

A property of higher order asymptotically sufficient statistics

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1. Introduction. Suppose that n -dimensional random variable $z_n = (x_1, x_2, \dots, x_n)$ is distributed according to a probability distribution $P_{\theta, n}$ parameterised by $\theta \in \Theta \subset \mathbb{R}^1$, and each x_i is independently and identically distributed. In LeCam[1] it was shown that every estimator t_n with the form $t_n = \hat{\theta}_n + n^{-1} \cdot I^{-1}(\hat{\theta}_n) \Phi_n^{(1)}(z_n, \hat{\theta}_n)$ ($I(\theta)$ means Fisher information number), which is constructed using a reasonable estimator $\hat{\theta}_n$ and the logarithmic derivative $\Phi_n^{(1)}(z_n, \theta)$ relative to θ of density of $P_{\theta, n}$, is asymptotically sufficient in the following sense; t_n is sufficient for a family $\{Q_{\theta, n}; \theta \in \Theta\}$ of probability distributions and that

$$\lim_{n \rightarrow \infty} \|P_{\theta, n} - Q_{\theta, n}\| = 0$$

uniformly on any compact set in Θ (where $\|\cdot\|$ means the totally variation of a measure). This implies that the statistic $(\hat{\theta}_n, \Phi_n^{(1)}(z_n, \hat{\theta}_n))$ is asymptotically sufficient up to order $o(1)$. As a refinement of this result it will be shown in this paper that for $k \geq 1$ the statistic $t_n^* = (\hat{\theta}_n, \Phi_n^{(1)}(z_n, \hat{\theta}_n), \dots, \Phi_n^{(k)}(z_n, \hat{\theta}_n))$, where $\Phi_n^{(i)}(z_n, \theta)$ means the $(i-1)$ -th derivative relative to θ of $\Phi_n^{(1)}(z_n, \theta)$, is asymptotically sufficient up to order $o(n^{-\frac{k-1}{2}})$ in the following sense; t_n^* is sufficient for a family $\{Q_{\theta, n}; \theta \in \Theta\}$ and

$$\lim_{n \rightarrow \infty} n^{\frac{k-1}{2}} \|P_{\theta, n} - Q_{\theta, n}\| = 0$$

uniformly on any compact subset of Θ . From our result it follows that if we use the maximum likelihood estimator $\hat{\theta}_n^*$ as the initial estimator $\hat{\theta}_n$ then the statistic $(\hat{\theta}_n^*, \Phi_n^{(2)}(z_n, \hat{\theta}_n^*), \dots, \Phi_n^{(k)}(z_n, \hat{\theta}_n^*))$ is

asymptotically sufficient up to order $o(n^{-\frac{k-1}{2}})$. In Ghosh and Subramanyam [4] it was mentioned that for exponential family of distributions, $(\theta_n^*, \Phi_n^{(2)}(z_n, \theta_n^*), \Phi_n^{(3)}(z_n, \theta_n^*), \Phi_n^{(4)}(z_n, \theta_n^*))$ is asymptotically sufficient up to order $o(n^{-1})$ in pointwise sense relative to θ . Our result is more general and accurate one.

As an application of our result we try to improve arbitrarily given statistical tests or estimators.

2. Notations and assumptions. Let $\mathcal{H}(\neq \emptyset)$ be an open set in $\mathbb{R}^1$. Suppose that for each $\theta \in \mathcal{H}$ there corresponds a probability measure P_θ defined on a measurable space $(X, \mathcal{A})$. For each $n \in N = \{1, 2, \dots\}$ let $(X^{(n)}, \mathcal{A}^{(n)})$ be the cartesian product of n copies of $(X, \mathcal{A})$, and $P_{\theta, n}$ the product measure of n copies of P_θ . For a function h and a probability measure P , $E[h; P]$ stands for the expectation of h under P .

