0$, the co-rotation radius is located inside $r = r_n$ if $\omega_{r*}/(m_*\Omega_n)$ is larger than unity, and vice versa. In general, the co-rotation radius is located close to (but not exactly at the same radius as) the background vortensity (q_0) local minimum with large concavity (q''_0). If there are several q_0 local minima with similar concavity, the co-rotation radius is located in the vicinity of one of them. For the GG type, there are two q_0 local minima with large q''_0 . The co-rotation radius actually changes from outside $r = r_n$ to inside $r = r_n$ at the black dotted line with the increase of $\mathcal{A}_0$ in case (vi) of Figure 3.9. For the other types, there is only one q_0 local minimum with large q''_0 , and the region which has the co-rotation radius is uniquely determined. Note that m_* is also changed at the transition line of the co-rotation radius in case (vi) of Figure 3.8.

Interestingly, in Figure 3.7, the line with $\gamma_*/\Omega_n = 0.2$ overlaps with the marginally stable line against the rotational instability. This may indicate the existence of some physical relationship between the rotational instability and the RWI. However, I focus on the marginally stable states against the RWI and do not discuss further the regime where the largest linear growth rate of the RWI is large or the Rayleigh's condition is violated. The effective potential D is very complicated in the regime where the Rayleigh's condition is strongly violated, and it is difficult to find the largest linear growth rate accurately in such a parameter regime. This issue should be discussed elsewhere.

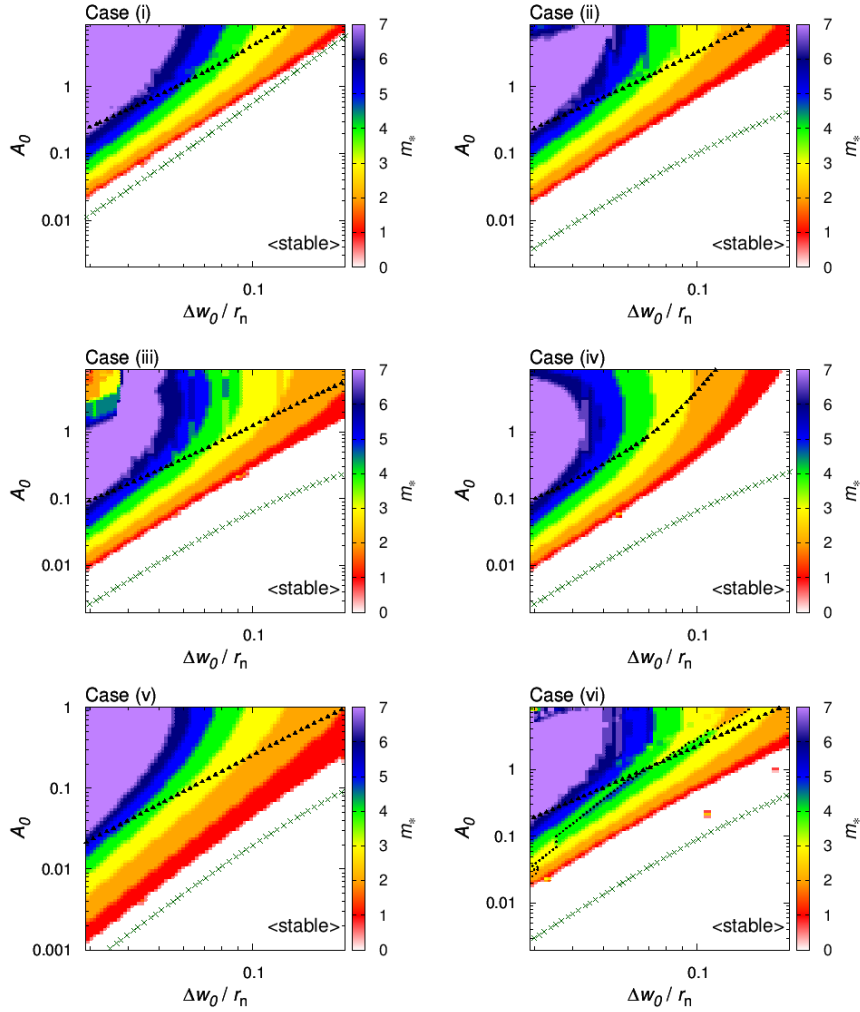


Figure 3.8: Similar to Figure 3.7, but the most unstable mode m_* is shown as color (red: 1, orange: 2, yellow: 3, green: 4, blue: 5, dark-blue: 6, purple: no less than 7). The white region for $m = 0$ corresponds to the parameters for the stable states against the RWI. The black dotted line in case (vi) is the transition line of the co-rotation radius.

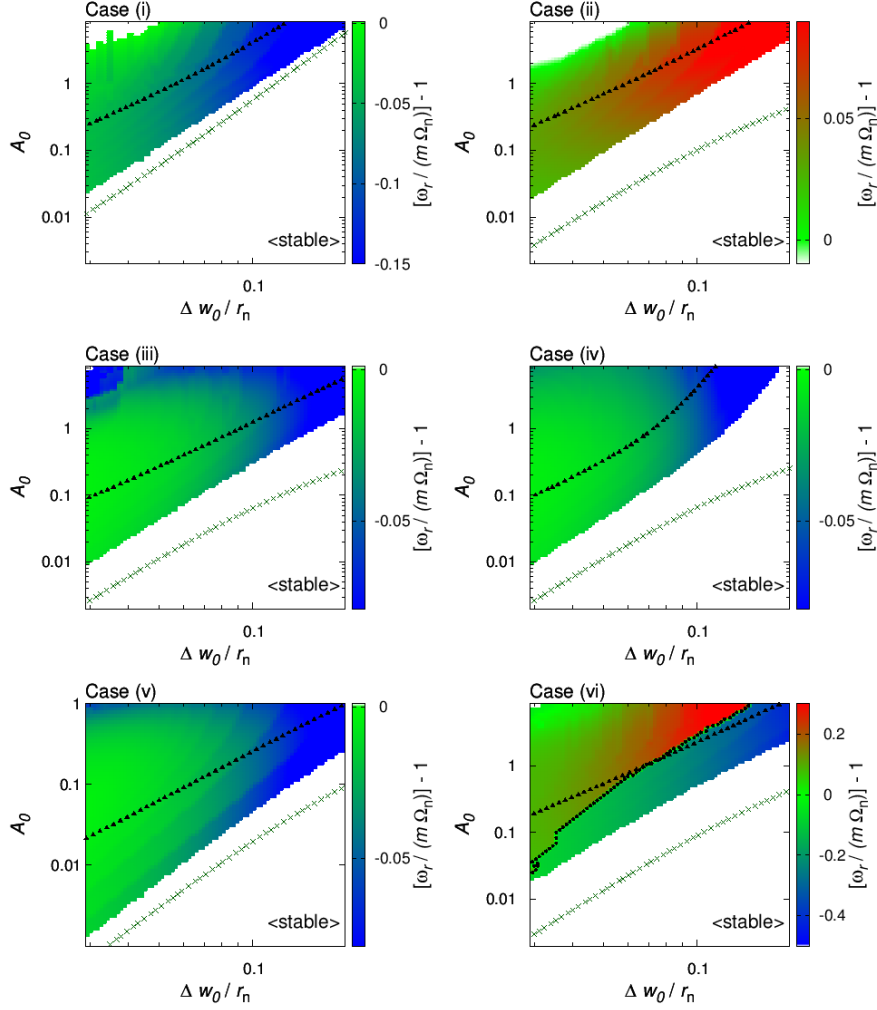


Figure 3.9: Similar to Figure 3.7, but ω_{r*}/m_* is shown as color. Due to $\Omega'_0 < 0$, the co-rotation radius is located the inside of $r = r_n$ for $\omega_{r*}/(m_*\Omega_n) > 1$, and vice versa. The black dotted line in case (vi) shows the transition of the co-rotation radius.

3.4 Marginally Stable States and Conditions for the RWI

3.4.1 Nearly Marginally Stable States

In Section 3.3, I have shown the mode frequencies and the azimuthal modes for the most unstable states against the RWI. I now investigate, in more detail, the marginally stable states. The key to understand the RWI is the effective potential D , which is introduced in Section 3.1.3. I focus on the calculations in case (iii) ($h = 0.1$, $\Gamma = 5/3$, the GB type) with $\Delta w_0 = 0.05r_n$ for $m = 1$ or 2. For each azimuthal mode ($m = 1$ or 2), the linear growth rate γ can be determined if I specify the contrast of the background bump $\mathcal{A}_0$. To put it in the other way around, the contrast of the background bump $\mathcal{A}_0$ can be evaluated using the specified linear growth rate γ . Therefore, in this section, I specify several values of the linear growth rate γ and compare the effective potentials and the eigenfunctions.

Figure 3.10 shows the real part of the effective potential $\text{Re}[D]$ (top panels) and Ξ (bottom panels) in the case of $\gamma/\Omega_n \simeq 10^{-3}$. These cases correspond to $\mathcal{A}_0 = 0.0400$ for $m = 1$ and $\mathcal{A}_0 = 0.0447$ for $m = 2$. The real part of the eigenvalue or the co-rotation radius is $\omega_r/\Omega_n = 0.983 \times 1$ or $r_c = 1.011r_n$ for $m = 1$ and $\omega_r/\Omega_n = 0.985 \times 2$ or $r_c = 1.010r_n$ for $m = 2$. From Figure 3.10, $\text{Re}[D]$ has three ($m = 1$) or four ($m = 2$) turning points, where $\text{Re}[D] = 0$ is satisfied. Compared to the case for the base disk model without a bump (Figure 3.5), there are two additional turning points other than ELRs. Here, I call the new turning points "Rossby turning points", and the region between Rossby turning points "Rossby region". Recalling that D is considered as the effective potential of the stationary Schrödinger equation with zero energy, the Rossby region is a wave propagation region. The Rossby region is regarded as a potential well, and therefore the real part of the eigenfunction $\text{Re}[\Xi]$ is trapped in the Rossby region. The co-rotation radius is very close to the radius of the q_0 local minimum. In producing the bottom panels of Figure 3.10, it is necessary to define a normalization constant of the perturbation with respect to the background. In this section, I define the dimensionless normalization

constant of the perturbation by $\bar{\Xi} \equiv \text{Re}[\Xi(r = r_{\Xi})]/(r_n^{5/2}\Sigma_n^{1/2}\Omega_n^{1/2})$, where r_{Ξ} is the radius where Ξ exhibits the maximum within the Rossby region. I choose $\bar{\Xi} = 0.025$ throughout this chapter.

Figure 3.11 shows radial profiles of the perturbations $(\Sigma_1, v_{r1}, v_{\varphi1}, q_1)$, where

$$q_1(r) \equiv \frac{1}{\Sigma_0} \left[\frac{1}{r} (rv_{\varphi1})' - ik_{\varphi} v_{r1} - q_0 \Sigma_1 \right] \quad (3.53)$$

is the vortensity perturbation, for the same background flow as that in the left panels of Figure 3.10 ($\gamma/\Omega_n \simeq 10^{-3}$ and $m = 1$). The imaginary parts of Σ_1 , $v_{\varphi1}$ and q_1 and the real part of v_{r1} are approximately equal to zero. The real part of Σ_1 and the imaginary part of v_{r1} have no node and peaks in the vicinity of the co-rotation radius. Meanwhile, the real part of $v_{\varphi1}$ has a node in the vicinity of the co-rotation radius. The real part of q_1 also has no node like $\text{Re}[\Sigma_1]$ and $\text{Im}[v_{r1}]$, but a spike at the co-rotation radius. Unlike other perturbations, $\text{Re}[q_1]$ is almost confined within the Rossby region. Perturbations have similar trends even for $m = 2$ to those for $m = 1$.

To look into the change of $\text{Re}[D]$ with varying $\mathcal{A}_0$ for $m = 1$, Figure 3.12 shows $\text{Re}[D]$ for $\gamma/\Omega_n \simeq 2 \times 10^{-4}$ ($\mathcal{A}_0 = 0.0392$), 10^{-3} ($\mathcal{A}_0 = 0.0400$) and 5×10^{-3} ($\mathcal{A}_0 = 0.0442$), respectively. I focus on the Rossby region in the left panel and on the vicinity the co-rotation radius in the right panel. It should be noted that $\text{Re}[D]$ almost diverges at the co-rotation radius since C defined in equation (3.30) has a term proportional to $1/\Delta\omega$. However, as seen in Figure 3.12, the width of the singularity at the co-rotation radius becomes narrower while the peak does not change very much as I approach the marginally stable states. This indicates that the singularity at the co-rotation radius becomes weaker as I approach the marginally stable states where $\gamma = 0$. It should be noted that I adopt $N = 20000$ in Figure 3.12 in order to resolve $\text{Re}[D]$ in the vicinity of the co-rotation radius. The change of $\text{Re}[D]$ with varying $\mathcal{A}_0$ is also similar for $m = 2$ to that for $m = 1$.

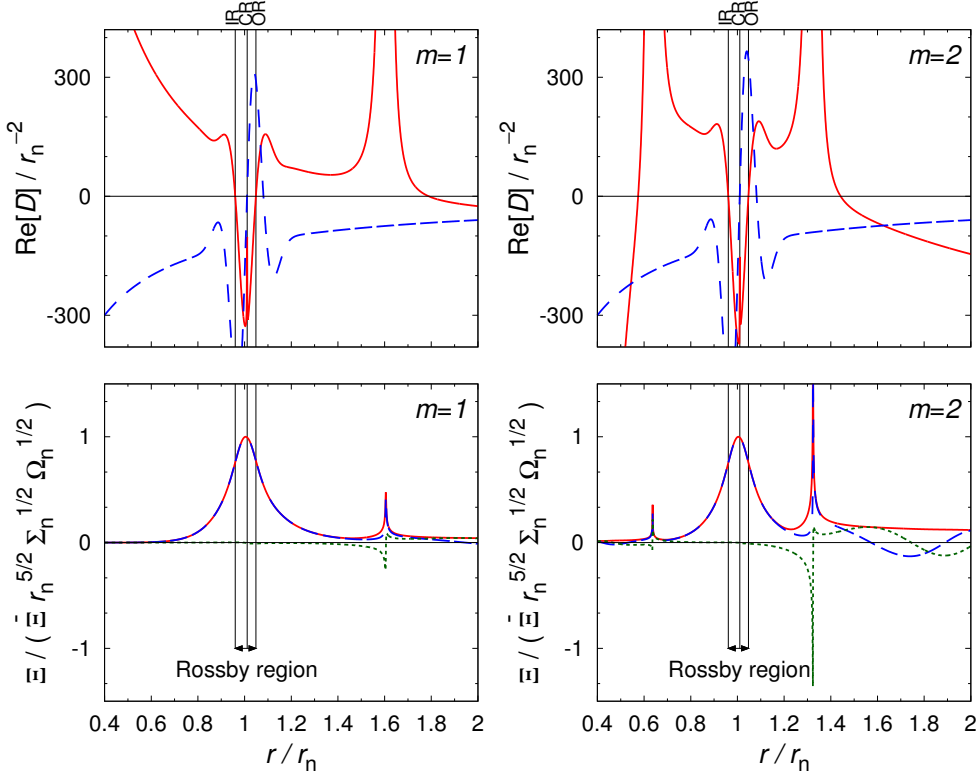


Figure 3.10: Real part of the effective potential $\text{Re}[D]$ (top panels) and the eigenfunction Ξ (bottom panels) are shown in case (iii) of $\gamma/\Omega_n \simeq 10^{-3}$ and $\Delta w_0 = 0.05r_n$. This corresponds to the case for $\mathcal{A}_0 = 0.04$, $\omega_r/\Omega_n = 0.983 \times 1$ and $r_c = 1.011r_n$ for $m = 1$ (left panels), and $\mathcal{A}_0 = 0.0447$, $\omega_r/\Omega_n = 0.985 \times 2$ and $r_c = 1.010r_n$ for $m = 2$ (right panels). In top panels, $\text{Re}[D]$ and q'_0/q_0 are represented with the red solid line and the blue dashed line, respectively. Bottom panels show $|\Xi|$ (red solid line), $\text{Re}[\Xi]$ (blue dashed line) and $\text{Im}[\Xi]$ (green dotted line), respectively. IR denotes the inner Rossby turning point, OR does the outer Rossby turning point, and CR does the co-rotation radius. The region between IR and OR is the Rossby region. Note that $\bar{\Xi}$ is defined as the dimensionless normalization constant of the perturbation by $\bar{\Xi} \equiv \text{Re}[\Xi(r = r_\Xi)]/(r_n^{5/2} \Sigma_n^{1/2} \Omega_n^{1/2}) = 0.025$, where r_Ξ is the radius where Ξ exhibits the maximum within the Rossby region.

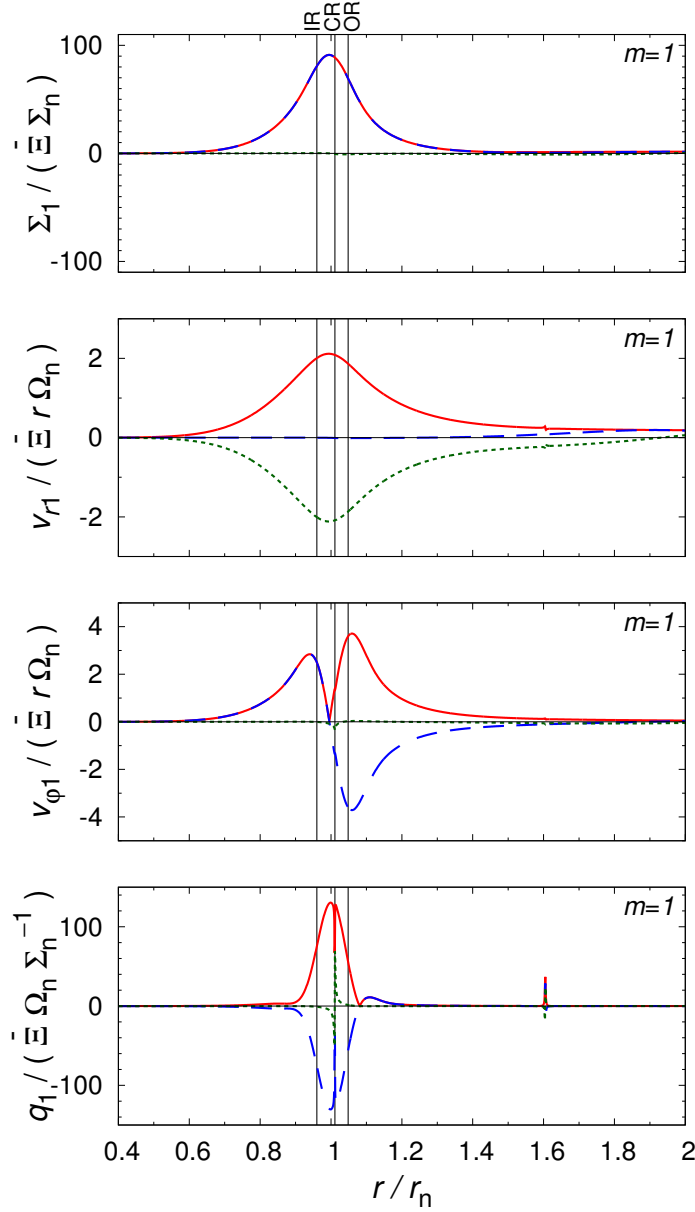


Figure 3.11: Radial profiles of the perturbations $(\Sigma_1, v_{r1}, v_{\phi1}, q_1)$ are shown for the same background flow as that in the left panels of Figure 3.10. Absolute values, real parts and imaginary parts of perturbations are represented with the red solid line, the blue dashed line and the green dotted line, respectively. IR denotes the inner Rossby turning point, OR does the outer Rossby turning point and CR does the co-rotation radius.

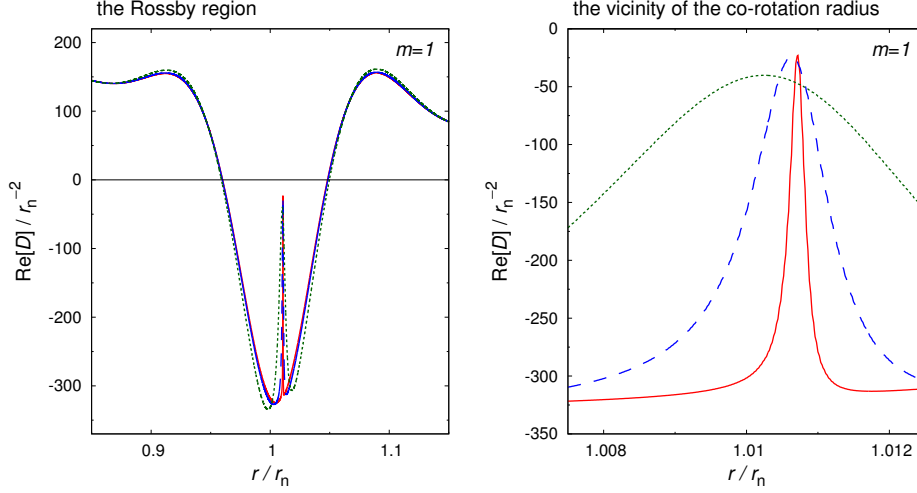


Figure 3.12: $\text{Re}[D]$ is shown in case (iii) of $\gamma/\Omega_n \simeq 2 \times 10^{-4}$ ($\mathcal{A}_0 = 0.0392$) with red solid line, 10^{-3} ($\mathcal{A}_0 = 0.0400$) with blue dashed line and 5×10^{-3} ($\mathcal{A}_0 = 0.0442$) with green dotted line for $\Delta w = 0.05r_n$. I focus on the Rossby region in left panel and on the vicinity of the co-rotation radius in right panel. It is noted that I adopt the grid $N = 20000$ in order to resolve the vicinity of the co-rotation radius.

3.4.2 Mode Frequency and Effective Potential for the Marginally Stable States

It is actually possible to derive the fact that there is no singularity at the co-rotation radius for the marginally stable states. For barotropic flow, vortensity is always conserved with respect to fluid elements as equation (3.5). The linearized form of equation (3.5) is written as

$$i\Delta\omega q_1 = v_{r1}q'_0. \quad (3.54)$$

Since $\Omega_0 = \omega_r(r_c)/m$ and $\gamma = 0$ should be satisfied for the marginally stable states against the RWI, $\Delta\omega$ must vanish at the co-rotation radius by definition. In other words, either v_{r1} or q'_0 should be equal to zero at the co-rotation radius for the marginally stable states. From Figure 3.11, it is expected that $\text{Im}[v_{r1}]$ has no node then. Therefore, the co-rotation radius has to locate at the radius of the q_0 local extremum for the marginally stable states.

In fact, the co-rotation radius is located at the radius of the q_0 local minimum rather than one of the q_0 local maximum as expected by Figure 3.10. I will explain a reason of this in Chapter 4.

Here, I define the radius of the q_0 local minimum by r_q ;

$$q'_0(r_q) = 0, \quad q''_0(r_q) > 0. \quad (3.55)$$

Since $r_c = r_q$ is satisfied for the marginally stable states, I can obtain the mode frequency from only the background profiles due to $\gamma = 0$. When the mode frequency for the marginally stable states is defined by

$$\omega_{\text{MS}}(m) \equiv m\Omega_0(r_q), \quad (3.56)$$

ω_{MS} is a real number.

For the marginally stable states, the effective potential D is also a real-number and obtained from only the background profiles as

$$\begin{aligned} D_{\text{MS}}(r, m) &\equiv D(r, \omega_{\text{MS}}(m), m) \\ &= \frac{B'_{\text{MS}}}{2} + \frac{B_{\text{MS}}^2}{4} - C_{\text{MS}}, \end{aligned} \quad (3.57)$$

where

$$B_{\text{MS}}(r, m) \equiv \frac{1}{r} + \frac{F'_{\text{MS}}}{F_{\text{MS}}} - \frac{\Omega'_0}{\Omega_0}, \quad (3.58)$$

$$C_{\text{MS}}(r, m) \equiv -k_\varphi^2 - \frac{\kappa_0^2 - m^2(\Omega_{0,v} - \Omega_0)^2}{c_{s0}^2} - \frac{2}{r} \frac{\Omega_0}{(\Omega_{0,q} - \Omega_0)} \frac{F'_{\text{MS}}}{F_{\text{MS}}}, \quad (3.59)$$

$$F_{\text{MS}}(r, m) \equiv \frac{\Sigma_0 \Omega_0}{\kappa_0^2 - m^2(\Omega_{0,q} - \Omega_0)^2}, \quad (3.60)$$

and the constant $\Omega_0(r = r_q)$ is denoted by $\Omega_{0,q}$. I have used $\omega_{\text{MS}} = m\Omega_{0,q}$ in equations (3.58)–(3.60) where appropriate. Since

$$\frac{F'_{\text{MS}}(r_q)}{F_{\text{MS}}(r_q)} = -\frac{q'_0(r_q)}{q_0(r_q)} = 0 \quad (3.61)$$

is satisfied, D_{MS} has no singularity at $r = r_q$ for the marginally stable states.

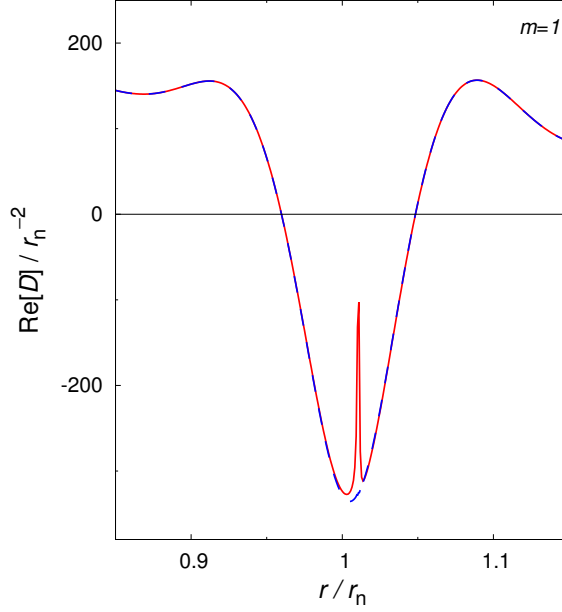


Figure 3.13: Radial profiles of $\text{Re}[D]$ (red solid line) and D_{MS} (blue dashed line) are shown for the same case as the case in the left panels of Figure 3.10.

This is in consistent with the results and the discussions in Section 3.4.1.

3.4.3 Search for Parameters of the Marginally Stable States

I have seen that, for the marginally stable states, it is possible to find the co-rotation radius and the effective potential from only the background profiles. I now discuss how I can find the set of the parameters that gives the marginally stable states. From equations (3.57)-(3.60), D_{MS} can be defined even if the system is not exactly in the marginally stable states. Figure 3.13 shows $\text{Re}[D]$ and D_{MS} for the same case as the case in left panels of Figure 3.10 ($\gamma/\Omega_n \simeq 10^{-3}$, for $m = 1$). The profile of D_{MS} approximately corresponds to that of $\text{Re}[D]$, but is smooth in the vicinity of the co-rotation radius.

I derive the parameters for the marginally stable states of the system against the RWI without solving equation (3.28) or (3.32). Since it is expected that

the azimuthal mode is unity for the marginally stable states of the system, I adopt $m = 1$ (see Figure 3.8 and Section 3.3). When I integrate

$$\mathfrak{E}'' - D_{\text{MS}}\mathfrak{E} = 0, \quad (3.62)$$

$\mathfrak{E}$ is obtained as a real number function of r because D_{MS} is real. I fix h , Γ , and Δw_0 , and look for the value of $\mathcal{A}_0$ where the solution $\mathfrak{E}$ satisfies the radiative boundary conditions at the outer and inner radii (see Section 3.1.4) by using a 1D shooting method. I denote $\mathcal{A}_0$ derived in this way by $\mathcal{A}_{0,\text{MS}}$.

