

Mixed type curves in Minkowski 3-space

Tongchang Liu and Donghe Pei

Abstract

In this paper, we study mixed type curves in Minkowski 3-space. Mixed type curves are regular curves, and there are both non-lightlike points and lightlike points in a mixed type curve. For non-lightlike curves and null curves in Minkowski 3-space, we can study them by a Frenet frame or a Cartan frame respectively. But for mixed type curves, the two frames will not work. And as far as we know, no one has yet given a frame to study them in Minkowski 3-space. So we give the lightcone frame in order to provide a tool for studying this type curves in mathematical and physical research. As an application of the lightcone frame, we define an evolute of a mixed type curve. And we also give some examples to show the evolutes.

1 Preliminaries

Let $\mathbb{R}^3 = \{(x_1, x_2, x_3) | x_1, x_2, x_3 \in \mathbb{R}\}$ be a real vector space. The Minkowski 3-space $\mathbb{R}_1^3$ is $\mathbb{R}^3$ endowed with the Lorentzian metric

$$\langle \cdot, \cdot \rangle = -dx_1^2 + dx_2^2 + dx_3^2.$$

A non-zero vector $\mathbf{v} \in \mathbb{R}_1^3$ is said to be spacelike, timelike or lightlike if $\langle \mathbf{v}, \mathbf{v} \rangle > 0$, $\langle \mathbf{v}, \mathbf{v} \rangle < 0$ or $\langle \mathbf{v}, \mathbf{v} \rangle = 0$, respectively. We usually consider the zero vector as a spacelike vector.

A curve $\gamma = \gamma(t)$ in $\mathbb{R}_1^3$ is said to be spacelike, timelike or null if its tangent vector field $\gamma'(t)$ is spacelike, timelike or lightlike, respectively, for all t .

But a regular curve in $\mathbb{R}_1^3$ may not be of one of the above three types. If there are both non-lightlike points and lightlike points in a regular curve in $\mathbb{R}_1^3$, we call it the mixed type curve.

Let γ be a spacelike or timelike curve in $\mathbb{R}_1^3$ parametrized by arc-length, we suppose that

$$\langle \gamma'', \gamma'' \rangle \neq 0.$$

Then there is a Frenet frame $\{\gamma; \mathbf{T} = \gamma', \mathbf{N} = \frac{\gamma''}{\|\gamma'', \gamma''\|^{\frac{1}{2}}}, \mathbf{B} = \mathbf{T} \wedge \mathbf{N}\}$ satisfying the following Frenet equations:

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\delta_1 \delta_2 \kappa & 0 & \delta_1 \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix},$$

where

Key Words and Phrases: mixed type curves, the lightcone frame, evolute.

$$\delta_1 = \langle \mathbf{T}, \mathbf{T} \rangle, \delta_2 = \langle \mathbf{N}, \mathbf{N} \rangle.$$

The vector fields $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ are called the tangent, principal normal and binormal of γ , respectively. The functions κ and τ are called the curvature and torsion of γ , respectively (see [5]).

As we all know, an evolute of a regular space curve γ in $\mathbb{R}^3$ (see [2]) is defined by

$$Ev(\gamma)(t) = \gamma(t) + \frac{1}{\kappa} \mathbf{N}(t) - \frac{\dot{\kappa}}{\kappa^2 \tau} \mathbf{B}(t).$$

By using the method, we can define an evolute of a non-lightlike curve γ in $\mathbb{R}_1^3$ by

$$Ev(\gamma)(t) = \gamma(t) + \delta_1 \delta_2 \frac{1}{\kappa} \mathbf{N}(t) + \delta_1 \delta_2 \frac{\dot{\kappa}}{\kappa^2 \tau} \mathbf{B}(t).$$

But for a mixed type curve, the frame will not work. We want to define an evolute of a mixed type curve, so we need a new frame. In the following work, we consider mixed type curves with isolated lightlike points and we suppose $\dot{\gamma} \wedge \ddot{\gamma} \neq 0$.