We assume that the map: $\theta \mapsto P_\theta$ is one to one, and that for each $\theta \in \mathcal{H}$ P_θ has a density $f(\cdot, \theta)$ relative to a σ -finite measure μ on $(X, \mathcal{A})$. We assume also that $f(x, \theta) > 0$ for every $x \in X$ and every $\theta \in \mathcal{H}$. We denote by μ_n the product measure of n copies of the same component μ . We define $\Phi(x, \theta) = \log f(x, \theta)$ for each $x \in X$ and $\theta \in \mathcal{H}$, and $\Phi_n(z_n, \theta) = \sum_{i=1}^n \Phi(x_i, \theta)$ for each $n \in N$, each $z_n = (x_1, x_2, \dots, x_n) \in X^{(n)}$ and each $\theta \in \mathcal{H}$. For a positive integer k we consider the following conditions which will be called Condition (C_k) in this paper.

Condition (C_k) . (1). $\Phi(x, \theta)$ is $(k+2)$ -times continuously differentiable with respect to θ in $\mathcal{H}$ for each $x \in X$. For each j ($1 \leq j \leq k+2$) we define $\Phi^{(j)}(x, \theta) = \partial^j \Phi(x, \theta) / \partial \theta^j$ and $\Phi_n^{(j)}(z_n, \theta) = \sum_{i=1}^n \Phi^{(j)}(x_i, \theta)$.
(2). For each $\theta \in \mathcal{H}$ there exists a positive number ε such that

- a. $\sup_{|z-\theta| \leq \varepsilon} E[\sup_{|\sigma-\theta| \leq \varepsilon} |\Phi^{(k+2)}(x, \sigma)|^2; P_z] < \infty$
 b. $\sup_{|z-\theta| \leq \varepsilon} E[|\Phi^{(k+1)}(x, z)| \cdot u_\varepsilon(x, z); P_z] < \infty$ and $E[u_\varepsilon(x, \theta); P_\theta] < \infty$
 where $u_\varepsilon(x, z) = \sup_{|\sigma-z| \leq \varepsilon} |f'(x, \sigma)/f(x, z)|$
 c. $\text{Var}(\Phi^{(k+1)}(x, z); P_z)$ are positive and finite uniformly for

every z satisfying $|z-\theta| \leq \varepsilon$.

(3). Define $\bar{Z}(x; \varepsilon', \sigma) = \sup\{\Phi^{(k+1)}(x, z) - E[\Phi^{(k+1)}(x, z); P_z]; z \in \mathcal{U}, |z-\sigma| \leq \varepsilon'\}$ and $Z^*(x; \varepsilon', \sigma) = -\inf\{\Phi^{(k+1)}(x, z) - E[\Phi^{(k+1)}(x, z); P_z]; z \in \mathcal{U}, |z-\sigma| \leq \varepsilon'\}$ for each $\varepsilon' > 0$ and $\sigma \in \mathcal{U}$. For each $\theta \in \mathcal{U}$ there exist positive numbers η and ρ such that for every $(t, \varepsilon', \sigma) \in (-\rho, \rho) \times (0, \eta] \times (\theta - \eta, \theta + \eta)$ the moment generating functions of $\bar{Z}(x; \varepsilon', \sigma)$ and $Z^*(x; \varepsilon', \sigma)$ exist and converge uniformly with respect to θ in $(\theta - \eta, \theta + \eta)$.

Remark 1. An example satisfying Condition(C_K) is the following one. Let μ be a σ -finite measure on $(\underline{X}, \underline{A})$ and the density function $f(x, \theta)$ of P_θ relative to μ be given by

