

ON A PROBLEM IN DIOPHANTINE APPROXIMATION

Saburô UCHIYAMA

Institute of Mathematics, University of Tsukuba

We are concerned in this article with a property of badly approximable real numbers. Professor W. M. Schmidt has made among many other things a number of interesting and important contributions in the study of such numbers.

An n -tuple $(\alpha_1, \dots, \alpha_n)$ of real numbers is, by definition, badly approximable if

$$(*) \quad |x|^{1/n} \cdot \max(\|\alpha_1 x\|, \dots, \|\alpha_n x\|) \geq \gamma$$

for some constant $\gamma > 0$, whenever $x \neq 0$ is an integer.

Here, $\|t\|$ denotes as usual the distance from the real number t to the nearest integer, so that we always have $0 < \|t\| \leq 1/2$.

We shall prove the following

THEOREM. Let $(\alpha_1, \dots, \alpha_n)$ be a badly approximable n -tuple of real numbers, i.e. an n -tuple satisfying $(*)$. Let $\beta_1, \dots, \beta_n$ be n arbitrary real numbers, and $X \geq 2$ an arbitrary integer. Put

$$D = \left\lceil \frac{(n+1)n^{n-1} X^{4n+2}}{2\gamma} \right\rceil + 1.$$

Then, for any integer $d \geq 0$ there is an integer x in the interval

$$d < x \leq d + D$$

such that

$$\|\alpha_i x - \beta_i\| < \frac{1}{x} \quad (i = 1, 2, \dots, n).$$

By a transference theorem (cf. e.g. [2; Chap. V, Corollary to Theorem II]), an n -tuple $(\alpha_1, \dots, \alpha_n)$ is badly approximable if, and only if for every integral n -vector $\underline{x} \neq \underline{0}$

$$\|\alpha_1 x_1 + \dots + \alpha_n x_n\| \geq \frac{\gamma'}{|\underline{x}|^n}$$

for some constant $\gamma' > 0$. Here

$$|\underline{x}| = \max(|x_1|, \dots, |x_n|)$$

if $\underline{x} = (x_1, \dots, x_n)$. We may take $\gamma' = n^{n+1}\gamma$ in case the n -tuple $(\alpha_1, \dots, \alpha_n)$ satisfies (*). Thus, in particular, if $(\alpha_1, \dots, \alpha_n)$ is badly approximable, then $1, \alpha_1, \dots, \alpha_n$ are linearly independent over the rationals.

Now, following H. Bohr and B. Jessen [1], we define

$$F(t) = 1 + e^{2\pi i(\alpha_1 t - \beta_1)} + \dots + e^{2\pi i(\alpha_n t - \beta_n)}$$

and, with Fejér's kernel

$$K_N(t) = \sum_{v=-N}^N \left(1 - \frac{|v|}{N}\right) e^{2\pi i v t} = \frac{1}{N} \left(\frac{\sin \pi N t}{\sin \pi t} \right)^2,$$

$$\mathbb{K}_N(t) = K_N(\alpha_1 t - \beta_1) \cdots K_N(\alpha_n t - \beta_n)$$

$$= 1 + \left(1 - \frac{1}{N}\right) (e^{-2\pi i(\alpha_1 t - \beta_1)} + \dots \\ + e^{-2\pi i(\alpha_n t - \beta_n)}) + R(t);$$

here, $R(t)$ is a trigonometric polynomial whose exponents, divided by 2π , are all different from the numbers $0, -\alpha_1, \dots, -\alpha_n$.

We have

$$F(t) K_N(t) = 1 + \left(1 - \frac{1}{N}\right) n + S(t),$$

where $S(t)$ is a trigonometric polynomial whose exponents are all different from zero modulo 2π . Hence

$$\frac{1}{D} \sum_{d < x \leq d+D} F(x) K_N(x) = 1 + \left(1 - \frac{1}{N}\right) n \\ + \frac{1}{D} \sum_{d < x \leq d+D} S(x),$$

where we find easily

$$\left| \frac{1}{D} \sum_{d < x \leq d+D} S(x) \right| \leq \frac{N^n}{D} \frac{N^n}{2\gamma'}.$$

(Note that the sum of the coefficients of $K_N(t)$ equals N .)

By the positivity of the kernel $K_N(t)$ we have, since

$$\frac{1}{D} \sum_{d < x \leq d+D} K_N(x) \leq 1 + \frac{N^n}{D} \frac{N^n}{2\gamma'},$$

$$\max_{d < x \leq d+D} |F(x)| \geq 1 + \left(1 - \frac{1}{N}\right) n - \frac{N^{2n}}{2\gamma' D}.$$

$$\cdot \left(1 + \frac{N^{2n}}{2\gamma'D} \right)^{-1}.$$

Taking

$$N = nX^2,$$

$$D = \left[\frac{(n+1)n^{2n}X^{4n+2}}{2\gamma'} \right] + 1,$$

we get

$$\max_{d < x \leq d+D} |F(x)| \geq n+1 - \frac{3}{X^2}.$$

Let α, β be any one of the pairs α_i, β_i ($1 \leq i \leq n$).

Then, since

$$|F(x)| \leq n-1 + |1 + e^{2\pi i(\alpha x - \beta)}|,$$

we have

$$|1 + e^{2\pi i(\alpha x - \beta)}| \geq 2 - \frac{3}{X^2}.$$

Noticing that $|1 + e^{2\pi i t}| = 2|\cos \pi t|$ and $|\sin \pi t| \geq 2\|t\|$, we deduce from the above inequalities for $|F(x)|$ that

$$\|\alpha x - \beta\| \leq \frac{\sqrt{3}}{2} \frac{1}{X} < \frac{1}{X}$$

for some integer x , independent of the particular α, β , in the interval $d < x \leq d+D$. This completes the proof of our theorem.

It should be observed that our method can be applied to any n -tuple of real numbers $\alpha_1, \dots, \alpha_n$ such that 1 and the α_i 's are linearly independent over the field of rational

numbers, obtaining a result similar to the theorem above with a suitably defined D in terms of M_n , where

$$\frac{1}{M_n} = \min_{0 < \underline{x} \leq n} \|\alpha_1 x_1 + \dots + \alpha_n x_n\|.$$

We thus have a sort of quantitative formulation of the (small) approximation theorem of Kronecker's.

References

- [1] H. Bohr and B. Jessen: One more proof of Kronecker's theorem. J. London Math. Soc., 7(1932), 274-275.
- [2] J. W. S. Cassels: An Introduction to Diophantine Approximation. Cambridge Tracts in Math. & Math. Phys. No. 45. Cambridge Univ. Press, 1957.