2 Lightcone frame

In this section, we will introduce the lightcone frame in Minkowski 3-space.

We denote

$$L_{\theta(t)}^+ = (1, \cos \theta(t), \sin \theta(t)),$$

$$L_{\theta(t)}^- = (1, -\cos \theta(t), -\sin \theta(t))$$

and

$$M_{\theta(t)} = L_{\theta(t)}^+ \wedge L_{\theta(t)}^- = (0, \sin \theta(t), -\cos \theta(t)),$$

where $\theta(t)$ is a smooth function. We call $\{L_{\theta(t)}^+, L_{\theta(t)}^-, M_{\theta(t)}\}$ a lightcone frame in $\mathbb{R}_1^3$. And we give a figure to show that (see Figure 1).

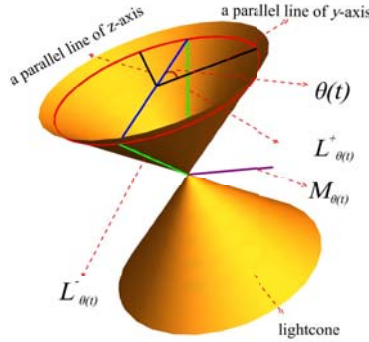


Figure 1: the lightcone frame

Let γ be a regular curve (or a mixed type curve) in $\mathbb{R}_1^3$. There exists a smooth function $(\alpha, \beta, \theta) : I \longrightarrow \mathbb{R}^3 \setminus \{(0, 0, \theta)\}$ such that

$$\dot{\gamma}(t) = \alpha(t)L_{\theta(t)}^+ + \beta(t)L_{\theta(t)}^-$$

for all $t \in I$. We say that a regular curve γ with the lightcone semi-polar coordinates (α, β, θ) if the above condition holds.

Since

$$\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle = -4\alpha(t)\beta(t),$$

$\gamma(t_0)$ is a

$$\begin{cases} \text{spacelike point} : & \alpha(t_0)\beta(t_0) < 0, \\ \text{timelike point} : & \alpha(t_0)\beta(t_0) > 0, \\ \text{lightlike point} : & \alpha(t_0)\beta(t_0) = 0. \end{cases}$$

We show that in the following figure (see Figure 2).

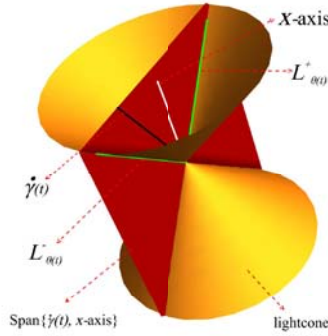


Figure 2: the lightcone semi-polar coordinates of $\dot{\gamma}(t)$

For convenience, let

$$\begin{aligned} \varepsilon_1(t) &= \langle \dot{\gamma}(t) \wedge \ddot{\gamma}(t), \dot{\gamma}(t) \wedge \ddot{\gamma}(t) \rangle \\ &= 4(\alpha(t)\dot{\beta}(t) - \beta(t)\dot{\alpha}(t))^2 + 4\dot{\theta}^2(t)\alpha(t)\beta(t)(\beta(t) - \alpha(t))^2, \end{aligned}$$

$$\begin{aligned} \varepsilon_2(t) &= \det(\dot{\gamma}(t), \ddot{\gamma}(t), \dddot{\gamma}(t)) \\ &= -2\dot{\theta}(t)(\beta(t) - \alpha(t))(\alpha(t)\ddot{\beta}(t) - \beta(t)\ddot{\alpha}(t)) \\ &\quad + 2(\alpha(t)\dot{\beta}(t) - \beta(t)\dot{\alpha}(t))(2\dot{\theta}(t)(\dot{\beta}(t) - \dot{\alpha}(t)) + \ddot{\theta}(t)(\beta(t) - \alpha(t))) \\ &\quad + \dot{\theta}^3(t)(\beta(t) - \alpha(t))^2(\beta^2(t) - \alpha^2(t)). \end{aligned}$$

Theorem 2.1. (The Existence Theorem) Let $(\alpha, \beta, \theta) : I \longrightarrow \mathbb{R}^3 \setminus \{(0, 0, \theta)\}$ be a smooth function. There exists a regular curve $\gamma : I \rightarrow \mathbb{R}_1^3$ with the lightcone semi-polar coordinates (α, β, θ) .