$$f(x, \theta) = h(x)c(\theta)\exp\left[\sum_{i=1}^m s_i(\theta)t_i(x)\right]$$

where $c(\theta), s_i(\theta)$ ($1 \leq i \leq m$) are $(k+2)$ -times continuously differentiable real valued functions of θ only, and $h(x), t_i(x)$ ($1 \leq i \leq m$) are real valued $\underline{A}$ -measurable functions of x independent of θ . Let $S = \{(s_1, s_2, \dots, s_m) \in \mathbb{R}^m; \int \exp[\sum_{i=1}^m s_i t_i(x)] h(x) d\mu(x) < \infty\}$ and $S(\mathcal{U}) = \{(s_1(\theta), \dots, s_m(\theta)); \theta \in \mathcal{U}\}$. If $S(\mathcal{U}) \subseteq \text{int } S$ (interior of S) and if $\sum_{i=1}^m \sum_{j=1}^m s_i^{(k+1)}(\theta) s_j^{(k+1)}(\theta) \text{Cov}(t_i, t_j; P_\theta) > 0$ for every $\theta \in \mathcal{U}$, then Condition(C_K) is satisfied by the family $\{P_\theta; \theta \in \mathcal{U}\}$. Here for each i $s_i^{(k+1)}$ means $(k+1)$ -th derivative of s_i .

3. Asymptotically sufficient statistics to higher orders. An estimator of θ depending on $z_n = (x_1, x_2, \dots, x_n) \in \underline{X}^{(n)}$ is an $\underline{A}^{(n)}$ -measurable function from $\underline{X}^{(n)}$ to $\mathbb{R}^1$. Such estimator will be called strict

if its range is a subset of $\mathcal{H}$. For each δ satisfying $0 < \delta < 1/2$ we denote by $\underline{C}(\delta)$ the class of all sequences $\{\hat{\theta}_n\}$ of strict estimators of θ such that for every compact subset K of $\mathcal{H}$

$$\sup_{\theta \in K} P_{\theta,n}(\sqrt{n}|\hat{\theta}_n - \theta| > n^\delta) = o(n^{-\frac{k-1}{2}}).$$

The notation $o(a_n)$ means that $\lim_{n \rightarrow \infty} o(a_n)/a_n = 0$.

Remark 2. For every δ ($0 < \delta < 1/2$) $\underline{C}(\delta)$ does not empty (cf. Pfanzagl[2], Lemma 2). The maximum likelihood estimator is contained in $\bigcap_{\delta > 0} \underline{C}(\delta)$ under suitable regularity conditions (cf. Pfanzagl[3], Lemma 3).

Let $\delta_0 = 1/[2(k+2)]$ and $C = \bigcup_{0 < \delta < \delta_0} \underline{C}(\delta)$.

Theorem 1. Suppose that Condition (C_K) is satisfied, and that $\{\hat{\theta}_n\} \in C$ then there exists a sequence $\{Q_{\theta,n}; \theta \in \mathcal{H}\}$, $n \in \mathbb{N}$, of families of probability measures on $(\underline{X}^{(n)}, \underline{A}^{(n)})$ with the following property:

(1) For each $n \in \mathbb{N}$, the statistic $t_n^* = (\hat{\theta}_n, \bar{\Phi}_n^{(1)}(z_n, \hat{\theta}_n), \dots, \bar{\Phi}_n^{(k)}(z_n, \hat{\theta}_n))$ is sufficient for $\{Q_{\theta,n}; \theta \in \mathcal{H}\}$. (2) For every compact set $K \subset \mathcal{H}$,

$$\sup_{\theta \in K} \|P_{\theta,n} - Q_{\theta,n}\| = o(n^{-\frac{k-1}{2}}).$$

The proof is omitted.

4. Tests based on asymptotically sufficient statistics. Let $\omega (\neq \emptyset)$ be a subset of $\mathcal{H}$. Suppose that it is desired to test the null hypothesis that $\theta \in \omega$ against the alternative that $\theta \in \mathcal{H} - \omega$. For a statistical test ϕ_n based on $z_n \in \underline{X}^{(n)}$ we denote by $\beta_n(\theta; \phi_n)$ the power function of ϕ_n , i.e., $\beta_n(\theta; \phi_n) = E[\phi_n; P_{\theta,n}]$. Let $\tilde{\mathcal{P}}(\alpha)$ be the class of all test sequences $\{\phi_n\}$ such that for every compact subset K of ω ,

$$\sup_{\theta \in K} |\beta_n(\theta; \phi_n) - \alpha| = o(n^{-\frac{k-1}{2}}).$$

In LeCam[1] such a test sequence, in the case of $k=1$, is called asymptotically similar of size α uniformly on compacts.