By the 1D shooting, I can obtain parameters for the marginally stable states of the system. Figure 3.14 shows $\mathcal{A}_{0,\text{MS}}$. It is clear that the line of $\mathcal{A}_{0,\text{MS}}$ is fully in accord with the line of $\gamma_*/\Omega_{\text{n}} = 10^{-3}$ in all the cases. The line of $\mathcal{A}_{0,\text{MS}}$ can be fitted by

$$\mathcal{A}_{0,\text{MS}} = \alpha \left(\frac{\Delta w_0}{r_{\text{n}}} \right)^\beta \quad (3.63)$$

for $0.02 \leq \Delta w_0/r_{\text{n}} \leq 0.05$ and $0.05 \leq \Delta w_0/r_{\text{n}} \leq 0.2$. I use two ranges of Δw_0 for fitting to investigate how the radial half width of the transition of physical parameters affects the stability against the RWI. The fitting parameters, α and β , in each case are shown in Table 2. I obtain $\beta \sim 3$ in all the cases, and in case (v), α is ~ 8 times smaller than other cases. Since h is twice as large as that in other cases, this suggests that $\mathcal{A}_{0,\text{MS}} \propto (\Delta w_0/h)^3$, especially for small Δw_0 . For large Δw_0 , β is decreased in cases (ii)-(iii) and (v)-(vi), and almost constant for case (i). On the other hand, β is increased in case (iv). Since the case (iv) is for isothermal disks ($\Gamma = 1$), these fitting parameters indicate that the effective adiabatic index of the gas Γ has a large effect on the stability against the RWI for large Δw_0 rather than for small Δw_0 . Note that the dependence of $\mathcal{A}_{0,\text{MS}}$ on Δw_0 ($\beta \sim 3$) can be captured using an analytic three layers model (see Section 4.4).

I note that the shooting method of solving equation (3.62) can be used to find the parameters $\mathcal{A}_0$ and Δw_0 , where the growth rate of the perturbations with a fixed m (not necessarily unity) vanishes. In general, when the perturbation with higher m ($m > 1$) is marginally stable, the perturbations with lower m are still unstable. Therefore, the stability of the system as a whole is

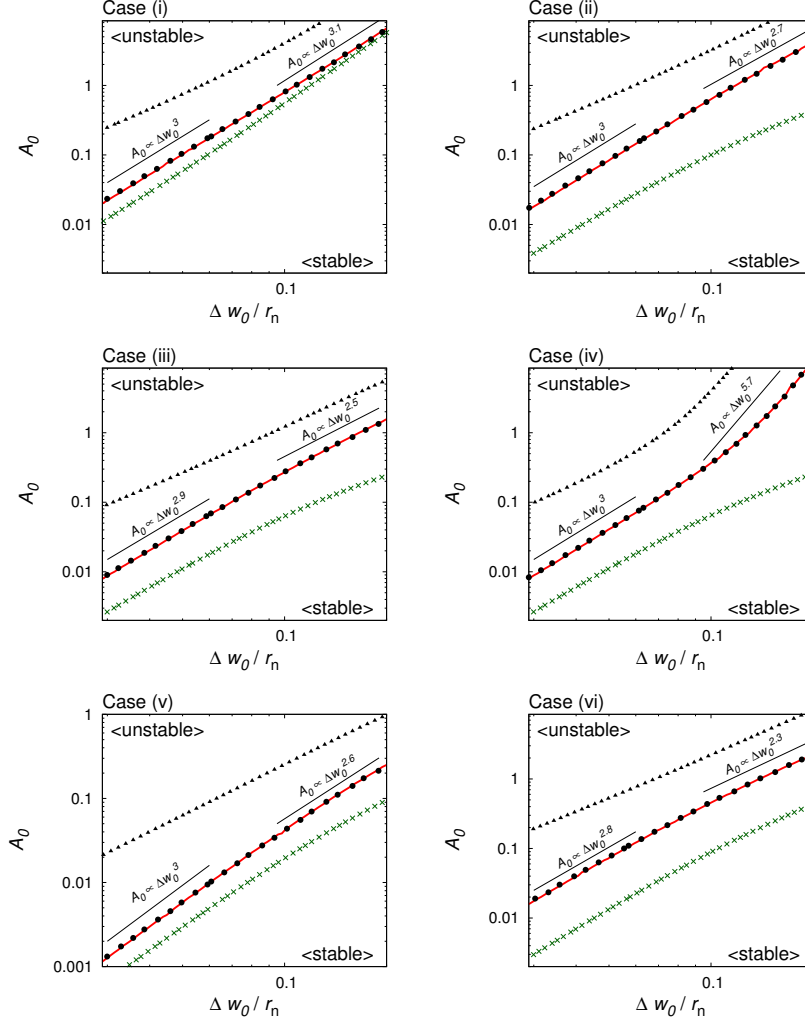


Figure 3.14: The line of $\gamma_*/\Omega_n = 10^{-3}$ (black dotted line) and the line of $\mathcal{A}_{0,\text{MS}}$ by the 1D shooting (red solid line) in each case. The line with $\gamma_*/\Omega_n \simeq 10^{-3}$ is fully in accord with the line of $\mathcal{A}_{0,\text{MS}}$. The lines of dark-green crosses and black triangles are same as those in Figure 3.7. For a comparison, the black solid lines with $\mathcal{A}_0 \propto (\Delta w_0 / r_n)^\beta$ are shown, where β is the fitting parameter (see Table 2).

Table 3.2: The fitting parameters by $A_{0,MS} = \alpha(\Delta w_0/r_n)^\beta$ for each case.

	$0.02 \leq \Delta w_0/r_n \leq 0.05$		$0.05 \leq \Delta w_0/r_n \leq 0.2$	
Case	α	β	α	β
i	8.08×10^2	3.00	9.07×10^2	3.06
ii	6.95×10^2	3.01	3.33×10^2	2.72
iii	2.58×10^2	2.93	9.32×10	2.53
iv	3.25×10^2	3.00	1.06×10^5	5.72
v	4.00×10	2.95	1.63×10	2.58
vi	3.29×10^2	2.80	9.31×10	2.31

determined by the stability of the modes with $m = 1$.

3.4.4 The Necessary & Sufficient Condition for the Onset of the RWI in a Semi-Analytic Form

Since equation (3.62) is similar to the stationary Schrödinger equation with zero energy, I attempt to analytically derive the condition for the RWI by using the Sommerfeld-Wilson quantization condition (Wilson, 1915; Sommerfeld, 1916), which is the condition for trapping eigenfunctions in the potential wells. Considering that Ξ has no node and assuming to be ground-state, the new condition is given as

$$\eta(m) \equiv \int_{r_{\text{IR}}}^{r_{\text{OR}}} \sqrt{-D_{\text{MS}}} \, dr \geq \eta_c, \quad (3.64)$$

where r_{IR} is the radius of the inner Rossby turning point, r_{OR} is the radius of the outer Rossby turning point, and η_c is a threshold for the onset of the RWI. The flow is unstable against the RWI for $\eta > \eta_c$ and is marginally stable for $\eta = \eta_c$. If the Sommerfeld-Wilson quantization condition is perfectly applicable, the value of η_c should equal to $\pi/(2\sqrt{2})$. The Sommerfeld-Wilson quantization condition requires the availability of the WKBJ approximation, but it is generally not available for ground states. However, the Sommerfeld-Wilson quantization condition is exceptionally applicable even for ground states when the potential well of the effective potential may be well approximated by a parabolic function with respect to r . I assume that equation (3.64) with

$\eta_c = \pi/(2\sqrt{2})$ and $m = 1$ can be used as the condition for the RWI because the potential well of D_{MS} with $m = 1$ typically looks parabolic as seen in Figure 3.13.

I confirm that the applicability of equation (3.64) with $\eta_c = \pi/(2\sqrt{2})$ and $m = 1$ by comparing with the parameters for marginally stable states of the system derived in Section 3.4.3. Figure 3.15 shows the line with $\eta_{m=1} = \pi/(2\sqrt{2})$ for each case in similar manner to Figure 3.14. The lines with $\eta_{m=1} = \pi/(2\sqrt{2})$ are very similar to the lines of $\mathcal{A}_{0,\text{MS}}$ in all the cases except for $\Delta w_0 \ll H_0(r_q)$. For $\Delta w_0 \ll H_0(r_q)$, it is inferred that the potential well of D_{MS} with $m = 1$ deviates from a parabolic potential well. I conclude that equation (3.64) can be used as the *necessary and sufficient* condition for the onset of the RWI in a semi-analytic form except for $\Delta w_0 \ll H_0(r_q)$.

If the potential well of D_{MS} is shallower than a parabolic potential well, it is expected that the value of η_c is larger than $\pi/(2\sqrt{2})$ from the knowledge of quantum mechanics. Otherwise, the value of η_c may be smaller than $\pi/(2\sqrt{2})$. The new condition is also applicable for other m . For higher azimuthal mode, the WKB approximation is generally more available, and η_c may be around $\pi/(2\sqrt{2})$ even for small Δw_0 . Note that the new condition is not applicable for the parameters where the Rayleigh's condition is violated because of the appearance of new singularities in D_{MS} (not discussed in detail in this thesis).

3.4.5 Relation of the RWI with Associated Known Conditions

From my linear stability analyses and analytic considerations, it is now clear that the Lovelace's condition is the necessary condition for the onset of the RWI. Within the parameter space which I have explored, it may also be possible to say that the Rayleigh's condition is the sufficient condition for the RWI. However, I actually find an example where the Rayleigh's condition is violated but the system is stable against the RWI. This is the case for the similarity

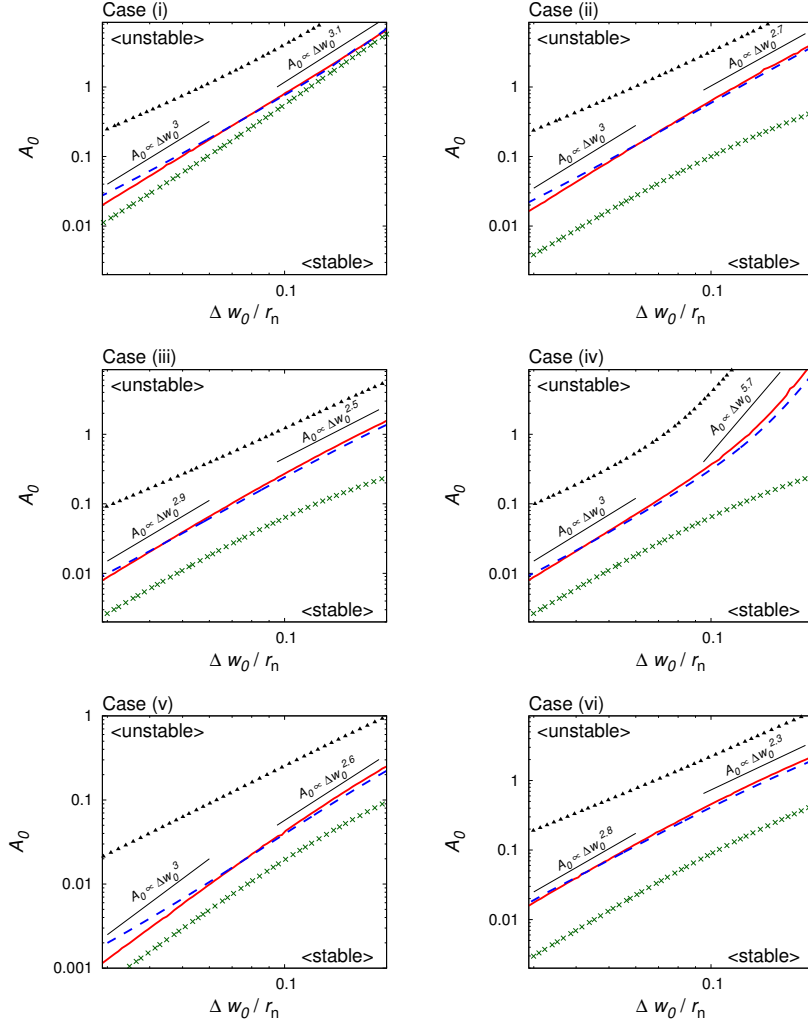


Figure 3.15: Similar to Figure 3.14, but the line with $\eta_{m=1} = \pi/(2\sqrt{2})$ is shown as blue dotted line instead of the line with $\gamma_*/\Omega_n = 10^{-3}$. The line with $\eta_{m=1} = \pi/(2\sqrt{2})$ and the line of $\mathcal{A}_{0,MS}$ are well fitted except for $\Delta w_0 \ll H_0(r_q)$.

solution of the accretion disk (Lynden-Bell & Pringle, 1974);

$$\frac{\Sigma_0}{\Sigma_n} = \frac{1}{r} \exp \left[- \left(\frac{r - r_n}{r_n} \right) \right] \quad (3.65)$$

with $h = 0.3$ and $\Gamma = 1$. The Rayleigh's condition is violated in this case (Ono et al., 2014). Figure 3.16 shows the background profiles of this similarity solution in a similar manner to Figure 3.1. Due to the absence of the q_0 local minimum, this similarity solution does not satisfy the Lovelace's condition, and is stable against the RWI. Therefore, the Rayleigh's condition is not the sufficient condition for the RWI.

3.5 Summary of Chapter 3

I perform linear stability analyses for representative 6 types of the background flow on a wide parameter space, and examine the marginally stable condition for the RWI. It is found that the co-rotation radius is located at the background vortensity local minimum with large concavity for marginally stable states against the RWI. Using these results, the stability against the RWI can be checked by at least the 1D-shooting. I newly derive the *necessary and sufficient* condition for the onset of the RWI in a semi-analytic form on the analogy of the quantum mechanics. The threshold of the new condition almost equals to $\eta_c = \pi/(2\sqrt{2})$ except for $\Delta w_0 \ll H_0(r_q)$. I also investigate the necessity and sufficiency of the associated known conditions to the RWI. It is verified that the Lovelace's condition is the necessary condition for the RWI. On the other hand, we prove that the Rayleigh's condition is not the sufficient condition for the RWI.

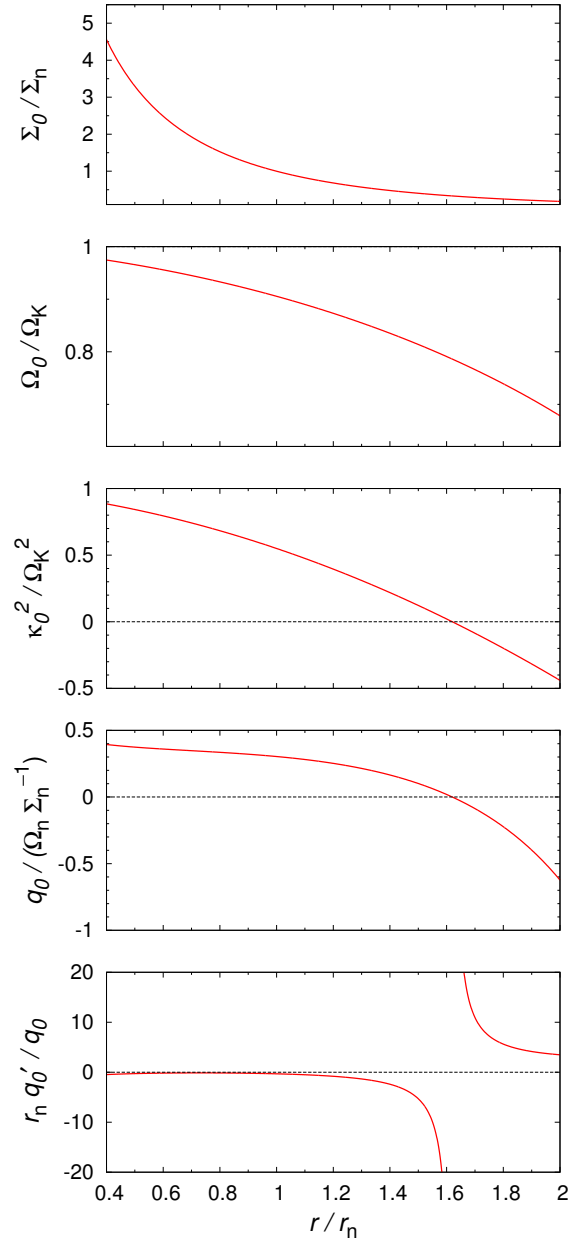


Figure 3.16: Similar to Figure 3.1, but for the similarity solution with $h = 0.3$ and $\Gamma = 1.0$. This profile violate the Rayleigh's condition, while has no q_0 local minimum.

Chapter 4

Physical Nature of the RWI

In this chapter, I focus on the physical nature of the RWI. The results of the linear stability analyses in Chapter 3 show that the RWI is closely related to the vortensity conservation law. This indicates that the physics of the RWI resembles that of the shear instability. In other hand, Tsang & Lai (2008) pointed out that the RWI resembles the Drury-Papaloizou-Pringle instability (DPPI: Drury, 1980, 1985; Papaloizou & Pringle, 1984, 1985, 1987; Goldreich et al., 1986; Narayan et al., 1987). It is known that the physical natures of both the shear instability and the DPPI are explained with the over reflection of waves at the co-rotation resonance (or the critical layer) (Narayan et al., 1987; Takehiro & Hayashi, 1992, e.g.). However, I attempt to understand the physics of the RWI from the slightly different point of the view. In geophysics, the shear instability is also explained by the interaction between two Rossby waves (Drazin & Reid, 1981; Hayashi & Young, 1987). Here, I show the interpretation of the RWI with the wave interaction.

Chapter 4 is organized as follows. Sections 4.1 and 4.2 are for introduction of the Rossby wave and for theorems of the shear instability, respectively. I show the interpretation of the shear instability and the RWI using the wave interaction in Section 4.3. For the verification of the discussions so far, I also show the analytic three layers model of the shear instability in the incompressible flow in Section 4.4. Section 4.5 is for summary.

4.1 Rossby Wave

In general, hydrodynamic fluid is described with five basic equations: one continuity equation, three equations of motion, and one energy equation. This indicates that the hydrodynamic fluid has at most five wave modes. Two of them are sound wave modes which describe longitudinal waves. If the fluid is incompressible, the sound wave modes are trivial. Other two of the wave modes are gravity wave modes which describe transverse waves and are induced by external force typified by gravity. Therefore, if there is no external force, the gravity wave modes are trivial. Both sound wave modes and gravity wave modes have two modes which propagate in the opposite direction, individually. The last one of the wave modes is Rossby wave mode. Note that I define that the “mode is trivial” if it is impossible to distinguish the mode from the mean flow.

The Rossby wave mode describes a transverse neutral wave induced by gradient of the specific vorticity. If the distribution of the specific vorticity is uniform, the Rossby wave mode is trivial. I use the term of “specific vorticity” to describe a quantity which is related to the vorticity and conserved with respect to fluid elements. As long as adiabatic flow is considered, I can define the specific vorticity. Therefore, the vortensity defined in Chapter 3 is specific vorticity for 2D barotropic flow. I denote the specific vorticity by q .

In order to understand the Rossby wave mode, I consider incompressible and barotropic flow in Cartesian 2D (x - y) coordinates without any external force. This is the simplest system to describe the Rossby wave mode and the shear instability. In this system, the sound wave modes and the gravity wave modes are trivial. Now, the basic equations are integrated into only the conservation equation of the specific vorticity $q = \text{rot } \mathbf{v}$:

$$\frac{dq}{dt} = \frac{\partial q}{\partial t} + v_x \frac{\partial q}{\partial x} + v_y \frac{\partial q}{\partial y} = 0, \quad (4.1)$$

where $\mathbf{v} = (v_x, v_y)$ is the velocity field. It should be noted that the specific vorticity is vorticity in incompressible flow.

To perform linear stability analyses, I use the background profiles that are

stationary ($\partial/\partial t$) and parallel ($v_x = 0$ and $\partial/\partial y = 0$). I use subscript “0” to denote the background profiles and subscript “1” to denote perturbation in the same manner as the linear stability analyses in Chapter 3. The first order form of equation (4.1) is

$$\frac{\partial q_1}{\partial t} + v_{x1}q'_0 + v_{y0}\frac{\partial q_1}{\partial y} = 0, \quad (4.2)$$

where the prime denotes d/dx . Equation (4.2) shows the time evolution of the vorticity perturbation q_1 . The second term can be interpreted as the advection of the background vorticity in the x direction by the perturbed velocity field, while the third term represents the advection of the vorticity perturbation in the azimuthal direction due to the background flow. Since the q_1 patterns are identical as the “Rossby wave” (Rossby, 1939), equation (4.2) gives us the qualitative explanation of the Rossby wave.

Here, I assume that the background flow satisfies $q'_0 > 0$, $q_0(x = 0) = 0$, and $v_{y0}(x = 0) = 0$. There are one Rossby wave and perturbed velocity fields induced by the Rossby wave. Figure 4.1 is the schematic figure of the Rossby wave in this situation. Since vorticity is conserved with respect to fluid elements and $q'_0 > 0$ is satisfied, incoming flows to the region of $x = 0$ from the region of $x > 0$ have positive vorticity and those from the region of $x < 0$ have negative vorticity. These flows induce phase velocity of the Rossby wave in the $+y$ direction as shown in Figure 4.1. On the other hand, the phase velocity is in the $-y$ direction for $q'_0 < 0$. Since the magnitude of the vorticity net flux to the region of $x = 0$ depends on $q'_0(x = 0)$, it is considered that the phase velocity also depends on $q'_0(x = 0)$. If the background flow does not have the vorticity gradient, the phase velocity of the Rossby wave vanishes and the Rossby wave mode becomes trivial. The Rossby wave is a neutral wave and non-trivial when the background vorticity profile has a gradient.

Next, I show the nature of the Rossby wave more quantitatively. For incompressible flow, I can define stream function Υ satisfying

$$v_x = -\frac{\partial \Upsilon}{\partial y}, \quad v_y = \frac{\partial \Upsilon}{\partial x}. \quad (4.3)$$

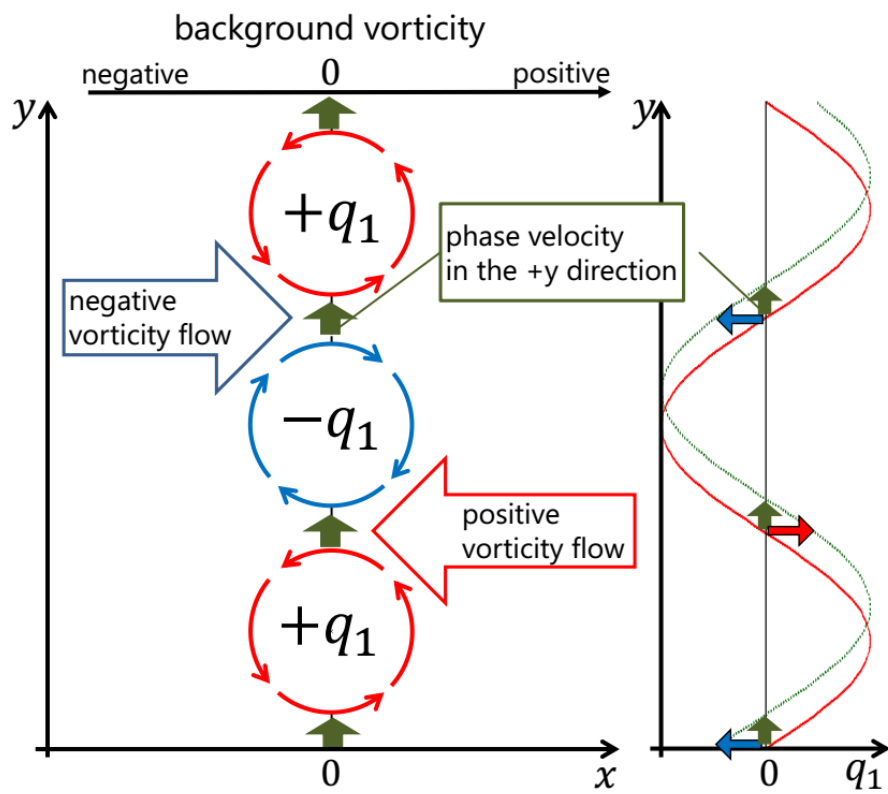


Figure 4.1: The schematic figure of the Rossby wave in incompressible flow.

Using the stream function, the specific vorticity is written as

$$q = \nabla^2 \Upsilon, \quad (4.4)$$

where ∇^2 denotes the Laplace operator. Substituting equations (4.3)–(4.4) into equation (4.1), I obtain

$$\frac{\partial(\nabla^2 \Upsilon)}{\partial t} = \frac{\partial \Upsilon}{\partial y} \frac{\partial(\nabla^2 \Upsilon)}{\partial x} - \frac{\partial \Upsilon}{\partial x} \frac{\partial(\nabla^2 \Upsilon)}{\partial y}. \quad (4.5)$$

The first order form of equation (4.5) is

$$\begin{aligned} \frac{\partial(\nabla^2 \Upsilon_1)}{\partial t} &= (\nabla^2 \Upsilon_0)' \frac{\partial \Upsilon_1}{\partial y} - \Upsilon_0' \frac{\partial(\nabla^2 \Upsilon_1)}{\partial y}, \\ &= q_0' \frac{\partial \Upsilon_1}{\partial y} - v_{y0} \frac{\partial(\nabla^2 \Upsilon_1)}{\partial y}. \end{aligned} \quad (4.6)$$

I consider perturbation as

$$\Upsilon_1(x, y, t) = \Upsilon_1(x) \exp(ik_y y - i\omega t), \quad (4.7)$$

where k_y is the wave number in the y direction and ω is the mode frequency. The linearized form of equation (4.6) is

$$\Upsilon_1'' - V_{\text{eff}} \Upsilon_1 = 0, \quad (4.8)$$

where

$$V_{\text{eff}}(x) = k_y^2 - \frac{k_y}{\omega - k_y v_{y0}} q_0'. \quad (4.9)$$

Since equation 4.8 is similar to the stationary Schrödinger equation with zero energy like equation (3.32) in Chapter 3, V_{eff} is also regarded as the effective potential.

If $V_{\text{eff}}'(x)$ is enough small, I can make the WKBJ approximation and assume that $\Upsilon_1(x)$ satisfies

$$\Upsilon_1(x) \propto \exp(ik_x x), \quad (4.10)$$

where k_x is the wave number in the x direction. Substituting equation (4.10) into equation (4.8), the dispersion relation of the Rossby wave mode is obtained

as

$$\omega = k_y v_{y0} + \frac{k_y}{k_x^2 + k_y^2} q'_0. \quad (4.11)$$

From equation 4.11, I can calculate the phase velocity of the Rossby wave mode as

$$v_{x,\text{phase}} = \frac{\omega}{k_x} = \frac{k_y}{k_x} v_{y0} + \frac{k_y}{k_x(k_x^2 + k_y^2)} q'_0, \quad (4.12)$$

$$v_{y,\text{phase}} = \frac{\omega}{k_y} = v_{y0} + \frac{q'_0}{(k_x^2 + k_y^2)}, \quad (4.13)$$

and the group velocity of the Rossby wave mode as

$$v_{x,\text{group}} = \frac{\partial \omega}{\partial k_x} = -\frac{2k_x k_y}{(k_x^2 + k_y^2)^2} q'_0, \quad (4.14)$$

$$v_{y,\text{group}} = \frac{\partial \omega}{\partial k_y} = v_{y0} + \frac{k_x^2 - k_y^2}{(k_x^2 + k_y^2)^2} q'_0. \quad (4.15)$$

Equations (4.12)–(4.15) show the dispersibility of the Rossby wave mode. From equation (4.13), I define the differential phase velocity of the Rossby wave in the y direction by

$$\delta v_{y,\text{phase}} \equiv v_{y,\text{phase}} - v_{y0} = \frac{q'_0}{(k_x^2 + k_y^2)}. \quad (4.16)$$

It is found that $\delta v_{y,\text{phase}}$ is proportion to q'_0 .

4.2 Theorems of the Shear Instability

There are two known theorems of the shear instability. One is the Rayleigh's inflection theorem (Rayleigh, 1880), and the other is the Fjørtoft's theorem (Fjørtoft, 1950). These theorems show the necessary condition of the onset for the shear instability, and are useful to understand the RWI. Here, I review these two theorems.

4.2.1 Rayleigh's Inflection Theorem

I consider the case for the unstable states against the shear instability ($\text{Im}(\omega) > 0$). I multiply equation (4.8) by Υ_1^* and integrate over $-\infty < x < \infty$, where Υ_1^* is the complex conjugate of Υ_1 . Assuming $\Upsilon_1^* \Upsilon_1 \rightarrow 0$ for $x \rightarrow -\infty, \infty$, I obtain

$$\int_{-\infty}^{\infty} dx [|\Upsilon_1'|^2 + V_{\text{eff}}|\Upsilon_1|^2] = 0. \quad (4.17)$$

The imaginary part of equation (4.17) is written as

$$\int_{-\infty}^{\infty} dx \left[-\frac{\text{Im}(\omega)k_y}{|\omega - k_y v_{y0}|^2} q_0' \right] = 0. \quad (4.18)$$

Due to $\text{Im}(\omega) > 0$, $-\text{Im}(\omega)k_y/|\omega - k_y v_{y0}|^2$ is always negative. To satisfy equation (4.20), the existence of a local extremum in q_0 profile is required. In other words, v_{y0} should have at least one inflection point for the onset of the shear instability (Rayleigh, 1880). This is a necessary condition for the shear instability. It is noticed that the Rayleigh's inflection theorem is almost same as the Lovelace's condition introduced in Section 3.2.2.