Theorem 2. Suppose that Condition (C_K) is satisfied and that $\{\hat{\theta}_n\}$ is a sequence of estimators belonging to $\underline{C}$. Then, for any sequence $\{\phi_n; n=1,2,\dots\}$ of statistical tests contained in $\tilde{\Phi}(\alpha)$ there exists a sequence $\{\psi_n; n=1,2,\dots\}$ of statistical tests contained in $\tilde{\Phi}(\alpha)$ with the following properties: (1) For every compact subset K of $\Theta - \omega$

$$\sup_{\theta \in K} |\beta_n(\theta; \phi_n) - \beta_n(\theta; \psi_n)| = o(n^{-\frac{k-1}{2}}).$$

The proof is omitted.

5. Estimates based on asymptotically sufficient statistics. For each positive number $s \geq 1$ we denote by $\underline{D}_s$ the class of all sequences $\{\tilde{\theta}_n\}$ of estimators of θ satisfying the following properties (1) and (2).

(1) For every $\rho > 0$ and every compact subset K of Θ ,

$$\sup_{\theta \in K} P_{\theta,n} (|\tilde{\theta}_n(z_n) - \theta| > \rho) = o(n^{-\frac{s-1}{2}})$$

(2) For each $\theta \in \Theta$ there exists a probability measure λ_θ on R^1 , which is weakly continuous relative to θ , such that $\lambda_\theta(\{0\}) \neq 1$ and that for any compact subset K of Θ the distribution of $n^{\frac{1}{2}}(\tilde{\theta}_n - \theta)$ converges weakly to λ_θ uniformly with respect to θ in K .

For any real number $p (p \geq 1)$ we define $k(p) = p+1$ if p is an integer, $k(p) = [p] + 2$ if p is not integer where $[p]$ means the maximum integer not exceeding p .

Theorem 3. Let $p \geq 1$ be any number. Let $\{\hat{\theta}_n\} \in \underline{C}$ and let $t_n^* = (\hat{\theta}_n, \tilde{\Phi}_n^{(1)}(z_n, \hat{\theta}_n), \dots, \tilde{\Phi}_n^{(K(p))}(z_n, \hat{\theta}_n))$. Suppose that $\Theta = \mathbb{R}^1$ and that Condition (C_k) is satisfied with $k=k(p)$. Then, for any $\{\tilde{\theta}_n\} \in \underline{D}_1$ there exists a sequence $\{\tilde{\theta}_n^*\}$ of estimates of θ satisfying the following properties (a), (b) and (c); (a). $\{\tilde{\theta}_n^*\}$ is locally uniformly consistent (i.e., for any $\beta > 0$ and for any compact subset K of Θ , $\lim_{n \rightarrow \infty} \sup_{\theta \in K} P_{\theta, n} (|\tilde{\theta}_n^*(z_n) - \theta| > \beta) = 0$) (b). For each $n \in \mathbb{N}$ $\tilde{\theta}_n^*$ is a function of t_n^* . (c). For any compact subset K of Θ

$$\lim_{n \rightarrow \infty} \sup_{\theta \in K} \left\{ \int_{X^{(n)}} |\tilde{\theta}_n^*(z_n) - \theta|^p dP_{\theta, n} / \int_{X^{(n)}} |\tilde{\theta}_n(z_n) - \theta|^p dP_{\theta, n} \right\} \leq 1.$$

The proof is omitted.

Remark 3. In the case where Θ is any open set in $\mathbb{R}^1$, we can conclude the same result as above theorem being exchanged the class $\underline{D}_1$ for $\underline{D}_{p+1}$.

6. Concluding remarks. A similar result to Theorem 1 is obtained in R. Michel [5]. His result asserts that the order of asymptotic sufficiency of t_n^* is of order $o(n^{-\frac{K-2}{2}})$.

References.

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