4.2.2 Fjørtoft's Theorem

I also consider the case for the unstable states against the shear instability ($\text{Im}(\omega) > 0$). The critical layer of the shear instability is defined by

$$v_{y0}(x_c) = \text{Re}(\omega)/k_y, \quad (4.19)$$

where x_c is the position of the critical layer. In addition, I assume that both q_0' and $v_{y0}(x) - \text{Re}(\omega)/k_y$ vanish only at $x = x_c$. The real part of equation (4.17) is written as

$$\int_{-\infty}^{\infty} dx \left[|\Upsilon_1'|^2 + k_y^2 |\Upsilon_1|^2 - \frac{k_y}{|\omega - k_y v_{y0}|^2} (\text{Re}(\omega) - k_y v_{y0}) q_0' \right] = 0. \quad (4.20)$$

Since $|\Upsilon_1|^2 + k_y^2 |\Upsilon_1|^2$ and $k_y^2/|\omega - k_y v_{y0}|^2$ is always positive,

$$\int_{-\infty}^{\infty} dx [(v_{y0} - \text{Re}(\omega)/k_y) q'_0] < 0 \quad (4.21)$$

is required to satisfy equation (4.20). From the assumption, the sign of $(v_{y0} - \text{Re}(\omega)/k_y) q'_0$ is invariant. Then, equation (4.21) at $x = x_c$ is written by

$$v'_{y0}(x_c) q''_0(x_c) (x - x_c)^2 < 0,$$

or

$$v'_{y0}(x_c) q''_0(x_c) < 0. \quad (4.22)$$

Equation (4.22) is a necessary condition for the onset of the shear instability (Fjørtoft, 1950).

Using the Fjørtoft's theorem, I can explain the reason why the co-rotation radius is located in the vicinity of the vortensity local minimum rather than the vortensity local maximum. The disks which I considered in Chapter 3 are almost Keplerian even though the disks have a rapid radial variation. The radial gradient of the rotation velocity at the co-rotation radius is always negative ($v'_{\varphi 0}(r_c) < 0$). From the Fjørtoft's theorem, q''_0 should be positive at the co-rotation radius. This indicates that the onset of the RWI requires the existence of the vortensity local minimum.

4.3 Interpretation of the Sher Instability and the RWI with the Wave Interaction

4.3.1 Interpretation of the Shear Instability

It is known that the shear instability is interpreted by the interaction between two Rossby waves (Drazin & Reid, 1981; Hayashi & Young, 1987). I consider the case where v'_{y0} is always negative, q''_0 is constant and positive, and both $v_{y0} = 0$ and $q'_0 = 0$ are satisfied at $x = 0$. Note that this system satisfies

both the Rayleigh's inflection theorem and the Fjørtoft's theorem. In this case, there are two Rossby waves at the regions of $x < 0$ and $x > 0$, respectively.

I pay my attention to the left Rossby wave ($x < 0$). In the region of $x < 0$, the background velocity v_{y0} is in the $+y$ direction. If the effects from the right Rossby wave are neglected, the differential phase velocity of the left Rossby wave $\delta v_{y,\text{phase}}^{\text{left}}$ is in the $-y$ direction from equation (4.16). In actuality, the differential phase velocity of the left Rossby wave $\delta v_{y,\text{phase}}^{\text{left}}$ is affected by the perturbed velocity fields induced by the right Rossby wave.

I define the phase difference normalized by $2\pi/k_y$ as $\Delta\theta$ ($-\pi \leq \Delta\theta < \pi$) as shown in top left panel of Figure 4.2. I consider four cases with $\Delta\theta = 0, \pi/2, \pi, -\pi/2$. For $\Delta\theta = 0$, the perturbed velocity fields induced by the right Rossby wave assist those induced by the left Rossby wave. Then, the speed of $\delta v_{y,\text{phase}}^{\text{left}}$ becomes faster. On the other hand, the perturbed velocity fields induced by the right Rossby wave disturb those induced by the left Rossby wave for $\Delta\theta = \pi$. The speed of $\delta v_{y,\text{phase}}^{\text{left}}$ becomes slower. For $\Delta\theta = \pm\pi/2$, the perturbed velocity fields induced by the right Rossby wave have no effect on those induced by the left Rossby wave and $\delta v_{y,\text{phase}}^{\text{left}}$ does not change.

The phase velocity of the left Rossby wave in the y direction, $v_{y,\text{phase}}^{\text{left}} \equiv v_{y0} + \delta v_{y,\text{phase}}^{\text{left}}$, has a minimum for $\Delta\theta = 0$ and a maximum for $\Delta\theta = \pi$. Due to the symmetry of the system, $v_{y,\text{phase}}^{\text{left}}(\Delta\theta) = v_{y,\text{phase}}^{\text{left}}(-\Delta\theta)$ should be satisfied. For $v_{y,\text{phase}}^{\text{left}}(\Delta\theta = 0) \cdot v_{y,\text{phase}}^{\text{left}}(\Delta\theta = \pi) < 0$, there is a phase difference $\Delta\theta_c (> 0)$ which satisfies $v_{y,\text{phase}}^{\text{left}}(\Delta\theta_c) = v_{y,\text{phase}}^{\text{left}}(-\Delta\theta_c) = 0$. At those times, the phase velocity of the right Rossby wave vanishes as well as that of the left Rossby wave due to the symmetry of the system. The two Rossby waves are stationary for $\Delta\theta = \pm\Delta\theta_c$.

The modes with $\Delta\theta = \pm\Delta\theta_c$ correspond to the stable and unstable modes against the shear instability, respectively. Since the Rossby wave is neutral wave, the perturbed velocity fields induced by itself have no effect on the amplitude of the Rossby wave. However, another Rossby wave can effects on the amplitude of the other Rossby wave. For $\Delta\theta = \Delta\theta_c$, the perturbed velocity fields induced the right Rossby wave conveys positive net flux of vorticity to the positive vorticity region of the left Rossby wave, and vice versa. This in-

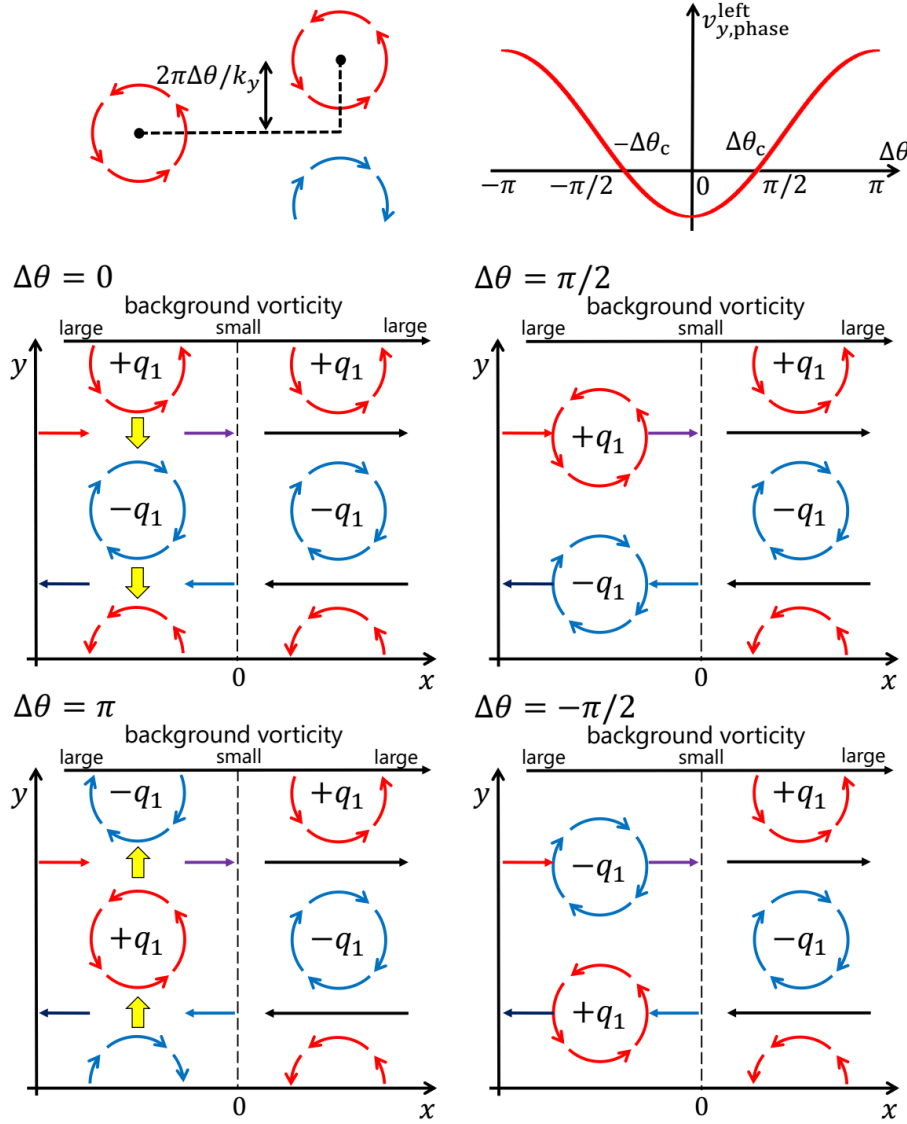


Figure 4.2: The schematic illustration of the effects of the right Rossby wave on the phase velocity of the left Rossby wave in the y direction. Top left panel shows the definition of the phase difference $\Delta\theta$. Top right panel shows the dependence of $v_{y,\text{phase}}^{\text{left}}$ on the phase difference. The other panels show the effects of the right Rossby wave on the phase velocity of the left Rossby wave in the y direction for $\Delta\theta = 0, \pi/2, \pi, -\pi/2$, respectively. Black arrows: the perturbed velocity fields induced by the right Rossby wave. Red, purple, and blue arrows show the flow induced by the right Rossby wave with large, intermediate, and small vorticity, respectively. Yellow arrows: the effects of the right Rossby wave on the phase velocity in the y direction of the left Rossby wave.

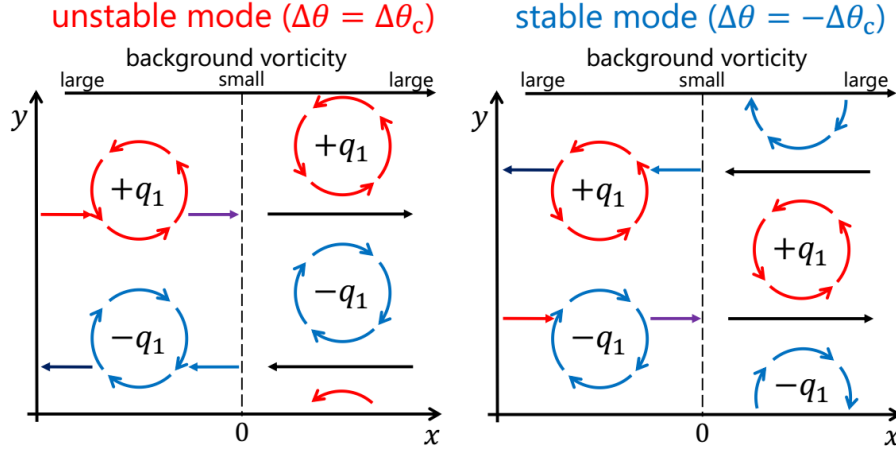


Figure 4.3: The schematic illustration of the effect of the right Rossby wave on the amplitude of the left Rossby wave for the $\Delta\theta = \Delta\theta_c$ (unstable mode) and for the $\Delta\theta = -\Delta\theta_c$ (stable mode), respectively. Black arrows: the perturbed velocity fields induced by the right Rossby wave. Red, purple, and blue arrows show the flow induced by the right Rossby wave with large, intermediate, and small vorticity, respectively.

indicates that the case with $\Delta\theta = \Delta\theta_c$ shows the unstable mode against the shear instability. On the other hand, the case $\Delta\theta = -\Delta\theta_c$ shows the stable mode against the shear instability. I show the schematic illustration describing the unstable state ($\Delta\theta = \Delta\theta_c$) and the stable state ($\Delta\theta = -\Delta\theta_c$) in Figure 4.3.

From the above, it is found that the *necessary and sufficient* condition for the onset of the shear instability is the existence of stationary two Rossby waves with positive phase difference ($\Delta\theta = \Delta\theta_c$). It is considered that the linear growth rate of the shear instability depends on the value of $\Delta\theta_c$. If the value of $\Delta\theta_c$ is smaller, the net vorticity flux conveyed by the perturbed velocity fields induced by the right Rossby wave is also smaller. Therefore, the case with $\Delta\theta_c = 0$ represents the marginally stable mode against the shear instability. On the other hand, the net vorticity flux is largest for $\Delta\theta_c = \pi/2$.

In the end of this section, I show the physical interpretations of the Rayleigh's inflection theorem and the Fjørtoft's theorem. Considering the case with negative v'_{y0} and no inflection point, the system cannot have a pair of the Rossby

waves whose differential phase velocity is in the opposite direction of the background flow. In this case, it is impossible to realize the existence of stationary two Rossby waves. Next, I consider the case with constantly negative q_0'' instead of constantly positive q_0'' . This system satisfies the Rayleigh's inflection theorem, but the Fjørtoft's theorem is not satisfied. The differential phase velocity of the Rossby wave is always in the same direction of the background flow. In this situation, the shear instability cannot occur.

4.3.2 Interpretation of the RWI

I turn my attention to the physical nature of the RWI. From the linear stability analyses in Chapter 3, the distributions of the vortensity perturbation and the surface density perturbation in the real space are given by $q_1(r, \varphi) = \text{Re}[q_1(r) \exp(im\varphi)]$ and $\Sigma_1(r, \varphi) = \text{Re}[\Sigma_1(r) \exp(im\varphi)]$, respectively. Figure 4.4 shows $q_1(r, \varphi)$ (top panels) and $\Sigma_1(r, \varphi)$ (bottom panels) for $\gamma/\Omega_n = 10^{-3}$ ($\mathcal{A}_0 = 0.04$; left panels), 0.01 ($\mathcal{A}_0 = 0.05$; middle panels) and 0.05 ($\mathcal{A}_0 = 0.123$; right panels) for case (iii) with $\Delta w_0 = 0.05r_n$ and $m = 1$ in the frame that rigidly rotates at the rotation velocity of the co-rotation radius. Note that the normalization constant of the perturbation $\bar{\Xi}$ is given by 0.025 (see Section 3.4.1 for the definition of $\bar{\Xi}$) in Figure 4.4. Since $r_c \approx r_n$ is satisfied in those cases, the background shear velocity is in the $-\varphi$ direction for $r \gtrsim r_n$ and in the $+\varphi$ direction for $r \lesssim r_n$. There are two perturbation patterns of both q_1 and Σ_1 in the φ direction, and it appears that the patterns hold the co-rotation radius in between. The patterns of the perturbations are stationary in the rotating frame. It should be noted that φ is shown in the range of $0 \leq \varphi \leq 3\pi$ for legibility.

I look at the q_1 patterns. Stationary two Rossby waves with a positive phase difference are observed in each case. It is found that the configurations of the q_1 patterns for the RWI are same as those of the vorticity perturbations for the shear instability in incompressible flow. This indicates that the RWI has common physical nature as the shear instability. As I predicted in Section 4.3.1, the phase difference of the two Rossby waves increases as the linear growth rate of the RWI increases. I summarize the interpretation for the

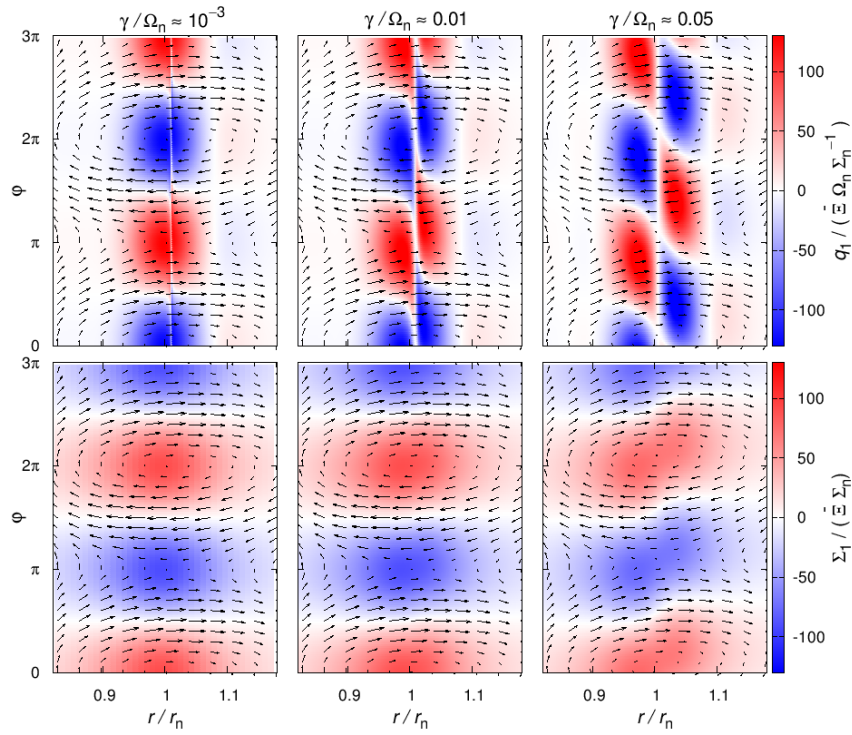


Figure 4.4: q_1 (top panels) and Σ_1 (bottom panels) are shown in the r - φ plane with $\gamma/\Omega_n \simeq 10^{-3}$ ($\mathcal{A}_0 = 0.04$; left panels), 0.01 ($\mathcal{A}_0 = 0.05$; middle panels) and 0.05 ($\mathcal{A}_0 = 0.123$; right panels) for $m = 1$ in the rotating frame with the co-rotation radius for case (iii) with $\Delta w_0 = 0.05 r_n$. Black arrows show velocity field $\mathbf{v}_0 + \xi \mathbf{v}_1$ and we adopt $\bar{\Xi} = 0.025$. For legibility, φ is shown in the range of $0 \leq \varphi \leq 3\pi$.

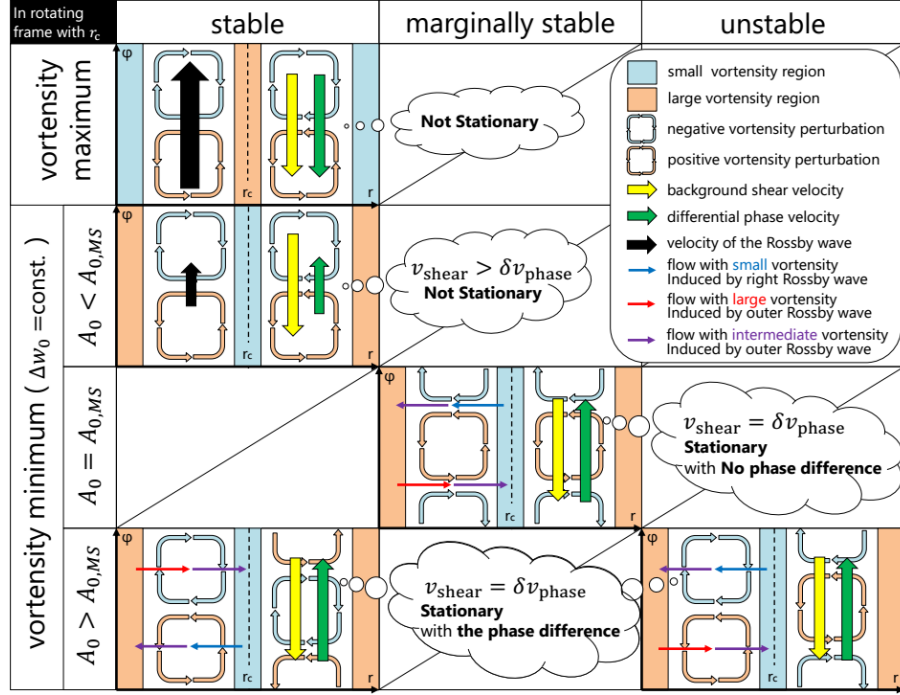


Figure 4.5: Schematic illustration of the physical nature of the RWI.

physical nature of the RWI using the interaction between two Rossby waves in Figure 4.5.

Here, I also discuss the Σ_1 patterns. Positive Σ_1 regions have clockwise velocity field around themselves, while negative Σ_1 regions have anti-clockwise velocity field. This indicates that positive q_1 regions roughly corresponds to the negative Σ_1 regions, and vice versa. In other words, q_1 has an anti-correlation with Σ_1 . It appears that the Σ_1 patterns get distorted due to the background shear, and the deformation gets larger with the linear growth rate. Due to the deformation, there is more incoming flow at the positive Σ_1 regions than outgoing flow. On the other hand, there is more outgoing flow at the negative Σ_1 regions than incoming flow. It is possible to confirm the unstable states using the configuration of the surface density perturbations.

4.4 Analytic Three Layers Model of the Shear Instability

In order to obtain the linear growth rate of the shear instability ($\text{Im}(\omega)$) and the distribution of the eigenfunction (Υ_1), it is required to solve equation (4.8). In general, it is difficult to solve equation (4.8) in an analytic way rather than a numerical calculation. Here, I introduce the three layers model in incompressible flow. In this model, it is possible to solve equation (4.8) analytically.

When I consider the background flow satisfying

$$v_{y0}(x) = \begin{cases} -\mathfrak{P}x + \mathfrak{Q}\mathfrak{L} & (x < -\mathfrak{L}), \\ -(\mathfrak{P} + \mathfrak{Q})x & (-\mathfrak{L} < x < \mathfrak{L}), \\ -\mathfrak{P}x - \mathfrak{Q}\mathfrak{L} & (\mathfrak{L} < x), \end{cases}$$

the background vorticity is given as

$$q_0(x) = \begin{cases} -\mathfrak{P} & (x < -\mathfrak{L}), \\ -(\mathfrak{P} + \mathfrak{Q}) & (-\mathfrak{L} < x < \mathfrak{L}), \\ -\mathfrak{P} & (\mathfrak{L} < x), \end{cases}$$

and the radial gradient of the background vorticity is given as

$$q'_0 = \mathfrak{P}\{-\delta(x + \mathfrak{L}) + \delta(x - \mathfrak{L})\}, \quad (4.23)$$

where $\delta(x)$ denotes the delta function. This background profile is divided into three regions (I) $x < -\mathfrak{L}$, (II) $-\mathfrak{L} < x < \mathfrak{L}$, and (III) $x > \mathfrak{L}$.

In this case, the effective potential is almost flat $V_{\text{eff}} = k_y^2$ except at $x = \pm\mathfrak{L}$. Assuming $\Upsilon_1 \rightarrow 0$ at $x = \pm\infty$, the eigenfunction Υ_1 is given as

$$\Upsilon_1(x) = \begin{cases} \Upsilon_{\text{I}}(x) = \mathfrak{R}_1 \exp(k_y x) & (x < -\mathfrak{L}), \\ \Upsilon_{\text{II}}(x) = \mathfrak{R}_2 \exp(k_y x) + \mathfrak{R}_3 \exp(-k_y x) & (-\mathfrak{L} < x < \mathfrak{L}), \\ \Upsilon_{\text{III}}(x) = \mathfrak{R}_4 \exp(-k_y x) & (\mathfrak{L} < x), \end{cases}$$

where $\mathfrak{R}_1$, $\mathfrak{R}_2$, $\mathfrak{R}_3$, and $\mathfrak{R}_4$ are the complex numbers and constant. The eigenfunction should be continuous and smooth. Using equation (4.8), the boundary conditions at $x = -\mathfrak{L}$ are derived as

$$\Upsilon_{\text{I}}(-\mathfrak{L}) = \Upsilon_{\text{II}}(-\mathfrak{L}), \quad (4.24)$$

$$\Upsilon'_{\text{II}}(-\mathfrak{L}) - \Upsilon'_{\text{I}}(-\mathfrak{L}) = \left[\frac{k_y \mathfrak{P}}{\omega - k_y v_{y0}(-\mathfrak{L})} \right] \Upsilon_{\text{I}}(-\mathfrak{L}). \quad (4.25)$$

In the same manner, the boundary condition at $x = \mathfrak{L}$ are derived as

$$\Upsilon_{\text{II}}(\mathfrak{L}) = \Upsilon_{\text{III}}(\mathfrak{L}), \quad (4.26)$$

$$\Upsilon'_{\text{III}}(\mathfrak{L}) - \Upsilon'_{\text{II}}(\mathfrak{L}) = \left[\frac{-k_y \mathfrak{P}}{\omega - k_y v_{y0}(\mathfrak{L})} \right] \Upsilon_{\text{III}}(\mathfrak{L}). \quad (4.27)$$

Equations (4.24)–(4.27) are rewritten as

$$\mathfrak{R}_1 = \mathfrak{R}_2 + \exp(2k_y \mathfrak{L}) \mathfrak{R}_3, \quad (4.28)$$

$$\left(1 + \frac{\mathfrak{P}}{\omega - (\mathfrak{P} + \mathfrak{Q})k_y \mathfrak{L}} \right) \mathfrak{R}_1 = \mathfrak{R}_2 - \exp(2k_y \mathfrak{L}) \mathfrak{R}_3, \quad (4.29)$$

$$\mathfrak{R}_4 = \exp(2k_y \mathfrak{L}) \mathfrak{R}_2 + \mathfrak{R}_3, \quad (4.30)$$

$$\left(\frac{\mathfrak{P}}{\omega + (\mathfrak{P} + \mathfrak{Q})k_y \mathfrak{L}} - 1 \right) \mathfrak{R}_4 = \exp(2k_y \mathfrak{L}) \mathfrak{R}_2 - \mathfrak{R}_3. \quad (4.31)$$

From equations (4.28)–(4.31), I eliminate $\mathfrak{R}_1$ and $\mathfrak{R}_4$, and obtain

$$\begin{pmatrix} \mathfrak{P} & [2\{\omega - (\mathfrak{P} + \mathfrak{Q})k_y \mathfrak{L}\} + \mathfrak{P}] \exp(-2k_y \mathfrak{L}) \\ [\mathfrak{P} - 2\{\omega + (\mathfrak{P} + \mathfrak{Q})k_y \mathfrak{L}\}] \exp(-2k_y \mathfrak{L}) & \mathfrak{P} \end{pmatrix} \begin{pmatrix} \mathfrak{R}_2 \\ \mathfrak{R}_3 \end{pmatrix} = 0.$$

In order to have non-trivial values of $\mathfrak{R}_2$ and $\mathfrak{R}_3$,

$$\det \begin{pmatrix} \mathfrak{P} & [2\{\omega - (\mathfrak{P} + \mathfrak{Q})k_y \mathfrak{L}\} + \mathfrak{P}] \exp(-2k_y \mathfrak{L}) \\ [\mathfrak{P} - 2\{\omega + (\mathfrak{P} + \mathfrak{Q})k_y \mathfrak{L}\}] \exp(-2k_y \mathfrak{L}) & \mathfrak{P} \end{pmatrix} = 0$$

is needed. Finally I obtain the dispersion relation as

$$\omega^2 = \frac{1}{4} [\{\mathfrak{P} - 2(\mathfrak{P} + \mathfrak{Q})k_y \mathfrak{L}\}^2 - \mathfrak{P}^2 \exp(-4k_y \mathfrak{L})]. \quad (4.32)$$

When the system is marginally stable against the shear instability, $\omega^2 = 0$

should be satisfied. Then, equation (4.32) is rewritten as

$$\mathfrak{P} - 2(\mathfrak{P} + \mathfrak{Q})k_y\mathfrak{L} = \pm\mathfrak{P}\exp(-2k_y\mathfrak{L}). \quad (4.33)$$

Assuming $k_y\mathfrak{L} \ll 1$, I obtain the condition for the marginally stable states against the shear instability as

$$\mathfrak{P} \approx k_y\mathfrak{L}\mathfrak{Q} \quad \text{for } \mathfrak{P} > 0, \quad (4.34)$$

$$\approx -\frac{\mathfrak{Q}}{k_y\mathfrak{L}} \quad \text{for } \mathfrak{P} < 0. \quad (4.35)$$

The parameter $\mathfrak{L}$ represents the representative length of the background variation and corresponds to Δw_0 in Chapter 3. In addition, the $\mathfrak{P}$ represents the background vorticity contrast. Comparing this three layers modes model with the disk model in Chapter 3, $\mathfrak{P}/\mathfrak{L}^2$ corresponds to $\mathcal{A}_0$ and is positive. From equation 4.34, I can estimate from the analytic model that the relation between $\mathcal{A}_0$ and Δw_0 :

$$\mathcal{A}_0 \propto \Delta w_0^3. \quad (4.36)$$

This estimation from the analytic model is consistent with the results of the linear stability analyses, especially for small Δw_0 (see Section 3.4.3).

Hereafter, I set $\mathfrak{Q}$ to zero to avoid complexity. From equation (4.32), the first derivative of ω^2 with respect to $k_y\mathfrak{L}$ is written as

$$\frac{\partial\omega^2}{\partial(k_y\mathfrak{L})} = \mathfrak{P}^2 [\exp(-4k_y\mathfrak{L}) - 1 + 2k_y\mathfrak{L}]. \quad (4.37)$$

Assuming $k_y\mathfrak{L} \ll 1$, I obtain

$$\frac{\partial\omega^2}{\partial(k_y\mathfrak{L})} \approx 8\mathfrak{P}^2 \left[\left(k_y\mathfrak{L} - \frac{1}{8} \right)^2 - \frac{5}{16} \right]. \quad (4.38)$$

This indicates that this model is most unstable against the shear instability for $k_y\mathfrak{L} \approx 1/8$ and the largest linear growth rate is about $\mathfrak{P}/8$.

Next, I turn my attention to the phase difference of two Rossby waves for the unstable states against the shear instability. From equation (4.23), it is expected that the two stationary Rossby waves are located at $x = \pm\mathfrak{L}$ for

the unstable states, respectively. Then, the phase difference $\Delta\theta$ is given as

$$\Delta\theta = \arg\left(\frac{\Upsilon_{\text{III}}(\mathfrak{L})}{\Upsilon_{\text{I}}(-\mathfrak{L})}\right) = \arg\left(\frac{\mathfrak{R}_4}{\mathfrak{R}_1}\right). \quad (4.39)$$

Using equations (4.28)–(4.31), I obtain

$$\frac{\mathfrak{R}_4}{\mathfrak{R}_1} = \left(2 + \frac{\mathfrak{P}}{\omega - k_y \mathfrak{L} \mathfrak{P}}\right) \left(\frac{\omega + k_y \mathfrak{L} \mathfrak{P}}{\mathfrak{P}}\right) \exp(2k_y \mathfrak{L}). \quad (4.40)$$

For the unstable states, the mode frequency ω is denoted by $\omega \equiv i\gamma$ because ω^2 is real number. When I consider the slightly unstable modes $0 < \gamma \ll \mathfrak{P}$, the real part and the imaginary part of $\mathfrak{R}_4/\mathfrak{R}_1$ are obtained as

$$\text{Re}\left[\frac{\mathfrak{R}_4}{\mathfrak{R}_1}\right] \approx \exp(2k_y \mathfrak{L})(2k_y \mathfrak{L} - 1), \quad (4.41)$$

$$\text{Im}\left[\frac{\mathfrak{R}_4}{\mathfrak{R}_1}\right] \approx \exp(2k_y \mathfrak{L}) \frac{2\gamma}{\mathfrak{P}} \left(1 - \frac{1}{k_y \mathfrak{L}}\right). \quad (4.42)$$

From equations (4.41)–(4.42), $\Delta\theta$ is derived as

$$\Delta\theta \approx \tan^{-1} \left[\frac{2\gamma}{\mathfrak{P}} \frac{(1 - 1/k_y \mathfrak{L})}{(2k_y \mathfrak{L} - 1)} \right] \approx \frac{2\gamma}{\mathfrak{P}} \frac{(1 - 1/k_y \mathfrak{L})}{(2k_y \mathfrak{L} - 1)}. \quad (4.43)$$

Assuming $\Delta\theta \ll 1$, it is found that the phase difference is proportion to the linear growth rate of the shear instability. This is consistent with the discussions in Section 4.3.

4.5 Summary of Chapter 4

In this chapter, I show the interpretation of the shear instability using the interaction between two Rossby waves. The Rossby wave is one of the neutral modes of the hydrodynamic fluid. When the system satisfy both the Rayleigh's inflection theorem and the Fjørtoft's theorem, there is a possibility that two stationary Rossby waves exist at the position between which the critical layer (the co-rotation radius). Then, there are two kinds of the configurations of the two stationary Rossby waves. For the positive phase difference, the configura-

tion describe the unstable modes. In the other hand, the system is stable for the negative phase difference. The linear growth rate of the shear instability depends on the phase difference between the two Rossby waves. Finally I confirm that the physical nature of the RWI can be explained in the same manner as that of the shear instability. I also show the the analytic three layers model in the incompressible flow. This analytic model helps us understand the shear instability and the RWI.

Chapter 5

Non-Linear Evolution of the RWI & RWI Vortices

I perform numerical simulations of the vortex formation by the RWI in 2D barotropic disks using the Athena++ code. As initial conditions, I consider axisymmetric disks with a Gaussian surface density bump of various radial contrasts and half-widths. Multiple vortices are formed initially in accordance with the most unstable azimuthal mode of the RWI. The vortices coalesce one after another, and only one quasi-stationary vortex (the RWI vortex) remains in the end, which then migrate to the inner part of the disk.

I investigate structures and motions of the RWI vortices formed in my numerical calculations. The RWI vortices have strong similarity with the steady solution of vortices (known as the Kida solution) in the Keplerian shear. I obtain empirical relations between the properties of the RWI vortices (the aspect ratio, the radial and azimuthal size, the migration speed, the surface density contrast at the vortex center) and the initial conditions.

Chapter 5 is organized as follows. I show a parameter space and numerical set-up in Section 5.1. In Section 5.2, I show results and analyses of numerical simulations and obtain empirical relations of the vortice formed by the RWI. Section 5.3 is for discussion to understand the vortices formed by the RWI, and Section 5.4 is the summary of this chapter.

5.1 Parameter Space and Numerical Method

5.1.1 Parameter Space of Numerical Calculations

I perform numerical calculations with various initial conditions, and investigate the time evolution of the disks and the vortices. I adopt the same disk model as that in Chapter 3 and the background disks with a Gaussian bump (case (iii)):

$$\frac{\Sigma_0}{\Sigma_n} = 1 + \mathcal{A}_0 \exp \left[-\frac{1}{2} \left(\frac{r - r_n}{\Delta w_0} \right)^2 \right], \quad (5.1)$$

as the initial conditions. Note that I consider only for $\Gamma = 5/3$ in my simulations. Hereafter, I normalize $r_n = \Sigma_n = \Omega_n = 1$. Considering the system is scale free with the disk surface density when disk self-gravity is neglected, I can set $\Sigma_n = 1$ without loss of generality.

Now, the initial conditions are characterized by three parameters: h , Δw_0 , $\mathcal{A}_0$. At first, I fix h and Δw_0 and vary $\mathcal{A}_0$. From the results of Chapter 3, the largest linear growth rate of the RWI, $\gamma_*(\mathcal{A}_0 : h, \Delta w_0)$, monotonically increases with $\mathcal{A}_0$ for the RWI unstable states. If, however, $\mathcal{A}_0$ exceeds a certain value, $\mathcal{A}_{0,\max}(h, \Delta w_0)$, the system violates the Rayleigh's condition and is prone to the rotational instability. Since the linear growth rate of the rotational instability is typically larger than that of the RWI, I expect that the system which is unstable against the rotational instability immediately gets to the marginally stable states of the rotational instability ($\mathcal{A}_0 = \mathcal{A}_{0,\max}(h, \Delta w_0)$). I therefore consider the cases where the system does not violate the Rayleigh's condition. In other words, I consider the cases with $\mathcal{A}_0 < \mathcal{A}_{0,\max}(h, \Delta w_0)$ as the initial conditions for numerical calculations. The maximum of the largest linear growth rate of the RWI is limited below the value of that with $\mathcal{A}_0 = \mathcal{A}_{0,\max}(h, \Delta w_0)$, which I denote by $\gamma_{*,\max}(\mathcal{A}_{0,\max} : h, \Delta w_0)$.

When three parameters (h , Δw_0 , γ_*) are given, $\mathcal{A}_0$ is uniquely determined. I vary h and Δw_0 in the ranges of $h = [0.05, 0.1, 0.15, 0.2]$ and $\Delta w_0 = [0.02, 0.0356, 0.0632, 0.112, 0.2]$. I also vary γ_* in the ranges of $\gamma_*/\Omega_n = [\gamma_{*,\max}/\Omega_n, 0.2, 0.15, 0.1, 0.05]$ for $h = 0.1$ or $\gamma_*/\Omega_n = [\gamma_{*,\max}/\Omega_n, 0.1]$ for $h \neq 0.1$. I run 54 models in total whose parameter sets are shown in Tables

5.1 and 5.2. It is noted that I do not have the “h10w5g2” model because $\gamma_{*,\max}/\Omega_n$ is smaller than 0.2 for $h = 0.1$ and $\Delta w_0 = 0.2r_n$. I show the most unstable azimuthal mode m_* as well as the largest linear growth rate γ_* in Tables 5.1 and 5.2. The set-up of the linear stability analyses and the linear growth rates for $1 \leq m \leq 10$ of the RWI are shown in Appendix C.1.

Before starting the main calculation of each run, I evolve the disk numerically using the axisymmetric filter (see Section 5.1.2) for 10 orbits at $r = r_n$ in order to relax the initial profile to a numerical equilibrium. I then impose a small initial perturbation to the radial momentum to trigger the RWI. The Fourier components of the radial momentum for an azimuthal mode m are defined by $\mathcal{F}(\mathcal{M}_r)_m$. I consider two types of initial perturbations. The first is a white noise perturbation. I set the amplitude of $\mathcal{F}(\mathcal{M}_r)_m$ as $|\mathcal{F}(\mathcal{M}_r)_m| = 10^{-6} |\mathcal{F}(\mathcal{M}_\varphi)_{m=0}| \exp[\{(r-1)/0.2\}^2/2]$ for $1 \leq m \leq 128$, where $\mathcal{F}(\mathcal{M}_\varphi)_{m=0}$ is the axisymmetric Fourier component of the azimuthal momentum. Note that I set a maximum azimuthal mode of the white noise perturbation to $m = 128$ in order to avoid the effects from the numerical resolution. The phase of $\mathcal{F}(\mathcal{M}_{r,m})$ is randomly varied for each m . The other type of perturbations is a single mode perturbation. I use this perturbation to study the linear regime of a certain azimuthal mode $m = k$. In this case, I set the amplitude of $\mathcal{F}(\mathcal{M}_r)_m$ as $|\mathcal{F}(\mathcal{M}_r)_m| = 10^{-6} |\mathcal{F}(\mathcal{M}_\varphi)_{m=0}| \exp[\{(r-1)/0.2\}^2/2]$ for $m = k$ and $|\mathcal{F}(\mathcal{M}_\varphi)_{m=0}| = 0$ for $m \neq 0$ and k .

5.1.2 Numerical Method

I use the Athena++ code (Stone et al. , in prep.), with the HLLC approximate Riemann Solver, the second-order piece-wise linear reconstruction, and the second-order van-Leer time integrator. I assume barotropic flow for simplicity, and therefore I overwrite the pressure after every time step so that equation:

$$P_0(r) = S_0 \Sigma_0^\Gamma, \quad (5.2)$$

is always satisfied. The computational domain extends $r_{\text{in}} < r < r_{\text{out}}$, where I set $r_{\text{in}} = 0.3$ and $r_{\text{out}} = 2.5$, in the radial direction and covers full 2π in the azimuthal direction in the 2D cylindrical coordinates. The radial extension of

Table 5.1: The parameter sets of the models for $h = 0.1$.

Name	h	$\Delta w_0/r_n$	γ_*/Ω_n	$\mathcal{A}_0$	m_*
h10w1g1	0.1	2.00E-2	0.227	4.04E-2	9
h10w1g2	0.1	2.00E-2	0.200	3.40E-2	8
h10w1g3	0.1	2.00E-2	0.150	2.41E-2	7
h10w1g4	0.1	2.00E-2	0.100	1.57E-2	6
h10w1g5	0.1	2.00E-2	0.050	8.70E-3	4
h10w2g1	0.1	3.56E-2	0.242	1.31E-1	6
h10w2g2	0.1	3.56E-2	0.201	1.06E-1	5
h10w2g3	0.1	3.56E-2	0.150	7.81E-2	5
h10w2g4	0.1	3.56E-2	0.100	5.46E-2	4
h10w2g5	0.1	3.56E-2	0.050	3.42E-2	3
h10w3g1	0.1	6.32E-2	0.246	4.39E-1	4
h10w3g2	0.1	6.32E-2	0.200	3.58E-1	4
h10w3g3	0.1	6.32E-2	0.150	2.77E-1	3
h10w3g4	0.1	6.32E-2	0.100	2.05E-1	3
h10w3g5	0.1	6.32E-2	0.050	1.42E-1	2
h10w4g1	0.1	1.12E-1	0.227	1.57E+0	3
h10w4g2	0.1	1.12E-1	0.200	1.38E+0	2
h10w4g3	0.1	1.12E-1	0.150	1.06E+0	2
h10w4g4	0.1	1.12E-1	0.100	8.02E-1	2
h10w4g5	0.1	1.12E-1	0.050	6.00E+1	2
h10w5g1	0.1	2.00E-1	0.191	5.66E+0	2
h10w5g3	0.1	2.00E-1	0.150	4.69E+0	2
h10w5g4	0.1	2.00E-1	0.100	3.39E+0	1
h10w5g5	0.1	2.00E-1	0.050	2.33E+0	1

NOTE. Name: the name of the model. h : the dimensionless disk aspect ratio. $\Delta w_0/r_n$: the radial half-width of the initial bump normalized by r_n . γ_*/Ω_n : the largest linear growth rate against the RWI normalized by Ω_n . $\mathcal{A}_0$: the radial surface density contrast of the initial bump. m_* : the most unstable azimuthal mode.

Table 5.2: The parameter sets of the models for $h \neq 0.1$.

Name	h	$\Delta w_0/r_n$	γ_*/Ω_n	$\mathcal{A}_0$	m_*
h20w1g1	0.2	2.00E-2	0.209	1.00E-2	8
h20w1g4	0.2	2.00E-2	0.100	3.97E-3	4
h20w2g1	0.2	3.56E-2	0.225	3.17E-2	5
h20w2g4	0.2	3.56E-2	0.100	1.23E-2	3
h20w3g1	0.2	6.32E-2	0.237	1.00E-1	3
h20w3g4	0.2	6.32E-2	0.100	4.15E-2	3
h20w4g1	0.2	1.12E-1	0.237	3.17E-1	2
h20w4g4	0.2	1.12E-1	0.100	1.53E-1	2
h20w5g1	0.2	2.00E-1	0.192	9.79E-1	2
h20w5g4	0.2	2.00E-1	0.100	5.38E-1	1
h15w1g1	0.15	2.00E-2	0.216	1.78E-2	8
h15w1g4	0.15	2.00E-2	0.100	6.89E-3	5
h15w2g1	0.15	3.56E-2	0.233	5.69E-2	5
h15w2g4	0.15	3.56E-2	0.100	3.56E-2	3
h15w3g1	0.15	6.32E-2	0.240	1.83E-1	4
h15w3g4	0.15	6.32E-2	0.100	8.13E-1	2
h15w4g1	0.15	1.12E-1	0.233	6.00E-1	2
h15w4g4	0.15	1.12E-1	0.100	2.96E-1	2
h15w5g1	0.15	2.00E-1	0.193	1.96E+0	2
h15w5g4	0.15	2.00E-1	0.100	1.13E+0	1
h05w1g1	0.05	2.00E-2	0.244	1.68E-1	11
h05w1g4	0.05	2.00E-2	0.100	7.14E-2	7
h05w2g1	0.05	3.56E-2	0.248	5.86E-1	7
h05w2g4	0.05	3.56E-2	0.100	2.77E-1	5
h05w3g1	0.05	6.32E-2	0.240	2.30E+0	4
h05w3g4	0.05	6.32E-2	0.100	1.18E+0	4
h05w4g1	0.05	1.12E-1	0.225	1.01E+1	3
h05w4g4	0.05	1.12E-1	0.100	5.28E+0	2
h05w5g1	0.05	2.00E-1	0.188	4.12E+1	2
h05w5g4	0.05	2.00E-1	0.100	2.52E+1	1

NOTE. Name: the name of the model. h : the dimensionless disk aspect ratio. $\Delta w_0/r_n$: the radial half-width of the initial bump normalized by r_n . γ_*/Ω_n : the largest linear growth rate against the RWI normalized by Ω_n . $\mathcal{A}_0$: the radial surface density contrast of the initial bump. m_* : the most unstable azimuthal mode.

the numerical domain is chosen so that all the effective Lindblad resonances (ELRs) and vortices reside within the computational domain.

I set the mesh size in such a way that the size of a cell is at least smaller than $0.04h$ in the vortex-forming region ($r \sim 1$). For $h = 0.1, 0.15$, and 0.2 , the mesh has 576 cells in the radial direction and 1596 grids in the azimuthal direction. For $h = 0.05$, the mesh has 1296 cells in the radial direction and 3744 cells in the azimuthal direction. From the resolution study (see Appendix D), the calculations with this mesh structure have sufficiently high resolution.

I adopt the non-reflective boundary conditions in the radial direction (Godon, 1996) and the periodic boundary conditions in the azimuthal direction. The non-reflective boundary conditions are designed to be non-reflective for one-dimensional simple waves only. Even for 2D nonlinear simulations, however, I have observed the strong reduction of the wave reflection at the radial boundaries. This non-reflective boundary conditions are also used in previous works (e.g., Paardekooper et al., 2010). Note that this non-reflective boundary conditions cannot vanish the wave reflection perfectly. However, the inner boundary does not have significant effects on the vortices formed by the RWI (see Appendix D).

I have further modified the original Athena++ code by introducing fast Fourier Transform (FFT) filters. I perform the Fourier transform of Σ , $\mathcal{M}_r$, $\mathcal{M}_\varphi$ in the azimuthal direction at every radius and every time step, where $\mathcal{M}_r$ is the momentum in the radial direction and $\mathcal{M}_\varphi$ is the momentum in the azimuthal direction. I denote the Fourier components for the azimuthal mode m by $\mathcal{F}(\Sigma)_m$, $\mathcal{F}(\mathcal{M}_r)_m$, $\mathcal{F}(\mathcal{M}_\varphi)_m$. I have developed two kinds of FFT filters, namely, “axisymmetric filter” and “single-mode filter”. For the relaxation of the initial conditions before the main calculations, I use the axisymmetric filter, where all the non-axisymmetric ($m \neq 0$) modes are filtered out. In the calculations where I investigate the linear regime of a specific azimuthal mode k (see Section 5.2.1), I use the single-mode filter, where all the non-axisymmetric components except for $m = k$ are set to zero. After these filters modify the Fourier components, I recalculate Σ , $\mathcal{M}_r$ and $\mathcal{M}_\varphi$ by the inverse Fourier transform, and update the quantities in the calculations.

5.2 Results and Analyses

5.2.1 Comparison with the Linear Stability Analyses

In this section, I verify the numerical calculations by comparing their results with those obtained by the linear stability analyses in Chapter 3. I perform two kinds of numerical calculations for the “h10w3g1” model ($h = 0.1$, $\Delta w_0 = 0.0632r_n$, $\gamma_*/\Omega_n = 0.246$, $\mathcal{A}_0 = 0.439$, and $m_* = 4$). I first perform the single mode calculations. I initially impose the single mode perturbation with only one specific azimuthal mode, which is *not* restricted to the most unstable azimuthal mode, and use the single-mode filter (Section 5.1.2) to filter out the other modes during calculations. The outcomes of the single mode calculations have only the m mode component and the axisymmetric component. The surface density of the axisymmetric component corresponds to the azimuthally averaged surface density $\langle \Sigma \rangle(r)$. The surface density of the m mode component $\Sigma_m(r, \varphi) \equiv \text{Real}[\mathcal{F}(\Sigma)_m \exp(im\varphi)]$ is obtained by subtracting $\langle \Sigma \rangle(r)$ from $\Sigma(r, \varphi)$. Second, I perform the white noise calculations in which I impose the white noise initially. The calculations run without the single-mode filter. As a post-process, I extract the profiles and time evolution of $\Sigma_m(r, \varphi)$ using the single mode filter.

Figure 5.1 shows the time evolution in $\Sigma_{m,\text{max}}$ of both the single mode calculation and the white noise calculation for each mode ($1 \leq m \leq 10$), where $\Sigma_{m,\text{max}}$ is the maximum value of Σ_m around $r = 1$. Here, $\tau \equiv t\Omega_n/2\pi$ is the time measured in the unit of the orbital period at $r = 1$. I also show the linear growth rates which are independently derived from the linear stability analyses. All the numerical calculations show excellent agreement with the linear analyses. The white noise calculations also show the weakly non-linear growth due to the two modes coupling for the mode with small γ_m . For example, I can observe the modes coupling regime between $m = 3$ and $m = 4$ to produce the $m = 1$ ($= 4 - 3$) mode in $4.4 \leq \tau \leq 6.9$. The weakly non-linear growth also accord with the growth estimated from the linear analyses. The last panel of Figure 5.1 shows the time evolution of the maximum value of the non-axisymmetric surface density ($\sum_{m \geq 1} \Sigma_m$) in the white noise calcu-

lation. Since the maximum value of $\sum_{m \geq 1} \Sigma_m$ is smaller than $\sum_{m \geq 1} \Sigma_{m,\max}$, the evolution line is below the sum of the linear growth lines for $1 \leq m \leq 8$. However, these two lines roughly match.

Figure 5.2 compares the distribution of $\Sigma_m(r, \varphi)$ normalized by $\Sigma_{m,\max}$ in the single mode calculations and the surface density perturbation normalized by the maximum value derived by the linear stability analyses for the $m = 1$ –4 modes. I match the azimuthal phase of them in such a way as to locate the point of $\Sigma_m = \Sigma_{m,\max}$ on the r -axis. Note that I also denote the surface density perturbation of the linear stability analyses by $\Sigma_m(r, \varphi)$. Except for the $m = 1$ mode in the white noise calculation, Σ_m profiles of the numerical calculations match the surface density perturbation of the linear stability analyses. The discrepancy of Σ_m profiles for $m = 1$ occurs because the $m = 1$ mode component of the white noise calculation is already in the weakly non-linear regime at $\tau = 5.6$ where two modes ($m = 3$ and 4) couple.

From Figures 5.1 and 5.2, my numerical calculations agree with the linear stability analyses in the linear regime and the weakly non-linear regime. These indicate that my results of both numerical calculations and linear stability analyses are reliable.

5.2.2 The Evolution of the "h10w3g1" Model

Here, I investigate the time evolution of the "h10w3g1" model ($h = 0.1$, $\Delta w_0 = 0.0632r_n$, $\gamma_*/\Omega_n = 0.246$, $\mathcal{A}_0 = 0.439$, and $m_* = 4$) in detail to describe typical evolution of the RWI and to introduce my analytical methods. Figure 5.3 shows the 2D distribution of Σ at $\tau = 0, 8, 11, 18, 20, 30, 50, 100$ and 150. The linear growth is saturated when $\max[\sum_{m \geq 1} \Sigma_m]$ approaches A_0 , and the system enters the fully non-linear phase at $\tau \sim 8$. At that time, four vortices are formed by fragmentation of the initial axisymmetric bump. The number of the vortices formed initially is in accordance with the most unstable azimuthal mode of the RWI, m_* . The vortices merge one after another ($4 \rightarrow 3$ at $\tau \sim 11$; $3 \rightarrow 2$ at $\tau \sim 18$; $2 \rightarrow 1$ at $\tau \sim 20$). In the end, the vortices settle into one quasi-stationary vortex (the RWI vortex) after the

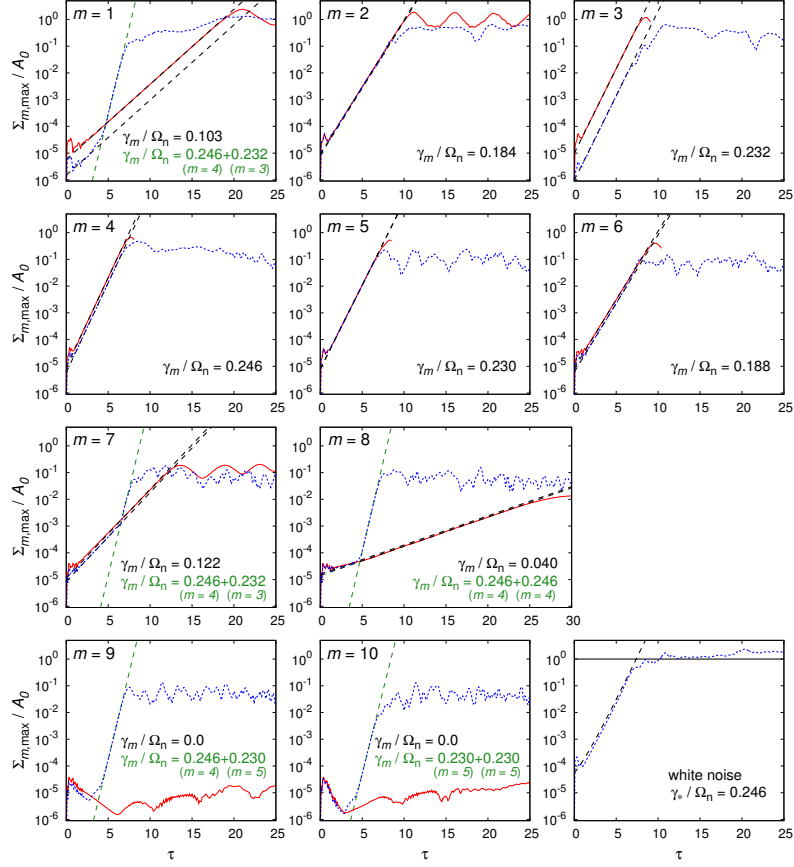


Figure 5.1: The linear growth of $\Sigma_{m,\max}$ in the single mode calculations (red lines) and the white noise calculations (blue dotted lines) of the “h10w3g1” model for each azimuthal mode m ($1 \leq m \leq 10$). Black dashed lines show the linear growth and green dashed lines show the growth expected from the mode-mode coupling based on the linear stability analyses. The last Panel shows the growth of the surface density perturbations in a real space in the white noise calculations.

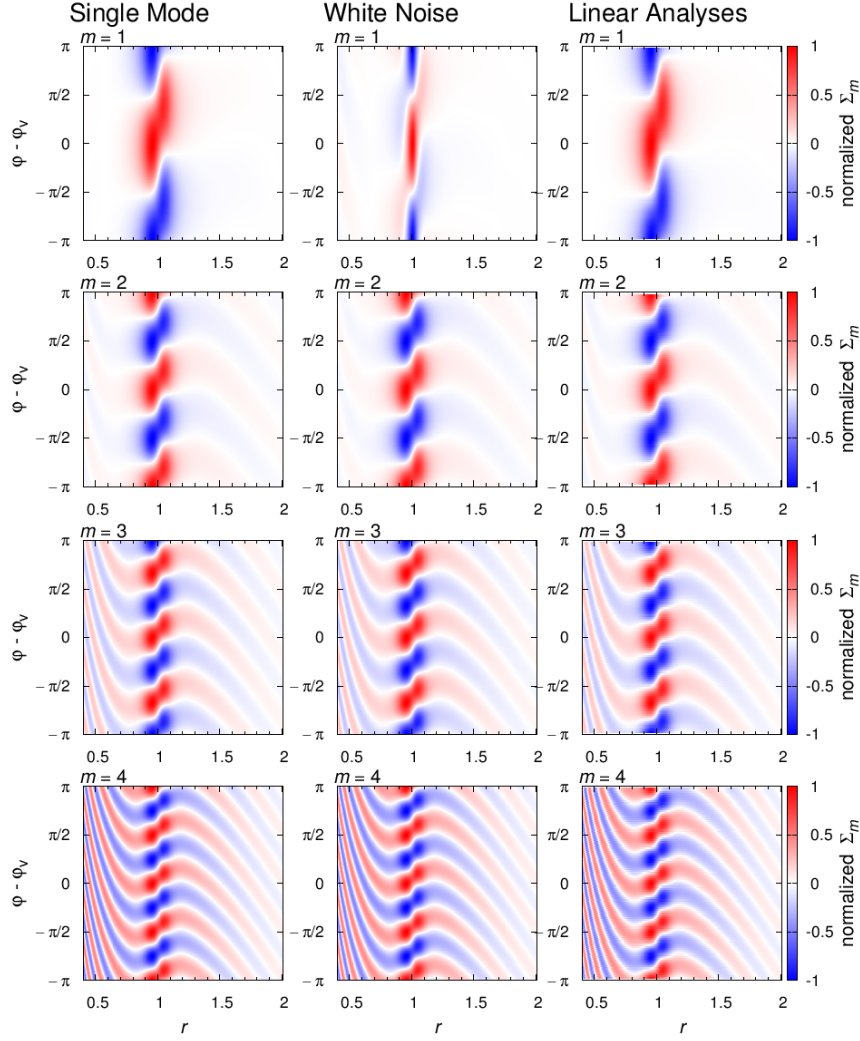


Figure 5.2: The distribution of the normalized surface density perturbations in the single mode calculations (the left column) and the white noise calculations (the middle column) of the “h10w3g1” model at $\tau = 5.6$ for each azimuthal mode m ($1 \leq m \leq 4$, from the top row to the bottom). The right panels show the distribution of the surface density eigenfunction derived from the linear stability analyses for each m .

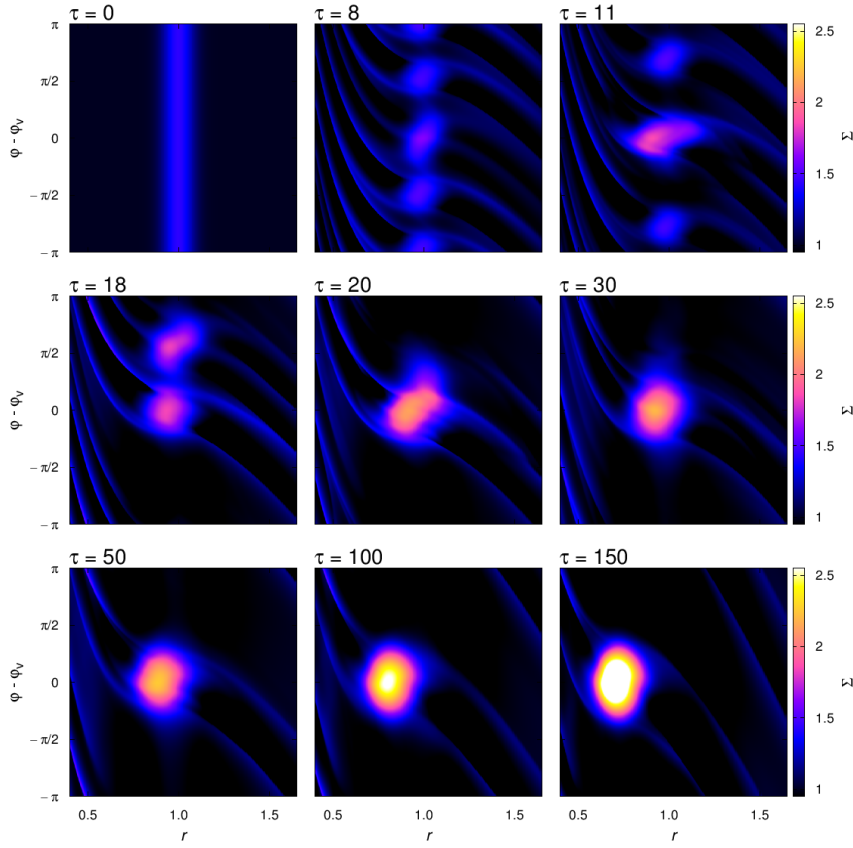


Figure 5.3: Snapshots of the surface density at $\tau = 0, 8, 11, 18, 20, 30, 50, 100$ and 150 in the white noise calculation of the "h10w3g1" model.

final merging of the vortices. Our calculations are terminated after sufficient time has passed since the RWI vortex formation. Note that we will discuss the saturation mechanism of the RWI in Section 5.3.2 and the timescale of the vortex merging in Section 5.3.3.

Structure of the RWI Vortex

I investigate the structure and the time evolution of the RWI vortex. I define the vortex center (r_v, φ_v) by

$$\delta v_\varphi(r_v, \varphi_v) = 0, \quad (5.3)$$

$$v_r(r_v, \varphi_v) = 0, \quad (5.4)$$

around the peak position of $\Sigma(r, \varphi)$, where $\delta v_\varphi(r, \varphi) \equiv v_\varphi(r, \varphi) - v_K(r)$. The vortex center roughly corresponds to the peak position of $\Sigma(r, \varphi)$. I do not discuss further the small difference between the location of the vortex center and the peak position of the surface density.

I calculate streamlines around the vortex center. As seen in Panel (a) of Figure 5.4, the streamlines which depart from $(r_v + b, \varphi_v)$ get back to their starting points, indicating that the flow is in quasi-steady state. The streamlines are almost elliptic in the r - r_v - φ plane and the semi-minor axes of the streamlines are aligned to the radial direction. I measure the semi-minor axis b (the radial direction), the semi-major axis a (the azimuthal direction), and the vortex aspect ratio $\chi \equiv a/b$ for each streamline. I also measure the turnover time of the streamline normalized by $2\pi/\Omega_v$, ψ :

$$\psi \equiv \frac{\Omega_v}{2\pi} \oint_{\mathcal{L}} \frac{d\ell}{\tilde{v}}, \quad (5.5)$$

where $\Omega_v \equiv \Omega_K(r_v)$, $\mathcal{L}$ is the one orbit length of the streamline, $d\ell$ is the line element along the streamline, $\tilde{v} = \sqrt{v_r^2 + (v_\varphi - r\Omega_v)^2}$ is the magnitude of the velocity field in the rotating frame with the vortex. Panels (b) and (c) of Figure 5.4 show the profiles of χ and ψ as functions of the normalized distance from the vortex center of each streamline, b/r_v , at $\tau = 100$. It is found that both χ and ψ are almost constant around the vortex center. When going beyond one disk scale height from the vortex center, these quantities are no longer constant and rapidly increase. In order to investigate the vortex aspect ratio and the turnover time around the vortex center, I define χ_2 and ψ_2 by χ and ψ at $b/r_v = 0.02$, and show the time evolutions of them in Panels (d) and (e) of Figure 5.4. I find that χ_2 and ψ_2 in the “h10w3g1” run are almost

constant ($\chi_2 \approx 4.34$, $\psi_2 \approx 2.36$).

Next, I investigate the distribution of the surface density Σ , the velocity field $\mathbf{v}$, and the pressure function Π , around the RWI vortex. From Panels (a), (b), (e), and (f) of Figure 5.5 and Panel (b) of Figure 5.6, I find that the surface density and the pressure function at the vortex center, $\Sigma_v \equiv \Sigma(r_v, \varphi_v)$ and $\Pi_v \equiv \Pi(r_v, \varphi_v)$, increase with time. From Panel (a) of Figure 5.6, the distance of the vortex location from the central star, r_v , decreases with time. These mean that the RWI vortex migrates toward the central star and the surface density contrast between the RWI vortex and the disk, $\zeta_v \equiv \Sigma_v/\Sigma_n$, increases with the vortex migration. I compare the RWI vortex with the analytic steady vortex (see Appendix A). If the RWI vortex is approximated by the steady vortex, it is expected that $\delta v_{\varphi v,r}/\Omega_v$, $v_{rv,\varphi}/(r_v\Omega_v)$, $\Pi_{v,rr}/\Omega_v^2$, and $\Pi_{v,\varphi\varphi}/(r_v\Omega_v)^2$ are almost constant because χ_2 and ψ_2 are approximately constant, where

$$\begin{aligned}\delta v_{\varphi v,r} &= \left[\frac{\partial \delta v_\varphi(r, \varphi_v)}{\partial r} \right]_{r=r_v}, \\ v_{rv,\varphi} &= \left[\frac{\partial v_r(r_v, \varphi)}{\partial \varphi} \right]_{\varphi=\varphi_v}, \\ \Pi_{v,rr} &= \left[\frac{\partial^2 \Pi(r, \varphi_v)}{\partial r^2} \right]_{r=r_v}, \\ \Pi_{v,\varphi\varphi} &= \left[\frac{\partial^2 \Pi(r_v, \varphi)}{\partial \varphi^2} \right]_{\varphi=\varphi_v}.\end{aligned}$$

Those are confirmed in Panels (c)–(f) of Figures 5.5 and 5.6 ($\delta v_{\varphi v,r}/\Omega_v \approx 0.344$, $v_{rv,\varphi}/(r_v\Omega_v) \approx 0.0966$, $\Pi_{v,rr}/\Omega_v^2 \approx 0.498$, and $\Pi_{v,\varphi\varphi}/(r_v\Omega_v)^2 \approx 0.0147$). I define the radial and azimuthal half widths Δr and $\Delta\phi$ by the half of the distance between two extrema of δv_ϕ and v_r (see Panels (c) and (d) of Figure 5.5). I find that Δr and $r_v\Delta\phi$ are also approximately constant from Panels (g) and (h) of Figure 5.6 ($\Delta r \approx 0.143r_n$, $r_v\Delta\phi \approx 0.581r_n$). In the vicinity of the vortex center, the radial profile of δv_ϕ and the azimuthal profile of v_r are almost on the straight lines as functions of $r - r_v$ and $\varphi - \varphi_v$, respectively. At the velocity extrema, the values of δv_ϕ and v_r are about two thirds times as large as $\delta v_{\varphi v,r}(r - r_v)$ and $\delta v_{rv,\varphi}(\varphi - \varphi_v)$, respectively.

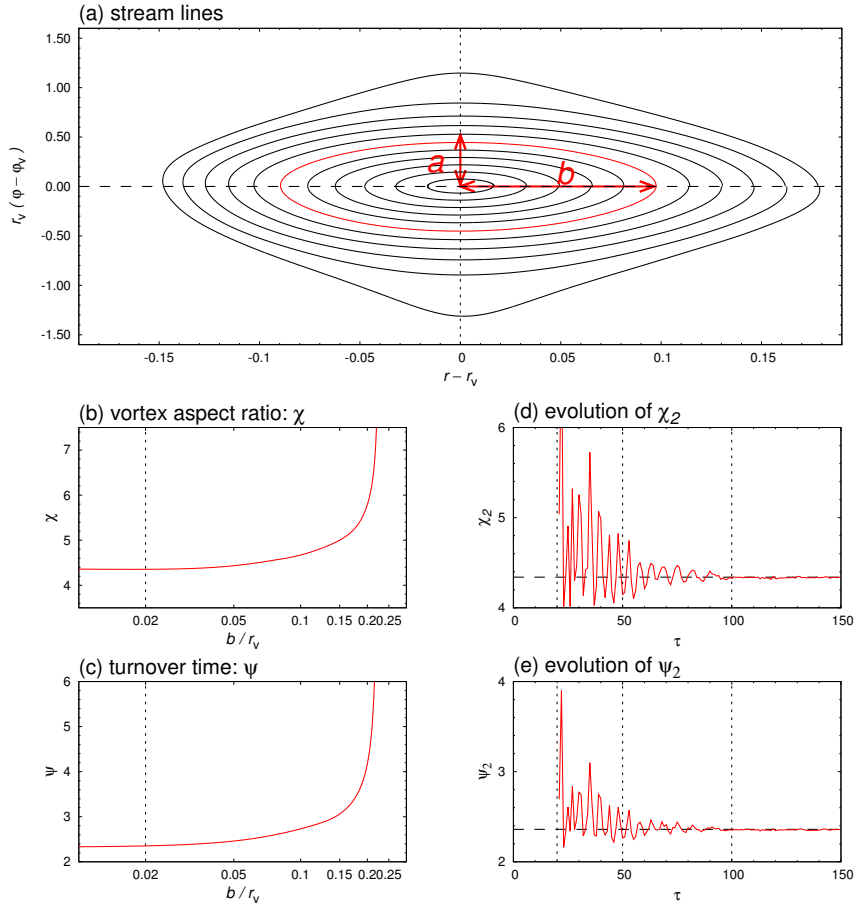


Figure 5.4: The structures and evolution of streamlines around the vortex center in the "h10w3g1" run. Panel (a) shows the streamlines around the RWI vortex at $\tau = 100$. Panels (b) and (c) show the profiles of χ and ψ at $\tau = 100$, respectively. The time evolution of χ_2 and ψ_2 are shown in Panels (d) and (e), respectively.

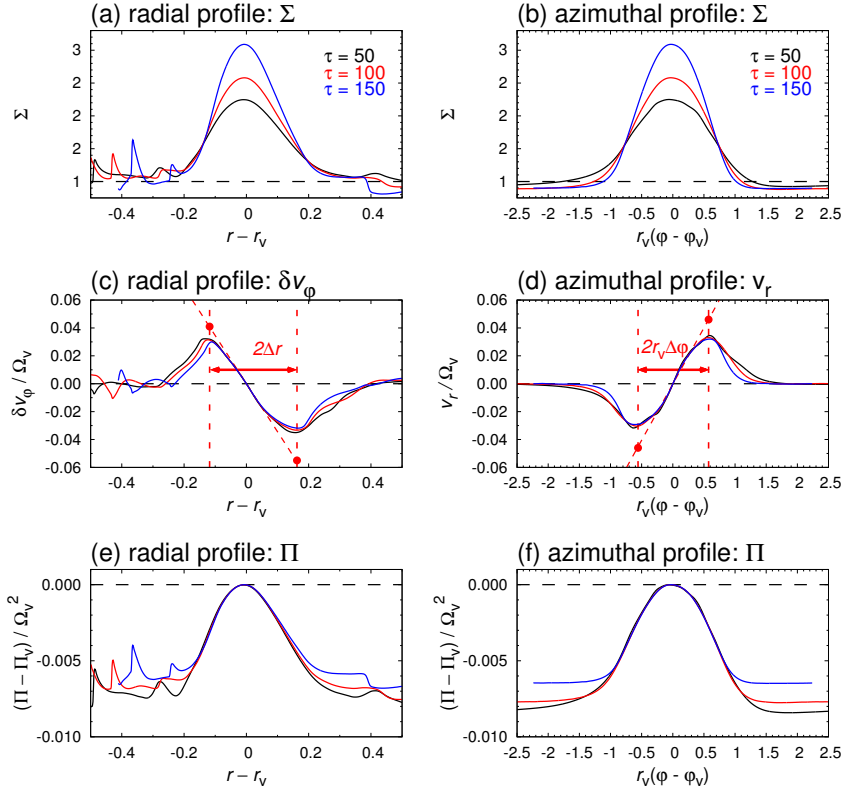


Figure 5.5: The structures of the RWI vortex in the "h10w3g1" run $\tau = 50$ (black), 100 (red), 150 (blue), respectively. Panels (a)–(f) show the profiles of $\Sigma(r, \varphi_v)$, $\Sigma(r_v, \varphi)$, $\delta v_\varphi(r, \varphi_v)/\Omega_v$, $v_r(r_v, \varphi)/\Omega_v$, $(\Pi(r, \varphi_v) - \Pi_v)/\Omega_v^2$, and $(\Pi(r_v, \varphi) - \Pi_v)/\Omega_v^2$. The red dashed line in Panels (c) (or (d)) shows the radial (azimuthal) gradient of δv_φ (v_r) at the vortex center. Red points show the values of $\delta v_\varphi(r, \varphi_v)/\Omega_v$ and $v_r(r_v, \varphi)/\Omega_v$ expected from the velocity gradient at the vortex center.

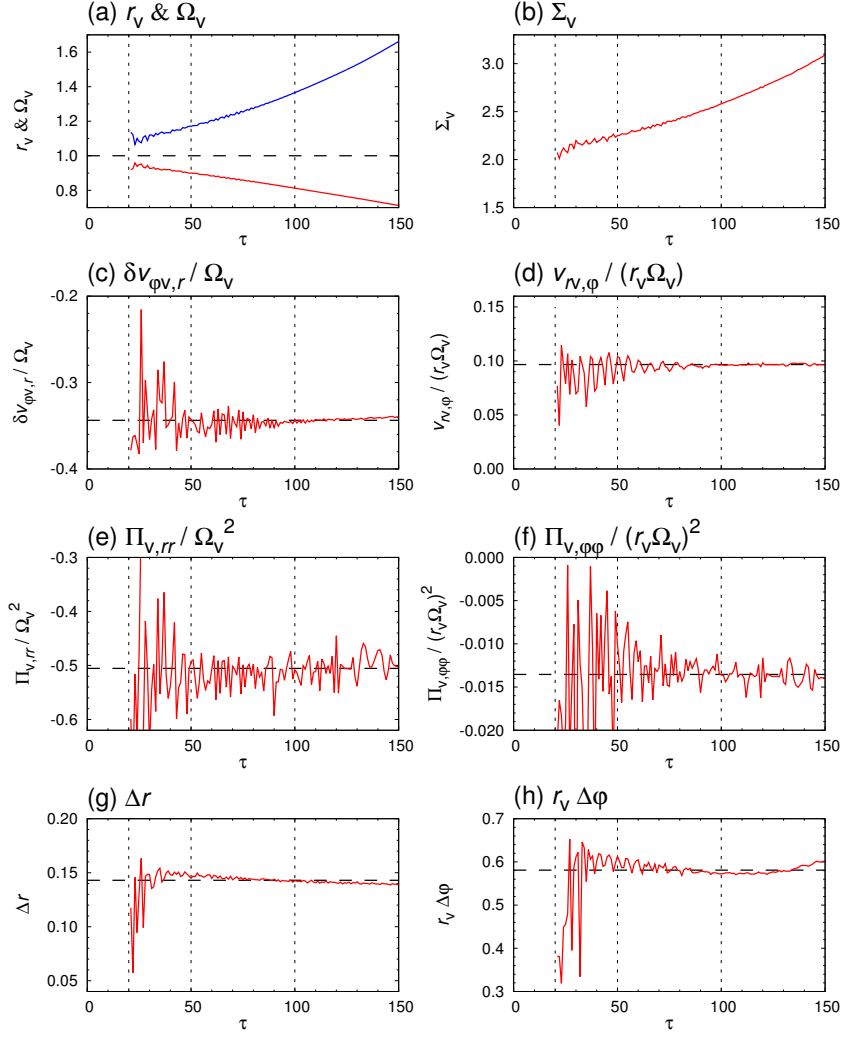


Figure 5.6: The time evolution of the approximately constant parameters of the RWI vortex in the "h10w3g1" run. Each panel shows the time evolution of (a) r_v (red) and Ω_v (blue), (b) Σ_v , (c) $\delta v_{\phi v, r} / \Omega_v$, (d) $v_{rv, \phi} / \Omega_v$, (e) $\Pi_{v, rr} / \Omega_v^2$, (f) $\Pi_{v, \phi \phi} / (r_v \Omega_v)^2$, Δr , and $r_v \Delta \phi$.

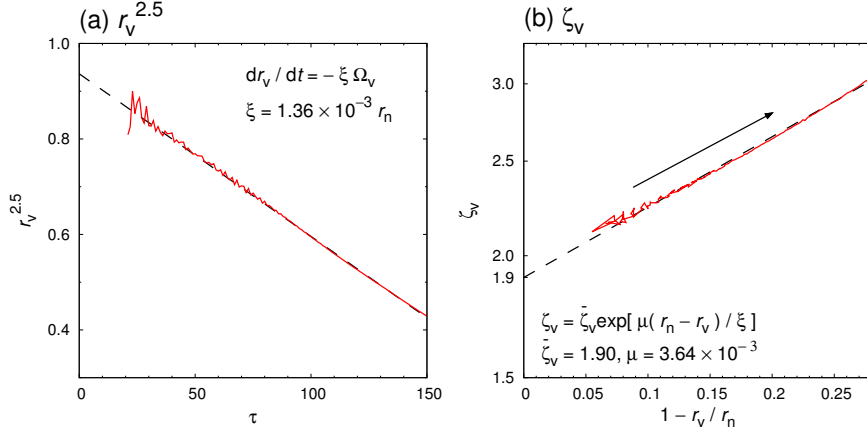


Figure 5.7: The vortex migration and the growth of the surface density contrast at the vortex center in the "h10w3g1" run. Red solid lines show the evolution (a) $r_v^{2.5}$ and (b) the surface density contrast at the RWI vortex center ζ_v , respectively. Black dashed lines show (a) $dr_v/dt = -\xi \Omega_v$ and $\zeta_v = \bar{\zeta}_v \exp[\mu(r_n - r_v)/\xi]$ with $\xi = 1.36 \times 10^{-3} r_n$, $\bar{\zeta}_v = 1.90$, and $\mu = 3.64 \times 10^{-3}$. Black arrow in Panel (b) indicates the time evolution of ζ_v .

Evolution of the RWI Vortex

I turn my attention to the vortex migration. I find that r_v satisfies

$$\frac{dr_v^{2.5}}{dt} = \text{constant} = -2.5\sqrt{GM}\xi, \quad (5.6)$$

where $\xi \approx 1.36 \times 10^{-3} r_n$ as shown in Panel (a) of Figure 5.7. From equation (5.6), the migration of the RWI vortex can be written as

$$\frac{dr_v}{dt} = -\xi \Omega_v, \quad (5.7)$$

due to $\Omega_v = \sqrt{GM/r_v^3}$. The migration speed of the RWI vortex gets faster as it migrates inward.

The surface density contrast at the vortex center $\zeta_v \equiv \Sigma_v(r_v, \varphi_v)/\Sigma_n$

evolves satisfying

$$\zeta_v = \bar{\zeta}_v \exp \left[\frac{\mu(1 - r_v/r_n)}{\xi/r_n} \right], \quad (5.8)$$

where $\bar{\zeta}_v \approx 1.90$ is the surface density contrast at $r = 1$ and $\mu \approx 3.64 \times 10^{-3}$ is an approximately constant parameter as shown in Panel (b) of Figure 5.7. I briefly comment on the physical meaning of equation (5.8). If the fluid particles at the center of the vortex migrates as the vortex migrates, i.e., if the RWI vortex is regarded as a fluid particle, the continuity equation at the vortex center can be written as

$$\frac{d\zeta_v}{dt} = -[\text{div } \mathbf{v}](r_v, \varphi_v)\zeta_v. \quad (5.9)$$

Using equations (5.7)–(5.9), it is found that μ satisfies

$$\mu = -\frac{[\text{div } \mathbf{v}](r_v, \varphi_v)}{\Omega_v}. \quad (5.10)$$

Therefore, the parameter μ represents the velocity divergence at the vortex center or the growth rate of the surface density contrast ζ_v . Note that $[\text{div } \mathbf{v}]$ vanishes at the vortex center if the vortex is strictly in the steady state. I shall look at the dynamics of fluid particles around the vortex center and discuss the validity of the assumption in Section 5.3.2.

5.2.3 Empirical Relations between the Vortex Properties and the Initial Conditions

I now summarize the results of all the runs. As mentioned in Section 5.2.2, the eleven parameters of the RWI vortex (χ_2 , ψ_2 , $\delta v_{\varphi v, r}/\Omega_v$, $v_{rv, \varphi}/(r_v \Omega_v)$, $\Pi_{v, rr}/\Omega_v^2$, $\Pi_{v, \varphi \varphi}/(r_v \Omega_v)^2$, Δr , $r_v \Delta \varphi$, ξ , $\bar{\zeta}_v$, and μ) are approximately constant in the “h10w3g1” run. I have measured those parameters in each run, and verify that they are almost constant. Table 5.3 is the list of the measurements of the RWI vortex. The values of the measurements are shown in Appendix C.2. Note that I am not able to have measured ξ , $\bar{\zeta}_v$, and μ in some runs because the vortex migration is too slow.

Table 5.3: List of the Measurements about the RWI Vortex.

Measurement	Meaning
χ_2	The vortex aspect ratio of the streamline which satisfies $b/r_v = 0.02$.
ψ_2	The turnover time normalized by $2\pi/\Omega_v$ of the streamline which satisfies $b/r_v = 0.02$.
$\delta v_{\varphi v,r}/\Omega_v$	The radial gradient of δv_φ at the vortex center divided by Ω_v .
$v_{rv,\varphi}/(r_v\Omega_v)$	The azimuthal gradient of v_r at the vortex center divided by $r_v\Omega_v$.
$\Pi_{v,rr}/\Omega_v^2$	The radial convexity of Π at the vortex center divided by Ω_v^2 .
$\Pi_{v,\varphi\varphi}/(r_v\Omega_v)^2$	The azimuthal convexity of Π at the vortex center divided by $r_v^2\Omega_v^2$.
Δr	The radial half width between two extrema of $\delta v_\varphi(r, \varphi_v)$.
$r_v\Delta\varphi$	The azimuthal half width between two extrema of $v_r(r_v, \varphi_v)$.
ξ	The constant describing the migration speed of the RWI vortex.
$\bar{\zeta}_v$	The surface density contrast of the RWI vortex at $r_v = r_n$.
μ	The constant considered to be $-[\text{div } \mathbf{v}]/\Omega_v$ at the vortex center.

Comparison between the RWI Vortices and the Steady Solution of Vortices

I compare the RWI vortices with the steady solution of vortices (see Appendix A). From Panel (a) of Figure 5.8, the aspect ratio χ_2 and the turnover time ψ_2 at $b/r_v = 0.02$ satisfy

$$(\chi_2 - 1) \approx 1.5\psi_2, \quad (5.11)$$

which corresponds to equation (A.7) of the steady solution of vortices. From Panels (b)-(e) of Figure 5.8, I also find that equations

$$\frac{\partial \delta v_\varphi}{\partial r} \approx (1.5 - \frac{\chi_2}{\psi_2})\Omega_v, \quad (5.12)$$

$$\frac{1}{r_v} \frac{\partial v_r}{\partial \varphi} \approx \frac{1}{\chi_2 \psi_2} \Omega_v, \quad (5.13)$$

$$\frac{\partial^2 \Pi}{\partial r^2} \approx (\frac{1}{\psi_2^2} - 2\frac{\chi_2}{\psi_2} + 3)\Omega_v^2, \quad (5.14)$$

$$\frac{1}{r_v^2} \frac{\partial^2 \Pi}{\partial \varphi^2} = (\frac{1}{\psi_2^2} - 2\frac{1}{\chi_2 \psi_2})\Omega_v^2, \quad (5.15)$$

are satisfied at least within a factor of 2. Equations (5.12)–(5.15) correspond to equations (A.5)–(A.6) and (A.8)–(A.9). From these results, I confirm that RWI vortices resemble the steady vortex solution in the Keplerian shear as shown in previous works (e.g., Surville & Barge, 2015).

Obtaining Empirical Relations

If the aspect ratio χ_2 is given, I can estimate the RWI vortex structures around the vortex center using the steady solution of vortices. Richard et al. (2013) derived a formula to estimate the vortex aspect ratio from the initial condition assuming that the vorticity of the non-Keplerian motion normalized by that of the background shear flow, $\tilde{\omega}_z$, is steady at the peak of the initial bump and the vortex center. For the initial conditions, $\tilde{\omega}_z$ at $r = r_n$ is written

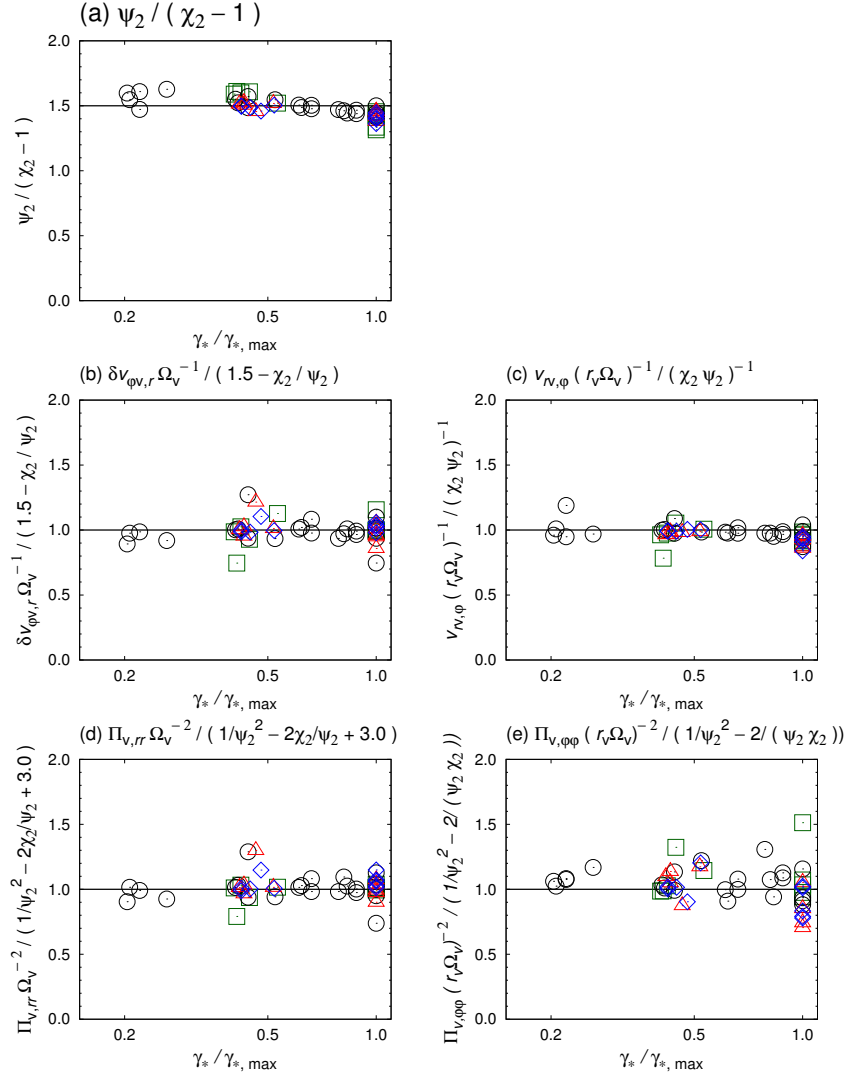


Figure 5.8: Comparing the RWI vortices with the analytic steady vortex. The values of (a) $\Psi_2/(\chi_2 - 1)$, (b) $\delta v_{\varphi v, r} \Omega_v^{-1}/(1.5 - \chi_2/\psi_2)$, (c) $v_{r v, \varphi} (r_v \Omega_v)^{-1}/(\chi_2 \psi_2)^{-1}$, (d) $\Pi_{v, rr} \Omega_v^{-2}/(1/\psi_2^2 - 2\chi_2/\psi_2 + 3.0)$, and (e) $\Pi_{v, \varphi \varphi} (r_v \Omega_v)^{-2}/(1/\psi_2^2 - 2/(\psi_2 \chi_2))$ are shown with green squares ($h = 0.05$), black circles ($h = 0.1$), red triangles ($h = 0.15$), and blue diamonds ($h = 0.2$). The value expected from the steady solution of vortices is shown with the solid black line in each panel.

as

$$\begin{aligned}\tilde{\omega}_z &= \left[\frac{d\{r(v_{\varphi 0} - v_K)\}}{dr} / \frac{\partial\{r(v_K - r\Omega_n)\}}{\partial r} \right]_{r=r_n} \\ &= \frac{h^2 \mathcal{A}_0 (1 + \mathcal{A}_0)^{\Gamma-2}}{3(\Delta w_0)^2} r_n^2.\end{aligned}\quad (5.16)$$

From the steady solution of vortices,

$$\frac{1 + \chi_r}{\chi_r(\chi_r - 1)} = \frac{h^2 \mathcal{A}_0 (1 + \mathcal{A}_0)^{\Gamma-2}}{3(\Delta w_0)^2} r_n^2 \quad (5.17)$$

is satisfied, where χ_r is the aspect ratio estimated by the method of Richard et al. (2013). From Panel (a) of Figure 5.9, χ_r matches well with χ_2 in the cases with large linear growth rate γ_* , but is smaller than χ_2 in the cases with small γ_* . These indicate that the normalized vorticity $\tilde{\omega}_z$ is almost conserved during the evolution of the RWI. In order to estimate the vortex aspect ratio more accurately, I give a modification of the Richard's formula as

$$\chi_2 \approx \left(\frac{\gamma_*}{\gamma_{*,\max}} \right)^{-0.05/h} \chi_r, \quad (5.18)$$

where $\gamma_*/\gamma_{*,\max}$ is the normalized linear growth rate of the initial condition against the RWI. The aspect ratio of the RWI vortex can be estimated from the initial conditions using equations (5.17)–(5.18). Since the RWI vortex resembles the steady solution of vortices, the turnover time ψ_2 and the vortex structure around the vortex center are determined from the vortex aspect ratio.

Next, I turn my attention to the parameters that characterize the vortex shape, namely, the radial and azimuthal extension of the vortex denoted by Δr and $r_v \Delta \varphi$, respectively. From Panels (b) and (c) of Figure 5.9, I obtain other empirical relations between Δr , $r_v \Delta \varphi$, and the initial conditions as

$$r_v \Delta \varphi \approx 1.1 \sqrt{\chi_2 \Delta w_0 r_n}, \quad (5.19)$$

$$\approx 0.87 \left(\frac{\gamma_*}{\gamma_{*,\max}} \right)^{0.2} \chi_2 \Delta r. \quad (5.20)$$

Here, I denote the maximum values of $|v_r|$ and $|\delta v_\varphi|$ by $|v_{r,\max}|$ and $|\delta v_{\varphi,\max}|$,

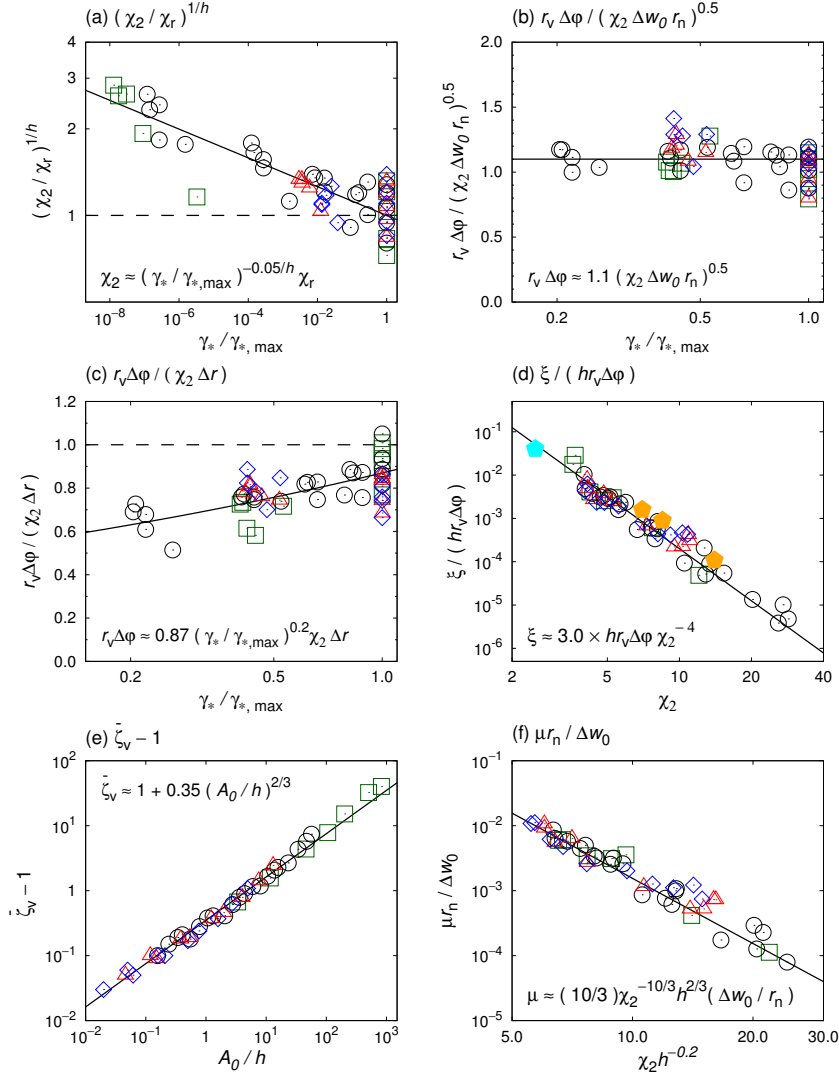


Figure 5.9: Empirical formulae to estimate the approximately constant parameters of the RWI vortices. The values of (a) $(\chi_2/\chi_r)^{1/h}$, (b) $r_v \Delta\varphi / \sqrt{\chi_2 \Delta w_0 r_n}$, (c) $r_v \Delta\varphi / (\chi_2 \Delta r)$, (d) $\xi / (hr_v \Delta\varphi)$, (e) $\bar{\zeta}_v - 1$, and (f) $\mu r_n / \Delta w_0$ are shown with green squares ($h = 0.05$), black circles ($h = 0.1$), red triangles ($h = 0.15$), and blue diamonds ($h = 0.2$), respectively, where χ_r is the vortex aspect ratio calculated from the method of Richard et al. (2013). The black solid lines show empirical relations, (a) $\chi_2 = (\gamma_*/\gamma_{*,\max})^{-0.05/h} \chi_r$, (b) $r_v \Delta\varphi = 1.1 \sqrt{\chi_2 \Delta w_0 r_n}$, (c) $r_v \Delta\varphi = 0.87 (\gamma_*/\gamma_{*,\max})^{0.2} \chi_2 \Delta r$, (d) $\xi = 3.0 \times hr_v \Delta\varphi \chi_2^{-4}$, (e) $\bar{\zeta}_v = 1 + 0.35 (\mathcal{A}_0/h)^{2/3}$, and (f) $\mu = (10/3) \chi_2^{-10/3} h^{2/3} (\Delta w_0 / r_n)$, respectively. Pentagons in Panel (d) show the values of ξ derived from the results of Paardekooper et al. (2010) (cyan) and Richard et al. (2013) (orange), respectively.

respectively. Since the values of v_r and δv_φ are about two thirds times as large as $\delta v_{r_v, \varphi}(\varphi - \varphi_v)$ and $\delta v_{\varphi_v, r}(r - r_v)$ at the velocity extrema, respectively, as mentioned in Section 5.2.2, I can estimate $|v_{r, \max}|$ and $|\delta v_{\varphi, \max}|$ as

$$\begin{aligned} |v_{r, \max}| &\approx \frac{2}{3} |\delta v_{r_v, \varphi} \Delta \varphi| \\ &\approx \frac{1.1}{\sqrt{\chi_2}(\chi_2 - 1)} \Omega_v \sqrt{\Delta w_0 r_n} \end{aligned} \quad (5.21)$$

$$\begin{aligned} |\delta v_{\varphi, \max}| &\approx \frac{2}{3} |v_{\varphi_v, r} \Delta r| \\ &\approx 1.26 \left(\frac{\gamma_*}{\gamma_{*, \max}} \right)^{-0.2} \frac{1}{\sqrt{\chi_2}(\chi_2 - 1)} \Omega_v \sqrt{\Delta w_0 r_n}, \end{aligned} \quad (5.22)$$

using equations (5.19) and (5.20).

In order to estimate the migration speed of the RWI vortex, I investigate the values of ξ . From Panel (d) of Figure 5.9, I find that ξ satisfies

$$\xi \approx 3.0 h r_v \Delta \varphi \chi_2^{-4}. \quad (5.23)$$

Paardekooper et al. (2010) reported the steady vortex with the vortex aspect ratio $\chi = 2.5$ and the radial vortex size $0.0496 r_n$ migrates inward with $\xi \approx 0.0005 r_n$ for $h = 0.1$ at $r_v = 1$. When I assume $r_v \Delta \varphi = \chi \times 0.0496 r_n = 0.124 r_n$, I find that equation (5.23) is consistent with their result (see the cyan pentagon in Panel (d) of Figure 5.9). Richard et al. (2013) also measured the vortex migration speed of the RWI vortices in their 3D calculations. If the azimuthal sizes of those RWI vortices can be estimated using equation (5.19), those RWI vortices almost satisfy equation (5.23) (see the orange pentagons in Panel (d) of Figure 5.9). I note that my numerical setup is somewhat different from that in Paardekooper et al. (2010) and Richard et al. (2013), where the surface density profile of the background disk has radial gradient and the disk is assumed to be locally isothermal.

I check the behavior of $\bar{\zeta}_v$ and μ in order to estimate the surface density contrast of the RWI vortex, ζ_v , from the initial conditions. From Panels (e) and (f) of Figure 5.9, empirical relations between $\bar{\zeta}_v$, μ and the initial conditions

are obtained as

$$\bar{\zeta}_v \approx 1 + 0.35 \left(\frac{\mathcal{A}_0}{h} \right)^{2/3}, \quad (5.24)$$

$$\mu \approx \frac{10}{3} \chi_2^{-10/3} h^{2/3} \frac{\Delta w_0}{r_n}. \quad (5.25)$$

Using equations (5.7), (5.8), (5.10)–(5.15), and (5.17)–(5.25), I can estimate the properties of the RWI vortex located at any distance from the central star (the aspect ratio, the turnover time, the velocity field and the convexity of the pressure function around the vortex center, the radial and azimuthal size, the migration speed, and the surface density contrast) from the initial conditions (h , Δw_0 , and $\gamma_*/\gamma_{*,\max}$). I note that $\mathcal{A}_0$ and $\gamma_{*,\max}$ can be calculated from h , Δw_0 and γ_* with the linear stability analyses. It is expected that the empirical relations also depend on Γ , but I do not focus on this parameter in this paper. Ultimately, I need radiation hydrodynamic simulations to model protoplanetary disks realistically.

5.3 Discussions on Outcomes of Numerical Calculations

5.3.1 Timescale of Multiple Vortices Regime

In this section, I focus on the vortex merger regime after the saturation. In all the runs, the regime with more than two vortices continues at most for a few orbits. On the other hand, it is found that the lifetime of the two vortices regime shows some variation. By visual inspection of the surface density distribution, I have measured the orbits τ_2 when the number of the vortices becomes two due to the vortex mergers, and τ_1 when the one last remaining vortex is formed at the final vortex merger. For all the models with $m_* = 1$ and some models with $m_* = 2$, it is so difficult to have measured τ_2 and τ_1 that I set τ_2 to no data and τ_1 to the orbit number when the RWI is saturated. I show the values of τ_2 and τ_1 in Appendix C.1.

The values of the lifetime of the two vortices regime, $(\tau_1 - \tau_2)$, seem to be random. As shown in Figure 5.10, however, I find that there is an upper limit in $(\tau_1 - \tau_2)$;

$$\begin{aligned}\tau_1 - \tau_2 &\lesssim 300 \exp(-11.5h), \\ &\approx 95 \exp[1 - h/0.1],\end{aligned}\tag{5.26}$$

within my parameter space. For $r_n = 100$ AU, one orbit corresponds to about 10^3 yrs. From equation (5.26), it is expected that the lifetime of the two vortices regime is up to about a few $\times 0.1$ Myr. The lifetime of the two vortices regime is one to two orders of magnitude shorter than the disk lifetime, which is 1–10 Myr (e.g., Haisch et al., 2001). I predict that it is difficult to observe protoplanetary disks with multiple vortices formed by the same RWI event except at outer disk.

5.3.2 Tracer Particle Analysis around the RWI Vortex

Here, I investigate the more detailed information of the RWI vortex from the tracer particle analysis. I calculate the path lines of the fluid particles for 30 orbits. 240×360 tracer particles are initially distributed uniformly within the range of $0.6 \leq r \leq 1.2$ and $-\pi \leq \varphi \leq \pi$ at $\tau = 55, 60, 65, 70, 75, 80$, and 85 in the “h10w3g1” run. According to the r evolutions of the fluid particles which are initially distributed at $\tau = 70$ (see Panel (a) of Figure 5.11), I categorize fluid particles into four groups (I, II, III, IV). The Group I represents the particles which compose the vortex at $\tau = 70$. All the group I particles remain in the vortex part for the 30 orbits. The fluid particles categorized into the group II are in the inner part of the disk initially, and move to the outer part of the disk within the 30 orbits. The group III particles and the group IV particles remain in the inner part and in the outer part for the 30 orbits respectively. The vortex captures a few particles which reside in the inner region, but those particles escape from the vortex to the outer part only after a few turnover motions and are categorized into the group II. I find that no particle moves inward (the outer part $\rightarrow$ the inner part, the outer part $\rightarrow$ the vortex part, and the vortex part $\rightarrow$ the inner part) during the 30 orbits.

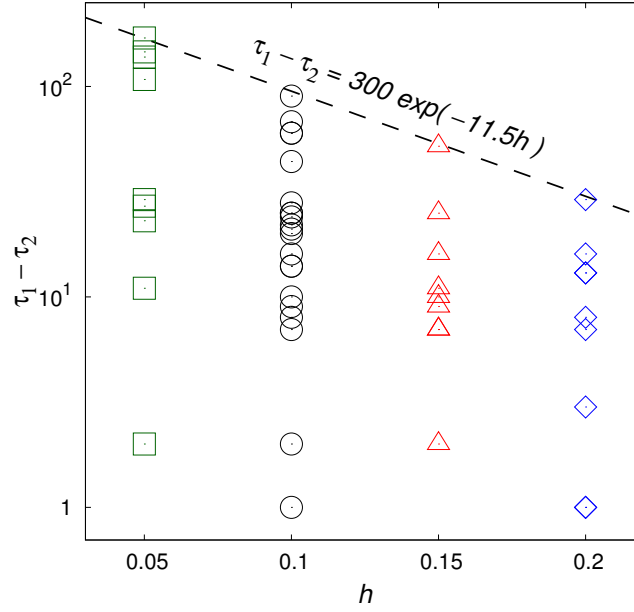


Figure 5.10: Life-time of the two vortices regime. The values of $\tau_1 - \tau_2$ are shown with green squares ($h = 0.05$), black circles ($h = 0.1$), red triangles ($h = 0.15$), and blue diamonds ($h = 0.2$), respectively. Black dashed line shows $\tau_1 - \tau_2 = 300 \exp(-11.5h)$

I show the distribution of the particle groups at $\tau = 70$ in Panel (b) of Figure 5.11.

To investigate the time evolution of the fluid particles close to the vortex center, I show the evolutions of r and Σ of three particles which are located near the vortex center at $\tau = 60, 70$, and 80 . From Panel (a) of Figure 5.12, those particles migrate inward with small oscillations. Here, I can define the real vortex center which is the center of the oscillations from the path lines. The vortex center defined by the path lines migrates at the same migration speed of the vortex center defined by the streamlines, r_v , but the distance from the central star is larger than r_v by about 2.2%. This fact can be also confirmed from the Panel (b) of Figure 5.11. The surface density of these

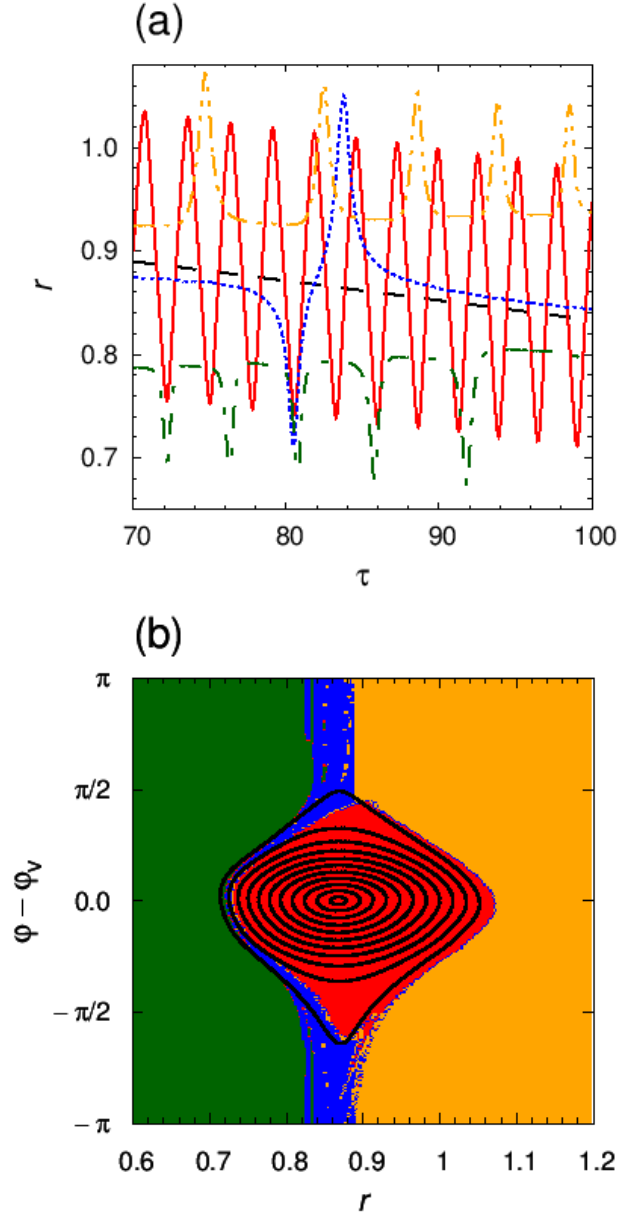


Figure 5.11: Categorization of the fluid particles into four groups and the distribution of particle groups. The r evolutions of (a) group I (red solid line), (b) group II (blue dotted line), (c) group III (green one dot chain line), and (d) group IV particles (orange two dots chain line) which depart from the starting point at $\tau = 70$ in the “h10w3g1” run are shown in Panel (a). Black dashed line shows the evolution of the vortex center. Panel (b) shows the distribution of the particle groups at $\tau = 70$ (red: group I, blue: group II, green: group III, and orange: group IV). Black lines show the streamlines around the vortex at $\tau = 70$.

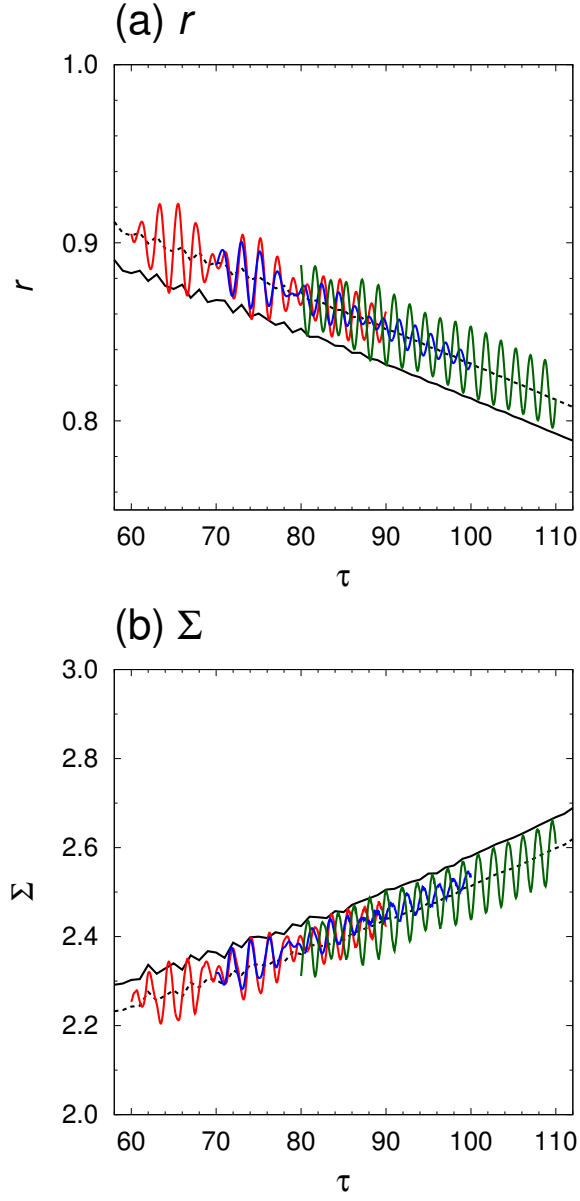


Figure 5.12: The r and Σ evolutions of the fluid particles which depart from the starting point near the vortex center at $\tau = 60$ (red line), 70 (blue line), and 80 (green line) are shown in Panels (a) and (b), respectively. In Panel (a), the black solid line shows the evolution of r_v , and the black dotted line shows the evolution of $1.022r_v$. In Panel (b), the black solid line shows the evolution of Σ_v , and the black dotted line shows the evolution of $0.974\Sigma_v$.

tracer particles increase with small oscillations at the same growth rate as Σ_v . The time averaged surface density of the three particles is smaller than Σ_v by about 2.6%.

In Section 5.2.2, I assumed that the vortex center are composed by the same fluid particles at any time. Even though the position of the vortex center defined by the path lines is slightly different from that of the vortex center defined by the streamlines, the vortex migration speed and the growth rate of surface density at the vortex center are essentially identical. I conclude that the assumption is appropriate, and the approximately constant parameter μ represents the time averaged velocity divergence at the vortex center normalized by the angular velocity of the vortex center.

I also calculate the total mass of the group I particles, M_v , which represents the vortex mass. From Figure 5.13, M_v is approximately constant in time and decreases only by 1% during the 30 orbits. I therefore expect that, in the course of the inward migration of the RWI vortex, it carries most of the fluid particles, which are originated from the initial location where the vortex is formed, down to the inner radii. In this sense, the RWI vortex can be considered as a physical entity.

5.3.3 Vortex Migration

In this section, I discuss the vortex migration. The migration speed of the RWI vortex is estimated from ξ (see Section 5.2.3). Using equations (5.7), (5.19) and (5.23), the timescale of the vortex migration can be written as

$$\begin{aligned} \left| \frac{r_v}{dr_v/dt} \right| &= \frac{r_v}{\xi \Omega_v} \approx \frac{\chi_2^4}{3h\Delta\varphi} \Omega_v^{-1}, \\ &\approx 1.96 \text{ Myr} \left(\frac{\chi_2}{6} \right)^4 \left(\frac{h}{0.1} \right)^{-1} \left(\frac{\Delta w_0}{0.1 r_n} \right)^{-0.5} \left(\frac{r_v}{100 \text{ AU}} \right)^{2.5} \end{aligned} \quad (5.27)$$

The migration timescale strongly depends on the vortex aspect ratio and the distance from the central star. As long as the vortex aspect ratio is larger than 5, the vortex migration timescale can be comparable to or larger than the life

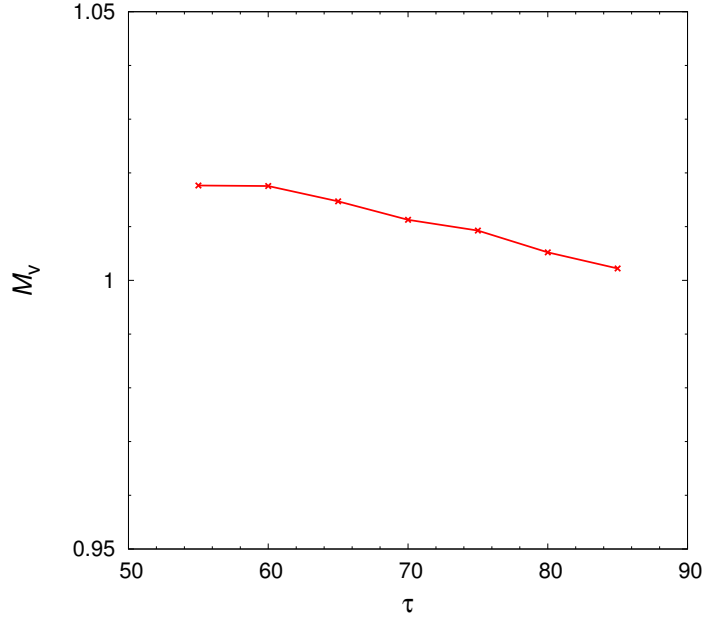


Figure 5.13: The time evolution of M_v calculated from the tracer particle analyses.

time of the protoplanetary disks ($\sim 1\text{--}10$ Myr). In other words, RWI vortices resulting from a weaker initial surface density bump migrate more slowly.

While the vortex migrates toward the central star, the vortex loses the angular momentum via density waves (Paardekooper et al., 2010). The velocity perturbations induced by the vortex motion excite density waves around the Lindblad resonances, which carry negative (inner spiral) or positive (outer spiral) angular momentum, causing the vortex to migrate. The positions of the Lindblad resonances of the m mode are as locations where the epicyclic frequency, κ , is the m times of the angular velocity of the fluid element in the frame co-rotating with the vortex (see Panel (a) of Figure 5.14);

$$\kappa(r, \varphi_v) = \pm m[\Omega_v - v_\varphi(r, \varphi_v)/r]. \quad (5.28)$$

As can be seen in Panels (b)–(e) of Figure (5.14), the density waves are indeed excited around the Lindblad resonances for each mode.

The fluid particles of the group II (see Section 5.3.32) also contribute to the angular momentum exchange between the disk and the vortex. When they move from the inner part to the outer part, they gain the angular momentum from the vortex. It is analogous to the co-rotation torque exerted on a planet in the planet-disk interaction (Balmforth & Korycansky, 2001). I calculate variation of the total angular momentum of the group I particles, ΔJ_1 , and that of group II particles, ΔJ_2 , over the 30 orbits. I find $|\Delta J_2/\Delta J_1| \sim 0.2$, indicating that 20% of the total torque exerted on the vortex comes from the contribution of the fluid elements passing through the vortex region. The remaining 80% of the total torque is interpreted to come from the density waves. We, however, find difficulty measuring the total torque which comes from the density waves directly, because it is difficult to distinguish the density waves from the vortex motion and to precisely measure the angular momentum flux of the density waves due to the numerical viscosity.

The azimuthal width, $r_v \Delta \varphi$, of the RWI vortex is almost steady even though the vortex migrates toward the central star as mentioned in Section 5.2.3. If the vortex continues to migrate with the constant azimuthal width, the angular extension of the vortex around the central star may be significant, and there is a possibility that the RWI vortex returns to the ring structure at the end. To study the long-term evolution of the vortices, numerical calculations which resolve the region close to the central star are needed.

5.3.4 Application of Empirical Relations to HD142527

In this section, I try to constrain the observation of HD142527, which has a lopsided disk, using the empirical relations. As shown in Section 2.2, It is estimated that $\zeta_v \sim 4$ and $h \approx 0.1$ at $r_v \sim 170$ AU in the disk of HD142527 (Muto et al., 2015; Boehler et al., 2017). It is so difficult to measure the vortex aspect ratio of gas, χ , via gas observations directly. However, Lyra & Lin (2013) reported that the vortex aspect ratio of the dust distribution trapped

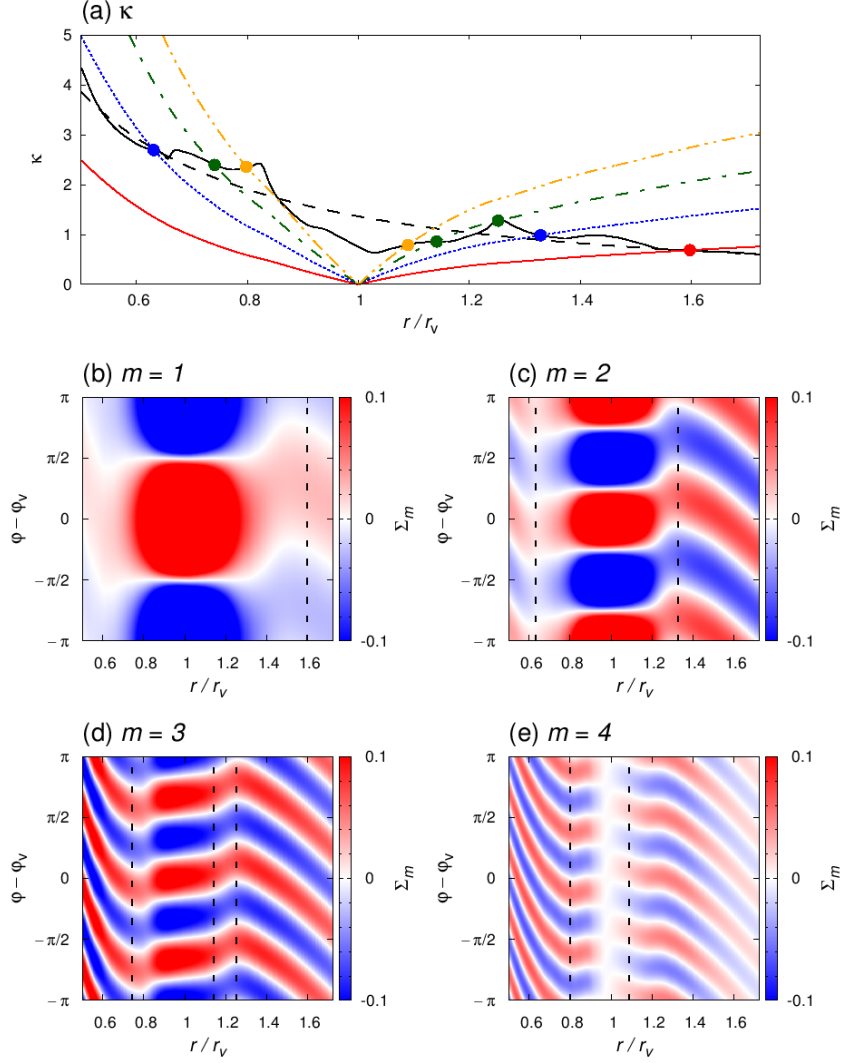


Figure 5.14: Lindblad resonances and surface density for each m mode. Panels (a) shows the epicyclic frequency κ in the “h10w3g1” run at $\tau = 100$. In Panel (a), $m(\Omega_v - v_\phi/r)$ is shown with red solid line ($m = 1$), blue dotted line ($m = 2$), green one dot chain line ($m = 3$), orange two dots chain line ($m = 4$) and Ω_K is shown with black dashed line. Panels (b)–(e) show the 2D distribution of Σ_m for $m = 1$ –4, respectively. The points in Panel (a) and the vertical dashed lines in Panels (b)–(e) indicate the Lindblad resonances.

by the gas vortex is similar to that of the gas vortex. Therefore, I can estimate the vortex aspect ratio of gas distribution as $\chi \sim 10$ using results of dust observations (e.g., Soon et al., 2017).

Here, I make assumption that there is a RWI vortex with $\zeta_v \sim 4$ and $\chi \sim 10$ at $r_v \sim 170$ AU formed from a Gaussian bump in HD142527. From equation (5.27), it is expected that the migration timescale of the vortex is so long ($\gtrsim 10$ Myr) and the vortex has experienced the almost in-situ formation. Therefore, I can assume $\bar{\zeta}_v \sim \zeta_v \sim 4$. Using equation (5.24), the surface density contrast of the initial Gaussian bump is $\mathcal{A}_0 \sim 2.5$. Since the normalized linear growth rate of the initial bump against the RWI is estimated from equation (5.18), I can also estimate the radial half-width of the initial bump is $\Delta w_0 \sim 0.15r_n$. This width corresponds to about one disk scale height.

I predict that HD142527 has a mechanism producing the axisymmetric structure of which the stability against the RWI is similar to a Gaussian bump with the contrast of $\mathcal{A}_0 \sim 2.5$ and the radial half width of about one disk scale height. However, I have made a number of simplifying assumptions. Further study concluding other physical processes ,such as viscosity, self-gravity, dust particles, should be needed in order to constrain the observations qualitatively.

5.4 Summary of Chapter 5

I perform numerical simulations of the RWI in 2D, barotropic and hydrodynamic disks using the Athena++ code. As the initial conditions, I consider axisymmetric disks with a Gaussian surface density bump which is characterized by three parameters: the dimensionless disk aspect ratio h , the radial half-widths Δw_0 , and the largest linear growth rate of the RWI γ_* . I vary these parameters in a wide parameter space and explore the non-linear evolutions for 54 models. Multiple vortices are formed initially in accordance with the most unstable mode of the RWI derived from the linear stability analyses. I confirm that the numerical simulations can reproduce the results of the linear stability analyses. After the linear growth is saturated, the vortices coalesce one after another, and only one quasi-stationary vortex (the RWI vortex) survives in the

end. After the saturation, the vortices start to migrate inward in the disks.

I confirm that those vortices almost correspond to the steady solution of vortices on the Keplerian shear as shown in previous works. I obtain the empirical relations between the properties of the RWI vortices (the aspect ratio: equation (5.18), the radial and azimuthal sizes: equations (5.19) and (5.20), the vortex migration speed: equation (5.23), the surface density contrast at the vortex center: equations (5.24) and (5.25)) and the initial Gaussian bump. Using these empirical relations and neglecting my over simplification, I predict that HD142527 has a mechanism producing the axisymmetric structure of which the stability against the RWI is same as a Gaussian bump with the contrast of $\mathcal{A}_0 \sim 2.5$ and the radial half width of about one disk scale height.

Comparing the period of the vortex merger regime with the disk life-time, it is found that the regime with multiple vortices continues up to about 100 orbits. I conclude that it is difficult to observe the disks with multiple vortices formed by the same RWI event except at outer disk. From the tracer particle analysis, I find that the RWI vortex can be considered as a physical entity like a large fluid particle. I also reveal that the fluid elements passing through the vortex region contribute to about 20% of the total torque exerted on the vortex. The remaining about 80% of the total torque is interpreted to come from the density waves.

My calculations have performed under a number of simplifying assumptions. However, I consider that I have captured some physical aspects of the non-linear evolution of the RWI, the vortices formed as a result of the RWI, and its migration within the disk. Future studies with different Γ and/or other physical processes would allow us to make the models for the vortices that can be compared with observations.

Chapter 6

Conclusion

6.1 Summary of Thesis Achievements

The Rossby wave instability (RWI) is one of the promising mechanisms of explaining the observed lopsided structures of protoplanetary disks. It is considered that the lopsided structure is one of the consequences of the disk evolution and planet formation. In addition, the lopsided structure is expected to effect on the planet formation, especially the planetesimal formation. Studies of the RWI are essential not only for the comprehension of the observed lopsided structure but also for the interpretation of the disk evolution and the planet formation.

The RWI has been studied in some previous works with linear stability analyses. Those have derived the necessary condition for the RWI (Lovelace's condition). However, the way to find accurately the parameters where the configuration is unstable against the RWI has not been confirmed and the *necessary and sufficient* condition for the onset of the RWI has been unknown. The previous works have explained the physical nature of the RWI using the over reflection of waves at the co-rotation radius. This explanation describes the RWI well, but it is difficult to grasp what happens in the real space. The outcomes of the RWI have been also investigated in a lot of previous works with numerical simulations. They reported that the vortices formed by the

RWI are consistent with the steady solution of vortices in Keplerian shear. However, it has been difficult to estimate the properties of the vortices formed by the RWI from the disk conditions before the onset of the RWI. It is needed to solve these unknown problems about the RWI.

In Chapter 3, I perform the linear stability analyses flow for representative six types of the barotropic background flow on a wide parameter space, and examine the stability against the RWI. As a result, the Lovelace's condition is certainly a necessary condition but is not a sufficient condition. It is also clarified that the Rayleigh's condition is not a sufficient condition. Using the vortensity conservation law, I establish the method to find the parameters where the configuration is unstable against the RWI. I also derive the *necessary and sufficient* condition for the onset of the RWI in a semi-analytic form on the analogy of the quantum mechanics. It is expected that the new condition is generally useful.

I show the interpretation of the RWI using the interaction between two stationary Rossby waves in Chapter 4. This mechanism explains what happens in the real space during the linear growth regime of the RWI. It is considered that the RWI is same as the shear instability. When the system satisfies the Rayleigh's inflection theorem and the Fjørtoft's theorem, there is a possibility that two stationary Rossby waves exist at the position between which the corotation radius. There are two kinds of the configurations of the two Rossby waves. For the positive phase difference, the configuration describes the unstable states. In the other hand, the system is stable for the negative phase difference. The linear growth rate of the RWI depends on the phase difference between the two Rossby waves. I also find that some properties of the RWI can be explained using the analytic three layers model in incompressible flow.

Finally in Chapter 5, I perform numerical simulations of the RWI in 2D, barotropic and hydrodynamic disks using the Athena++ code. It is confirmed that the numerical simulations can reproduce the results of the linear stability analyses. In all my calculations, the final outcome of the RWI is a quasi-stationary vortex (RWI vortex). The RWI vortices in my calculations are similar to the steady solution of vortices as mentioned in previous works. However, I find that the RWI vortices are different from the steady vortices in

some aspects. The RWI vortices migrate toward the central star by exciting the density waves on the disk. The surface density contrast of the RWI increases with the vortex migration. This indicates that the velocity divergence does not vanish at the vortex center. I newly find approximately constant parameters which describe the radial and azimuthal vortex sizes, the vortex migration, and the surface density contrast at the vortex center. I also obtain empirical relations between the properties of the RWI vortices and the initial conditions. Using these relations, I can predict the outcomes of the RWI without numerical calculation.

These results provide a solid theoretical ground for quantitative interpretation of the observed lopsided structures in protoplanetary disks. I expect that this thesis is a help reveal the formation mechanisms of the transitional disks as well as the lopsided disks.

6.2 Future Work

In this thesis, I consider the simplest disk model to investigate the RWI more theoretically. Various, potentially important, physical effects on the RWI, such as baroclinicity, viscosity, self-gravity and 3D, are not included. Here, I briefly discuss how these effects may affect on vortex formation by the RWI and the properties of the RWI vortex within the disk.

It is reported that the effects of baroclinicity, viscosity, and 3D are not significant in the formation of vortices by the RWI (Lin, 2012a,b, 2013; Miranda et al., 2016). On the other hand, self-gravity prevents the onset of the RWI (Lovelace & Hohlfeld, 2013; Yellin-Bergovoy et al., 2016). The vortex formation by the RWI, therefore, may not be affected unless the disk is massive.

As mentioned in Chapter 2, it is known that the effects of viscosity and 3D can destroy the vortices (Lin, 2014; Lesur & Papaloizou, 2009; Richard et al., 2013). The viscosity problem is still serious, but the vortex destruction by the 3D effects is not so serious problem. In order to survive for a long time in a 3D compressible disk, the vortex should satisfy that the vortex aspect

ratio χ is more than 4 (Richard et al., 2013). In all my numerical simulations except for the “h05w4g1” and “h05w5g1” models in Chapter 5, the vortex aspect ratio χ_2 is always larger than 4. This is because I only consider the initial conditions that do not violate Rayleigh’s criterion. In fact, RWI vortices formed from the initial conditions that violate the Rayleigh’s condition have vortex aspect ratios smaller than 4. I consider that the initial conditions that violate Rayleigh’s criterion is not realistic. From the results of Richard et al. (2013), I expect that RWI vortices formed in my calculations survive for a long time even in the 3D frameworks.

It is known that the effects of dusts can also destroy the vortices if the high dust-to-gas mass ratio is realized at the vortex center (Fu et al., 2014b; Crnkovic-Rubsamen et al., 2015). On the other hand, the vortex is a possible birthplace of planetesimals (e.g., Heng & Kenyon, 2010). Since the vortex can capture and concentrate dust particles, it is expected that dust particles can rapidly coagulate each other avoiding the radial drift problem of dust particles (Adachi et al., 1976).

In order to compare the results of this thesis with the observations more realistically and to constrain the environment of the observed lopsided disks, future studies with other physical processes and a larger parameter space are needed. Particularly, effects of viscosity and dust particles would be important to understand the disk evolution and the planet formation as well as the formation mechanisms of transitional disks and lopsided disks. I will investigate them in future work

Appendix A

Local Steady Vortex Model

The steady solution of vortices in shearing flows (Kida , 1981; Goodman et al., 1987) is useful to understand the structures of the RWI vortices. In this section, I introduce the steady solution of vortices before showing results.

I consider a 2D vortex orbiting a central star at angular velocity $\Omega_v = \Omega(r_v)$ under the shearing box approximation, where r_v is the distance between the central star and the center of the vortex. The shearing box is a rotating Cartesian box centered at $r = r_v$ with the angular velocity of Ω_v . I define $x = r - r_v$ and $y = r_v\varphi$ and neglect the terms arising from disk curvature. In this rotating frame, the equation of motions are written as

$$\frac{d\bar{v}_x}{dt} = \frac{\partial\bar{v}_x}{\partial t} + \bar{v}_x \frac{\partial\bar{v}_x}{\partial x} + \bar{v}_y \frac{\partial\bar{v}_x}{\partial y} = 2\mathfrak{S}\Omega_v^2 x + 2\Omega_v \bar{v}_y - \frac{\partial\Pi}{\partial x}, \quad (\text{A.1})$$

$$\frac{d\bar{v}_y}{dt} = \frac{\partial\bar{v}_y}{\partial t} + \bar{v}_x \frac{\partial\bar{v}_y}{\partial x} + \bar{v}_y \frac{\partial\bar{v}_y}{\partial y} = -2\Omega_v \bar{v}_x - \frac{\partial\Pi}{\partial y}, \quad (\text{A.2})$$

where $\bar{\mathbf{v}} = (\bar{v}_x, \bar{v}_y)$ is the velocity field in the local shearing box. In equation (A.1), I have defined the mean shear without a vortex $\mathfrak{S} \equiv -r\Omega_v^{-1}[\partial\Omega / \partial r]_{r=r_v}$, which is $\mathfrak{S} = 1.5$ for a Keplerian disk.

The steady vortex is assumed to have uniform vorticity $\omega_z = (\text{rot}\bar{\mathbf{v}})_z$. In this steady vortex, the fluid particles orbit the origin at constant angular velocity Ω_v/ψ and the shape of the trajectories are elliptic, where ψ is the turnover time of the vortex normalized by $2\pi/\Omega_v$. I denote the semi-minor axis of the

vortex as b , the semi-major axis of the vortex as a , and the vortex aspect ratio as $\chi \equiv a/b$. According to Kida (1981) (the equations (3.2) and (3.3)), the elliptic trajectories have to satisfy following two conditions to be steady. First, the semi-minor axis should be aligned with x - or y -axis. Otherwise, the elliptic trajectory would precess. When I assume the semi-minor axis is aligned with the x -axis (radial direction), the velocity field can be written as

$$\bar{v}_y = -\frac{\chi}{\psi}\Omega_v x, \quad (\text{A.3})$$

$$\bar{v}_x = \frac{1}{\chi\psi}\Omega_v y. \quad (\text{A.4})$$

Defining the velocity field in the inertial frame by $\mathbf{v} = (v_x, v_y)$, I can write

$$\frac{\partial(v_y - v_K)}{\partial x} = (\mathfrak{S} - \frac{\chi}{\psi})\Omega_v, \quad (\text{A.5})$$

$$\frac{\partial v_x}{\partial y} = \frac{1}{\chi\psi}\Omega_v. \quad (\text{A.6})$$

from equations (A.3) and (A.4), where $v_K = r_v\Omega_v + (1 - \mathfrak{S})\Omega_v x$ is the Keplerian rotation velocity. Second, the aspect ratio χ and the turnover time ψ should satisfy

$$(\chi - 1) = \mathfrak{S}\psi, \quad (\text{A.7})$$

for the vortex aspect ratio to be steady.

Substituting equations (A.3) and (A.4) into equations (A.1) and (A.2) and assuming $\partial/\partial t = 0$ (steady state), it is found that the pressure function Π should satisfy

$$\frac{\partial^2 \Pi}{\partial x^2} = \left(\frac{1}{\psi^2} - 2\frac{\chi}{\psi} + 2\mathfrak{S}\right)\Omega_v^2, \quad (\text{A.8})$$

$$\frac{\partial^2 \Pi}{\partial y^2} = \left(\frac{1}{\psi^2} - 2\frac{1}{\chi\psi}\right)\Omega_v^2. \quad (\text{A.9})$$

Substituting equations (A.8) and (A.9) into the continuity equation assuming the divergence of the velocity field to be zero, another relation between χ and ψ ,

$$(\chi^2 - 1) = 2\mathfrak{S}\psi^2, \quad (\text{A.10})$$

is obtained (Goodman et al., 1987). From equations (A.7) and (A.10), the aspect ratio and the turnover time are determined as

$$\chi = \frac{2 + \mathfrak{S}}{2 - \mathfrak{S}} = 7, \quad (\text{A.11})$$

$$\psi = \frac{2}{2 - \mathfrak{S}} = 4. \quad (\text{A.12})$$

Here, I have assumed $\mathfrak{S} = 1.5$ (Keplerian shear) in the last equality. Numerical simulations, however, show that the RWI vortices have various values of the vortex aspect ratio. Since the velocity divergence in the RWI vortex is non-zero as shown in Chapter 5, the assumption which is used to derive equation (A.10) is broken. Therefore, I assume the steady solution of vortices to satisfy equations (A.4)-(A.9) and not necessarily to satisfy equation (A.10) in this thesis. Note that the steady solution gives us the profiles of the velocity field and the gradient of the pressure function around the vortex center, but does not give any information about the size or density of the vortices.

Appendix B

Evolution of Azimuthally Averaged Surface Density

I investigate the axisymmetric component of the surface density $\langle \Sigma \rangle(r)$. I show the profiles of $\langle \Sigma \rangle(r)$ at $\tau = 0, 8, 20, 50, 100$, and 150 in Panel (a) of Figure B.1. The $\langle \Sigma \rangle(r)$ profiles resemble a Gaussian bump at any time. Therefore, I measure the location r_p , the radial contrast $\mathcal{A}$ and the radial half-width Δw of the bump by fitting with $\langle \Sigma \rangle = \mathcal{A} \exp[-\{(r - r_p)/\Delta w\}^2/2] + 1$ every orbit. I show the time evolutions of the three parameters of the Gaussian function ($r_p, \mathcal{A}, \Delta w$) in the Panels (b)–(d) of Figure B.1. Before the saturation of the RWI, $\mathcal{A}$ decreases and Δw increases rapidly. After the saturation at $\tau \sim 8$, this trend continues but at slower rate until $\tau \sim 50$. For $\tau \gtrsim 50$, $\mathcal{A}$ increases and Δw decreases. On the other hand, r_p monotonically decreases after the saturation. I see that the axisymmetric profiles approach the configuration that is stable against the RWI after the saturation.

As a result obtained in Chapter 3, I derived the condition for the onset of the RWI in a semi-analytic form for each azimuthal mode m as

$$\eta_m \equiv \int_{r_{\text{IR}}}^{r_{\text{OR}}} \sqrt{-D_{\text{MS},m}(r)} dr \gtrsim \eta_c \equiv \frac{\pi}{2\sqrt{2}}, \quad (\text{B.1})$$

where $D_{\text{MS},m}$ is the effective potential of the m mode if the system is assumed to be marginally stable against the RWI ($\gamma_m = 0$), r_{IR} and r_{OR} are the

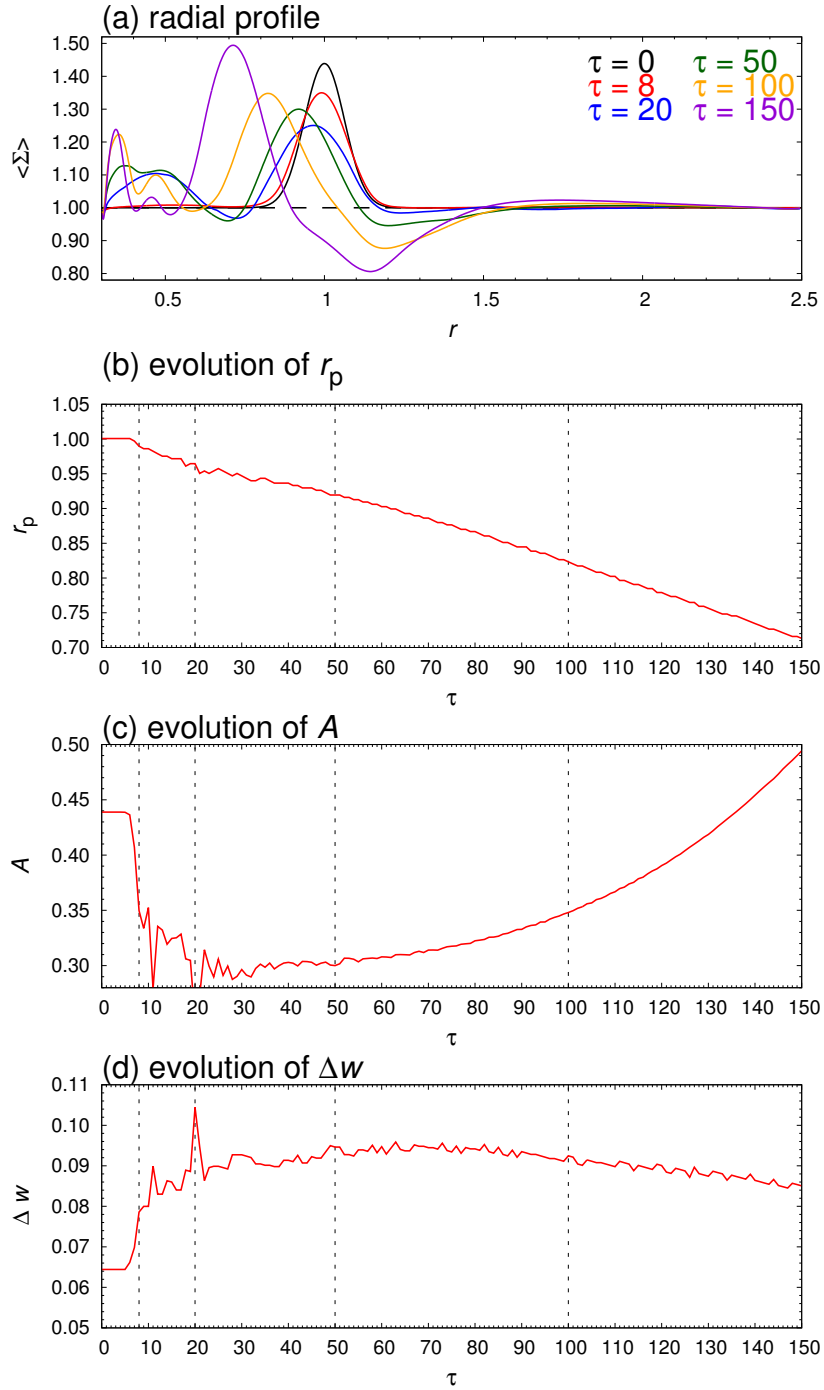


Figure B.1: The time evolution of the azimuthally averaged surface density in the “h10w3g1” run. Panel (a) shows the radial profiles of the azimuthally averaged surface density at $\tau = 0$ (black), 8 (red), 20 (blue), 50 (green), 100 (orange), 150 (purple). The time evolution of r_p , A , and Δw are shown in Panels (b), (c), and (d), respectively.

radii of the Rossby wave resonances (see Section 3.4.4 for details). When the system is marginally stable or unstable against the RWI, equation (B.1) with $m = 1$ should be satisfied at least. I calculate the time evolution of η_1 using the azimuthally averaged profiles. In order to calculate η_1 , the azimuthally averaged rotation velocity $\langle v_\varphi \rangle$ is required. For simplicity, I assume that the azimuthally averaged disk is in a dynamical equilibrium, and satisfies

$$\langle v_\varphi \rangle = \sqrt{v_K^2 + r \frac{d\langle \Pi \rangle}{dr}}, \quad (\text{B.2})$$

where $\langle \Pi \rangle \equiv \Gamma S_0 \langle \Sigma \rangle^{\Gamma-1} / (\Gamma - 1)$ is the azimuthally averaged pressure function. Note that I numerically calculate the higher order differentials of $\langle \Sigma \rangle$ and $\langle v_\varphi \rangle$ needed to obtain η_1 .

I show the time evolution of η_1 for the “h10w3g1” run (high γ_*) and the “h10w3g4” run (low γ_*) in Panel (a) of Figure B.2. In the “h10w3g1” run, η_1 decreases until the final vortex merger, and η_1 is smaller than η_c after the RWI vortex is formed. This indicates that $\langle \Sigma \rangle$ profile becomes stable against the RWI in the RWI vortex stage. In the “h10w3g4” run, η_1 also decreases until the final vortex merger, but η_1 is slightly larger than η_c after the RWI vortex is formed.

The slightly larger value of η_1 in the “h10w3g4” run than that of η_c does not necessarily indicate that the $\langle \Sigma \rangle$ profile is unstable against the RWI even in the RWI vortex stage. The value of η_c is derived under the assumption that the shape of the $D_{\text{MS},m}$ potential well can be approximated by a parabolic function. In fact, there are deviations of the potential well from the parabolic function in both the “h10w3g1” run and the “h10w3g4” run as shown in Panels (b) and (c) of Figure B.2. If the potential well is shallower than the parabolic function as in the “h10w3g4” case, larger η_1 than η_c is required for the onset of the RWI from the knowledge of quantum mechanics. Since the excess of η_1 over η_c in the RWI vortex stage for the “h10w3g4” model is small, I expect that the $\langle \Sigma \rangle$ profile of the “h10w3g4” run is also stable against the RWI after the RWI vortex formation.

I investigate the time evolution of η_1 in the other runs. For the models with $\gamma_*/\Omega_n \gtrsim 0.15$, the evolution of η_1 resembles that in the “h10w3g1” run.

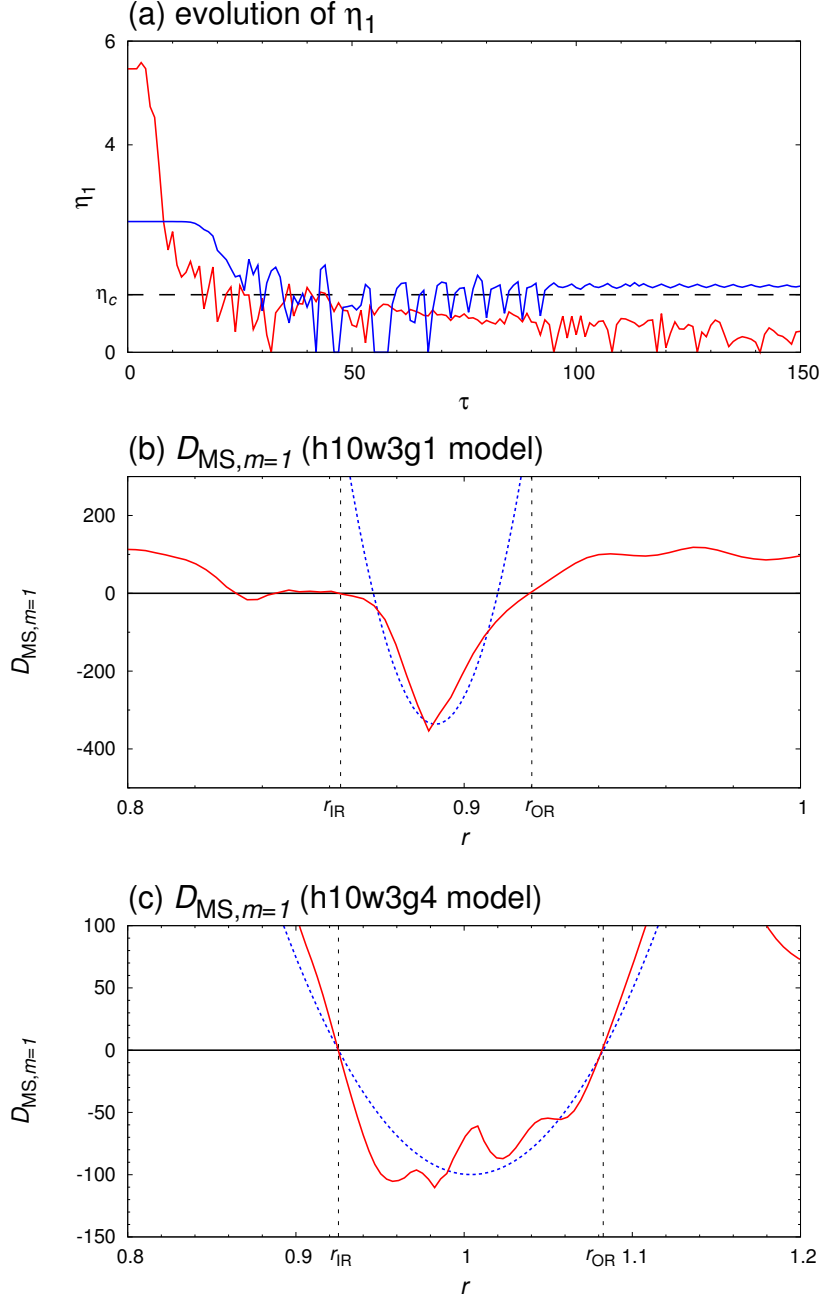


Figure B.2: Stability of the azimuthally averaged surface density profile against the RWI in a large growth rate case and a small growth rate case. Panel (a) shows the time evolution of η_1 for the "h10w3g1" run (red) and the "h10w3g4" run (blue). Black dashed lines show $\eta_c = \pi/(2\sqrt{2})$. $D_{MS,m=1}$ for the "h10w3g1" model at $\tau = 100$ and the "h10w3g4" model at $\tau = 130$ are shown in Panels (b) and (c), respectively. The black vertical dotted lines in Panels (b) and (c) indicate the turning points of $D_{MS,m=1}$, r_{IR} and r_{OR} . The blue dotted line shows the parabolic functions for a comparison.

Otherwise, the time evolution of η_1 resembles that in the “h10w3g4” run for the models with $\gamma_*/\Omega_n \lesssim 0.1$. I therefore expect that disks will be stable or marginally stable against the RWI after the RWI vortex formation in general.

I can observe a spike of the $\langle \Sigma \rangle$ profile close to the inner boundary. It is considered that the spike is caused by my numerical treatment at the inner boundary. I also investigate the stability of the spike against the RWI for all the models using equation (B.1). I find that the spike is always stable against the RWI. This can be reasonably explained that this is because the spike is too close to the central star for the onset of the RWI.

Appendix C

Supplementary Data in Chapter 5

C.1 The Linear Growth Rate of the Initial Conditions of Each Azimuthal Mode

In Tables C.1 and C.2, I show the most unstable azimuthal mode m_* and the linear growth rates γ_m for $1 \leq m \leq 10$ modes.

For all the linear stability analyses, I adopt the inner radius $r_{\text{in}} = 0.3$, the outer radius $r_{\text{out}} = 2.5$. I set the radial grid number N as $N = 1024$ for $h = 0.1, 0.15, 0.2$, and $N = 2048$ for $h = 0.05$.

C.2 Table of the Results

I have performed numerical calculations of 54 models and analysed the outcomes in Chapter 5. Here, I exhibit the supplementary data measured from those calculations. The values of τ_2 , τ_1 , and $\tau_2 - \tau_1$ are shown in Table C.3. For the “h10w5g3”–“h10w5g5”, “h20w5g4”, “h15w5g4”, and “h05w5g4” models, I am not able to have measured τ_2 and τ_1 with visual inspection due to low m_* .

Table C.1: The most unstable azimuthal mode and the linear growth rates of each mode for the models with $h = 0.1$.

Name	m_*	linear growth rate each azimuthal mode (γ_m/Ω_n)									
		1	2	3	4	5	6	7	8	9	10
h10w1g1	9	0.053	0.100	0.141	0.172	0.195	0.210	0.221	0.226	0.227	0.225
h10w1g2	8	0.053	0.095	0.131	0.158	0.177	0.190	0.197	0.200	0.199	0.194
h10w1g3	7	0.042	0.079	0.108	0.128	0.141	0.148	0.150	0.148	0.141	0.131
h10w1g4	6	0.033	0.061	0.081	0.094	0.0996	0.100	0.096	0.087	0.074	0.054
h10w1g5	4	0.021	0.038	0.047	0.050	0.047	0.039	0.026	0.008	–	–
h10w2g1	6	0.074	0.140	0.189	0.220	0.238	0.242	0.237	0.222	0.199	0.168
h10w2g2	5	0.067	0.123	0.165	0.190	0.201	0.200	0.190	0.170	0.142	0.106
h10w2g3	5	0.054	0.100	0.131	0.147	0.150	0.142	0.124	0.098	0.062	0.012
h10w2g4	4	0.042	0.075	0.095	0.100	0.094	0.077	0.052	0.017	–	–
h10w2g5	3	0.026	0.044	0.050	0.043	0.025	0.002	–	–	–	–
h10w3g1	4	0.103	0.184	0.232	0.246	0.230	0.188	0.122	0.040	–	–
h10w3g2	4	0.089	0.158	0.195	0.200	0.177	0.130	0.059	0.001	–	–
h10w3g3	3	0.072	0.126	0.150	0.145	0.113	0.059	0.002	–	–	–
h10w3g4	3	0.054	0.090	0.100	0.089	0.041	–	–	–	–	–
h10w3g5	2	0.033	0.050	0.042	0.009	–	–	–	–	–	–
h10w4g1	3	0.140	0.224	0.227	0.149	0.005	–	–	–	–	–
h10w4g2	2	0.126	0.200	0.195	0.115	0.002	–	–	–	–	–
h10w4g3	2	0.098	0.150	0.131	0.044	–	–	–	–	–	–
h10w4g4	2	0.070	0.100	0.065	–	–	–	–	–	–	–
h10w4g5	2	0.043	0.050	0.002	–	–	–	–	–	–	–
h10w5g1	2	0.174	0.191	0.002	–	–	–	–	–	–	–
h10w5g3	2	0.146	0.150	0.001	–	–	–	–	–	–	–
h10w5g4	1	0.100	0.080	–	–	–	–	–	–	–	–
h10w5g5	1	0.050	0.001	–	–	–	–	–	–	–	–

NOTE. Name: the name of the model. m_* : the most unstable azimuthal mode. γ_m/Ω_n : the linear growth rate against the RWI for each azimuthal mode m normalized by Ω_n .

Table C.2: The most unstable azimuthal mode and the linear growth rates of each mode for the models with $h \neq 0.1$.

Name	m_*	linear growth rate each azimuthal mode (γ_m/Ω_n)									
		1	2	3	4	5	6	7	8	9	10
h20w1g1	8	0.064	0.118	0.153	0.177	0.191	0.201	0.206	0.209	0.208	0.206
h20w1g4	4	0.044	0.076	0.093	0.100	0.099	0.096	0.086	0.076	0.061	0.048
h20w2g1	5	0.093	0.162	0.201	0.220	0.224	0.220	0.205	0.186	0.159	0.132
h20w2g4	3	0.056	0.091	0.100	0.093	0.072	0.046	0.005	—	—	—
h20w3g1	3	0.126	0.207	0.237	0.229	0.195	0.140	0.063	0.001	—	—
h20w3g4	2	0.069	0.100	0.087	0.045	—	—	—	—	—	—
h20w4g1	2	0.163	0.237	0.213	0.118	—	—	—	—	—	—
h20w4g4	2	0.087	0.100	0.030	—	—	—	—	—	—	—
h20w5g1	2	0.191	0.192	0.020	—	—	—	—	—	—	—
h20w5g4	1	0.100	0.049	—	—	—	—	—	—	—	—
h15w1g1	8	0.059	0.112	0.150	0.177	0.195	0.207	0.213	0.216	0.215	0.212
h15w1g4	5	0.039	0.070	0.089	0.098	0.100	0.097	0.089	0.079	0.065	0.049
h15w2g1	5	0.086	0.155	0.199	0.224	0.233	0.231	0.219	0.200	0.175	0.144
h15w2g4	3	0.050	0.085	0.100	0.098	0.084	0.060	0.028	—	—	—
h15w3g1	4	0.117	0.200	0.238	0.240	0.213	0.162	0.092	0.007	—	—
h15w3g4	2	0.064	0.100	0.099	0.084	0.060	0.028	—	—	—	—
h15w4g1	2	0.153	0.233	0.222	0.136	0.003	—	—	—	—	—
h15w4g4	2	0.079	0.100	0.048	—	—	—	—	—	—	—
h15w5g1	2	0.183	0.193	0.025	—	—	—	—	—	—	—
h15w5g4	1	0.100	0.066	—	—	—	—	—	—	—	—
h05w1g1	11 ¹	0.048	0.078	0.116	0.149	0.176	0.199	0.216	0.229	0.238	0.243
h05w1g4	7	0.023	0.044	0.063	0.079	0.090	0.097	0.100	0.099	0.095	0.088
h05w2g1	7	0.057	0.112	0.160	0.198	0.225	0.242	0.248	0.244	0.232	0.211
h05w2g4	5	0.030	0.058	0.080	0.094	0.100	0.098	0.087	0.067	0.039	—
h05w3g1	4	0.085	0.159	0.213	0.240	0.237	0.206	0.147	0.064	0.011	—
h05w3g4	4	0.044	0.079	0.100	0.240	0.237	0.206	0.147	0.064	0.010	—
h05w4g1	3	0.128	0.214	0.225	0.151	0.001	—	—	—	—	—
h05w4g4	2	0.064	0.100	0.082	0.001	—	—	—	—	—	—
h05w5g1	2	0.168	0.188	0.001	—	—	—	—	—	—	—
h05w5g4	1	0.100	0.088	—	—	—	—	—	—	—	—

NOTE. Name: the name of the model. m_* : the most unstable azimuthal mode. γ_m/Ω_n : the linear growth rate against the RWI for each azimuthal mode m normalized by Ω_n .

[1] The largest linear growth rate (γ_*/Ω_n) is 0.244 for the "h05w1g1" model.

In those cases, I set τ_2 to no data and τ_1 to the orbit number at the saturation. I also show that the values of the eleven measurements (see Table 5.3) in Table C.4. I am not able to have measured ξ for the “h05w1g4”, “h05w2g4”, and “h05w3g4” models, and $\bar{\zeta}_v$ and μ for the “h10w1g5”, “h10w2g5”, “h10w3g5”, “h05w1g4”, “h05w2g4”, and “h05w3g4” models due to too slow vortex migration. In those cases, I set these parameters to no data.

Table C.3: Values of τ_2 , τ_1 , $\tau_1 - \tau_2$.

Name	τ_2	τ_1	$\tau_1 - \tau_2$	Name	τ_2	τ_1	$\tau_1 - \tau_2$
h10w1g1	23	45	22	h20w1g1	24	25	1
h10w1g2	33	57	24	h20w1g4	30	59	29
h10w1g3	36	96	60	h20w2g1	14	27	13
h10w1g4	52	66	14	h20w2g4	21	37	16
h10w1g5	54	98	44	h20w3g1	12	15	3
h10w2g1	24	84	60	h20w3g4	29	42	13
h10w2g2	21	46	25	h20w4g1	8	16	8
h10w2g3	21	111	90	h20w4g4	19	26	7
h10w2g4	28	49	21	h20w5g1	11	12	1
h10w2g5	41	109	68	h20w5g4	–	23	–
h10w3g1	18	20	2	h15w1g1	37	62	25
h10w3g2	20	45	28	h15w1g4	30	82	52
h10w3g3	22	42	20	h15w2g1	16	27	11
h10w3g4	25	35	10	h15w2g4	25	32	7
h10w3g5	47	72	25	h15w3g1	14	21	7
h10w4g1	10	17	7	h15w3g4	19	35	16
h10w4g2	12	20	8	h15w4g1	8	17	9
h10w4g3	14	28	14	h15w4g4	19	29	10
h10w4g4	20	36	16	h15w5g1	11	13	2
h10w4g5	36	45	9	h15w5g4	–	23	–
h10w5g1	12	13	1	h05w1g1	69	177	108
h10w5g3	–	15	–	h05w1g4	57	195	138
h10w5g4	–	22	–	h05w2g1	29	199	170
h10w5g5	–	52	–	h05w2g4	36	182	146
				h05w3g1	16	45	29
				h05w3g4	26	49	23
				h05w4g1	10	21	11
				h05w4g4	19	46	27
				h05w5g1	13	15	2
				h05w5g4	–	24	–

NOTE. Name: the name of the model. τ_2 : orbit number when the two vortices are formed after vortex mergers. τ_1 : orbit number when the one last remaining vortex is formed after the final vortex merger. ($\tau_1 - \tau_2$): the life time of the two vortices regime.

Table C.4: Values of the approximately constant parameters of the RWI vortices.

Name	χ_2	ψ_2	$\delta v_{\varphi v,r}/\Omega_v$	$v_{rv,\varphi}/(r_v\Omega_v)$	$\Pi_{v,rr}/\Omega_v^2$	$\Pi_{v,\varphi\varphi}/(r_v\Omega_v)^2$	$\Delta r/r_n$	$r_v\Delta\varphi/r_n$	ξ/r_n	ζ_v	μ
h10w1g1	5.13	2.86	3.23E-1	7.09E-2	5.24E-1	1.24E-2	0.067	0.281	8.91E-5	1.21	6.37E-5
h10w1g2	5.97	3.39	2.59E-1	4.88E-2	4.36E-1	1.33E-2	0.066	0.298	7.01E-5	1.19	5.19E-5
h10w1g3	8.12	4.82	2.00E-1	2.60E-2	3.53E-1	8.05E-3	0.061	0.370	3.15E-5	1.15	2.15E-5
h10w1g4	12.69	7.87	1.43E-1	1.09E-2	2.69E-1	3.85E-3	0.053	0.510	1.06E-5	1.10	5.86E-6
h10w1g5	27.13	17.75	9.19E-2	2.47E-3	1.80E-1	1.06E-3	0.040	0.736	7.46E-7	–	–
h10w2g1	4.81	2.67	3.06E-1	7.20E-2	4.79E-1	1.41E-2	0.101	0.419	1.42E-4	1.41	1.78E-4
h10w2g2	5.65	3.22	2.57E-1	5.23E-2	4.23E-1	1.27E-2	0.095	0.467	1.01E-4	1.38	1.13E-4
h10w2g3	8.09	4.77	2.00E-1	2.53E-2	3.58E-1	7.16E-3	0.087	0.582	3.34E-5	1.28	3.54E-5
h10w2g4	13.38	8.13	1.47E-1	9.23E-3	2.85E-1	3.29E-3	0.074	0.765	6.80E-6	1.18	8.09E-6
h10w2g5	28.51	17.78	1.01E-1	1.99E-3	2.07E-1	8.01E-4	0.057	1.181	5.65E-7	–	–
h10w3g1	4.34	2.36	3.44E-1	9.66E-2	4.98E-1	1.47E-2	0.143	0.581	2.20E-4	1.90	3.64E-4
h10w3g2	5.56	3.12	2.74E-1	5.63E-2	5.05E-1	1.35E-2	0.136	0.670	1.07E-4	1.79	1.61E-4
h10w3g3	7.91	4.59	2.25E-1	2.71E-2	4.05E-1	7.60E-3	0.125	0.809	2.73E-5	1.61	3.88E-5
h10w3g4	12.89	7.67	1.81E-1	1.01E-2	3.49E-1	3.34E-3	0.107	1.047	5.41E-6	1.41	8.01E-6
h10w3g5	25.90	15.58	1.45E-1	2.38E-3	2.90E-1	8.88E-4	0.084	1.502	5.76E-7	–	–
h10w4g1	4.04	2.15	3.55E-1	1.08E-1	5.15E-1	1.61E-2	0.209	0.749	3.66E-4	3.28	7.13E-4
h10w4g2	4.67	2.55	3.20E-1	8.12E-2	4.96E-1	1.54E-2	0.201	0.819	2.55E-4	3.07	5.00E-4
h10w4g3	6.67	3.77	2.63E-1	3.88E-2	4.60E-1	9.90E-3	0.187	1.033	5.71E-5	2.67	9.67E-5
h10w4g4	10.49	6.04	2.22E-1	1.54E-2	4.19E-1	4.71E-3	0.162	1.269	1.18E-5	2.18	1.95E-5

Table C.4 continued on next page

Table C.4 (continued)

Name	χ_2	ψ_2	$\delta v_{\varphi v,r}/\Omega_v$	$v_{rv,\varphi}/(r_v\Omega_v)$	$\Pi_{v,rr}/\Omega_v^2$	$\Pi_{v,\varphi\varphi}/(r_v\Omega_v)^2$	$\Delta r/r_n$	$r_v\Delta\varphi/r_n$	ξ/r_n	$\bar{\zeta}_v$	μ
h10w4g5	20.21	11.94	1.90E-1	3.93E-3	3.75E-1	1.37E-3	0.136	1.675	2.25E-6	2.16	3.58E-6
h10w5g1	4.00	2.00	3.73E-1	1.09E-1	5.54E-1	1.96E-2	0.254	1.067	1.10E-3	8.40	1.72E-3
h10w5g3	5.06	2.76	3.12E-1	6.99E-2	5.26E-1	1.56E-2	0.299	1.162	3.33E-4	6.76	6.80E-4
h10w5g4	7.62	4.28	2.62E-1	3.02E-2	4.77E-1	8.22E-3	0.261	1.470	8.78E-5	5.34	1.53E-4
h10w5g5	15.35	8.82	2.21E-1	7.16E-3	4.33E-1	2.24E-3	0.230	1.816	9.91E-6	3.68	1.59E-5
h20w1g1	5.57	3.16	2.67E-1	4.77E-2	4.56E-1	1.05E-2	0.079	0.292	1.13E-4	1.06	5.19E-5
h20w1g4	10.82	6.73	1.19E-1	1.38E-2	2.22E-1	4.87E-3	0.064	0.485	4.17E-5	1.03	1.49E-5
h20w2g1	4.86	2.70	3.11E-1	7.12E-2	4.91E-1	1.27E-2	0.118	0.426	1.98E-4	1.10	1.70E-4
h20w2g4	10.29	6.25	1.44E-1	1.54E-2	2.68E-1	5.58E-3	0.096	0.774	7.21E-5	1.05	4.34E-5
h20w3g1	4.51	2.51	3.07E-1	8.40E-2	4.47E-1	1.42E-2	0.165	0.602	2.81E-4	1.17	3.90E-4
h20w3g4	9.17	5.46	1.77E-1	1.98E-2	3.22E-1	6.52E-3	0.130	0.982	8.18E-5	1.10	7.02E-5
h20w4g1	4.04	2.23	3.30E-1	1.01E-1	4.85E-1	2.14E-2	0.227	0.793	6.90E-4	1.37	1.21E-3
h20w4g4	8.13	4.76	2.08E-1	2.57E-2	3.75E-1	7.57E-3	0.187	1.348	1.18E-4	1.24	1.42E-4
h20w5g1	4.13	2.22	3.53E-1	1.02E-1	5.24E-1	1.54E-2	0.308	0.966	1.21E-3	2.08	2.23E-3
h20w5g4	7.02	4.00	2.54E-1	3.57E-2	4.51E-1	1.05E-2	0.257	1.529	2.53E-4	1.63	4.04E-4
h15w1g1	5.27	2.93	2.83E-1	5.57E-2	4.78E-1	1.11E-2	0.072	0.260	9.59E-5	1.10	5.45E-5
h15w1g4	10.91	6.81	1.24E-1	1.32E-2	2.37E-1	4.68E-3	0.061	0.506	3.23E-5	1.05	1.46E-5
h15w2g1	4.83	2.66	3.03E-1	7.41E-2	4.78E-1	1.51E-2	0.110	0.395	2.20E-4	1.18	2.26E-4
h15w2g4	11.05	6.68	1.47E-1	1.31E-2	2.76E-1	4.92E-3	0.085	0.757	3.48E-5	1.09	2.58E-5
h15w3g1	4.38	2.38	3.44E-1	8.87E-2	5.24E-1	1.08E-2	0.152	0.574	2.33E-4	1.37	3.52E-4

Table C.4 continued on next page

Table C.4 (continued)

Name	χ_2	ψ_2	$\delta v_{\varphi v, r} / \Omega_v$	$v_{rv, \varphi} / (r_v \Omega_v)$	$\Pi_{v, rr} / \Omega_v^2$	$\Pi_{v, \varphi \varphi} / (r_v \Omega_v)^2$	$\Delta r / r_n$	$r_v \Delta \varphi / r_n$	ξ / r_n	$\bar{\zeta}_v$	μ
h15w3g4	10.31	6.15	1.77E-1	1.52E-2	3.31E-1	5.60E-3	0.119	0.955	3.22E-5	1.19	3.31E-5
h15w4g1	4.12	2.26	3.40E-1	1.03E-1	4.83E-1	1.41E-2	0.210	0.723	5.24E-4	1.79	1.03E-3
h15w4g4	9.53	5.59	2.10E-1	1.85E-2	3.90E-1	6.30E-3	0.171	1.332	4.38E-5	1.46	5.85E-5
h15w5g1	4.12	2.14	3.65E-1	9.88E-2	5.69E-1	2.41E-2	0.296	1.029	1.22E-3	3.44	2.14E-3
h15w5g4	7.30	4.15	2.62E-1	3.26E-2	4.67E-1	9.33E-3	0.256	1.391	1.40E-4	2.42	2.31E-4
h05w1g1	5.29	2.95	2.91E-1	6.31E-2	4.69E-1	1.42E-2	0.063	0.309	4.62E-5	1.67	7.12E-5
h05w1g4	20.52	12.12	1.44E-1	3.15E-3	3.00E-1	1.22E-3	0.043	0.648	—	—	—
h05w2g1	4.87	2.71	2.99E-1	6.97E-2	4.76E-1	1.45E-2	0.093	0.457	5.87E-5	2.57	1.11E-4
h05w2g4	19.74	11.79	1.72E-1	4.14E-3	3.45E-1	1.38E-3	0.063	0.903	—	—	—
h05w3g1	4.23	2.29	3.44E-1	9.36E-2	5.11E-1	1.54E-2	0.143	0.594	1.21E-4	5.38	2.18E-4
h05w3g4	16.56	9.73	2.07E-1	6.06E-3	4.06E-1	1.90E-3	0.101	1.028	—	—	—
h05w4g1	3.59	1.94	3.50E-1	1.38E-1	4.40E-1	2.22E-2	0.189	0.648	5.72E-4	16.20	6.67E-4
h05w4g4	12.06	6.88	2.35E-1	1.27E-2	4.54E-1	3.94E-3	0.177	1.242	3.01E-6	8.82	1.26E-5
h05w5g1	3.67	2.03	3.56E-1	1.30E-1	3.77E-1	3.90E-2	0.241	0.680	9.73E-4	41.23	1.22E-3
h05w5g4	7.72	4.42	2.78E-1	2.95E-2	4.49E-1	8.50E-3	0.287	1.587	4.83E-5	33.38	8.32E-5

Appendix D

Numerical Test

I can observe a spike of the $\langle \Sigma \rangle$ profile close to the inner boundary Panel (a) of Figure 5.4 due to my numerical treatment at the inner boundary. In order to check the convergence of my numerical calculations, I have additionally performed a high-resolution calculation, and a low-resolution calculation of the "h10w3g1" model. The high-resolution calculation has twice as many cells as the fiducial calculation does in each direction. In contrast, the low-resolution calculation has half cells compared to the fiducial calculation in each direction. I also perform wide domain calculations in order to check the effects of the inner boundary on the RWI vortices. In the wide domain calculations, the numerical setting is the same as that of the fiducial calculation but the radius of the inner boundary is set at $r_{\text{in}}/r_{\text{n}} = 0.2, 0.1$, and 0.03 instead of 0.3 .

In all the calculations, the RWI vortices are formed around $\tau \approx 20$. I have measured six parameters that characterize the vortex and its migration ($\chi_2, \Delta, r_v \Delta \varphi, \xi, \bar{\zeta}_v$, and μ) in each additional calculation. Figure D.1 shows the deviation of the parameters from those of the fiducial calculation. Since the RWI vortices in the high resolution calculation and the wide domain calculations have almost same parameters as those in the fiducial calculation with 5% accuracy, I conclude that the fiducial calculation has sufficiently high resolution and the results are converged.

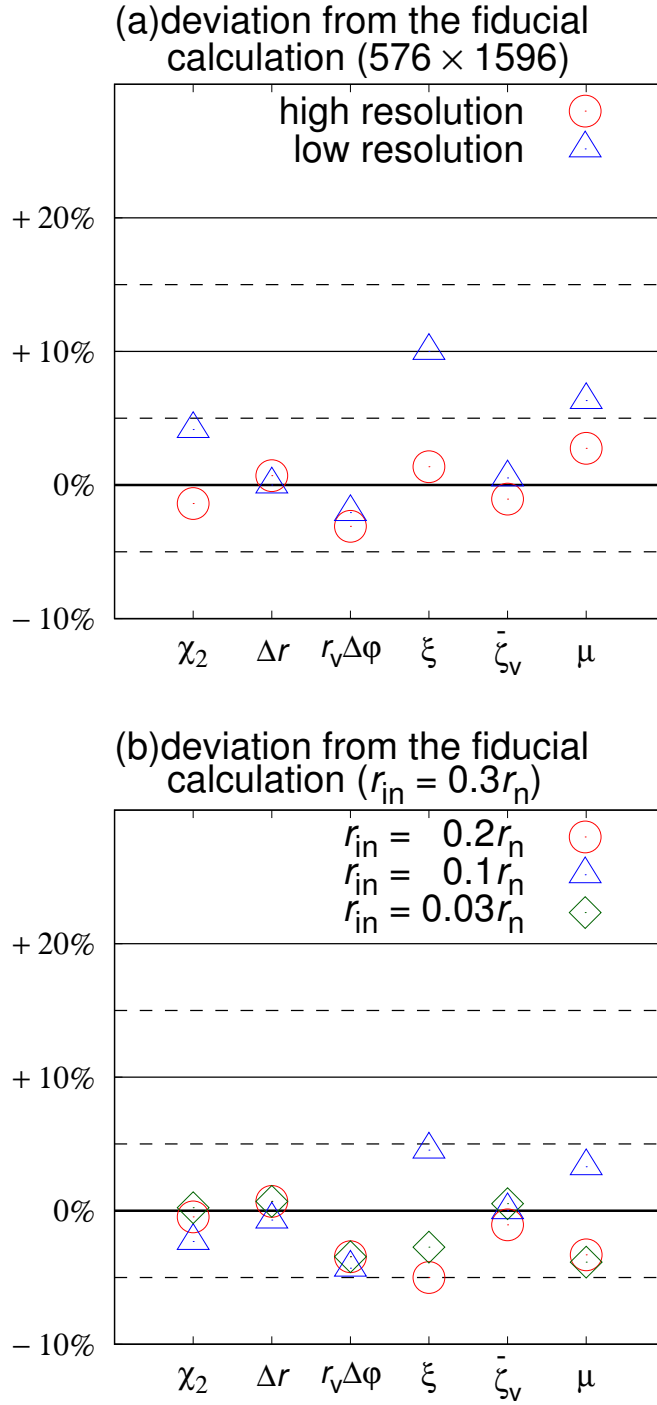


Figure D.1: Panel (a): The deviations of parameters (χ_2 , δr , $r_v \delta \varphi$, ξ , $\bar{\zeta}_v$, and μ) from the fiducial calculation in the high resolution calculation (red circles) and the low resolution calculation (blue triangles). Panel (b): The deviations of parameters from the fiducial calculation in the wide domain calculations with $r_{\text{in}}/r_n = 0.2$ (red circles), 0.1 (blue triangles), and 0.03 (green diamonds).

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