Remark 2.2. If γ and $\tilde{\gamma} : I \longrightarrow \mathbb{R}_1^3$ are regular curves with the same lightcone semi-polar coordinates (α, β, θ) , then there exists a constant vector $\mathbf{c} \in \mathbb{R}_1^3$ such that $\tilde{\gamma}(t) = \gamma(t) + \mathbf{c}$.

Before we give the uniqueness theorem, we need to make some preparations.

Definition 2.3. Let γ and $\tilde{\gamma} : I \longrightarrow \mathbb{R}_1^3$ be regular curves. We say that γ and $\tilde{\gamma}$ are congruent through a Lorentz motion if there exist a matrix $\mathbf{A}$ and a constant $\mathbf{c} \in \mathbb{R}_1^3$ such that $\tilde{\gamma}(t) = \mathbf{A}(\gamma(t)) + \mathbf{c}$ for all $t \in I$, where $\mathbf{A}$ satisfies

$$\mathbf{A}^T \mathbf{G} \mathbf{A} = \mathbf{G}, \det(\mathbf{A}) = 1, \mathbf{G} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For any vector $\mathbf{v} \in \mathbb{R}_1^3$ and $\mathbf{w} \in \mathbb{R}_1^3$, we can calculate

$$\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{A}(\mathbf{v}), \mathbf{A}(\mathbf{w}) \rangle,$$

$$\mathbf{v} \wedge \mathbf{w} = \mathbf{A}(\mathbf{v}) \wedge \mathbf{A}(\mathbf{w}).$$

So we have

$$\alpha(t)\beta(t) = \tilde{\alpha}(t)\tilde{\beta}(t), \quad \varepsilon_1(t) = \tilde{\varepsilon}_1(t), \quad \varepsilon_2 = \tilde{\varepsilon}_2(t).$$

Proposition 2.4. If $\gamma : I \longrightarrow \mathbb{R}_1^3$ is a non-lightlike curve, then

$$\kappa(t) = \frac{(-\delta_1 \delta_2 \varepsilon_1(t))^{\frac{1}{2}}}{8(-\delta_1 \alpha(t) \beta(t))^{\frac{3}{2}}}, \quad \tau(t) = \delta_1 \frac{\varepsilon_2(t)}{\varepsilon_1(t)}.$$

So we have

$$\kappa(t) = \tilde{\kappa}(t), \quad \tau(t) = \tilde{\tau}(t).$$

The fundamental theorem of non-lightlike curves has been given in [1, 4]. Using them, we get the uniqueness theorem.

Theorem 2.5. (The Uniqueness Theorem) Let γ and $\tilde{\gamma} : I \longrightarrow \mathbb{R}_1^3$ be regular curves with the lightcone semi-polar coordinates (α, β, θ) and $(\tilde{\alpha}, \tilde{\beta}, \tilde{\theta})$. Suppose the lightlike points are isolated. If

$$\alpha(t)\beta(t) = \tilde{\alpha}(t)\tilde{\beta}(t), \quad \varepsilon_1(t) = \tilde{\varepsilon}_1(t), \quad \varepsilon_2 = \tilde{\varepsilon}_2(t)$$

for all $t \in I$, then γ and $\tilde{\gamma}$ are congruent through a Lorentz motion.

3 Evolutes of mixed type curves

In this section we give the definition of mixed type curves. In the following work, we suppose $\varepsilon_2(t) \neq 0$. Firstly, we define an evolute of a mixed type curve with $\varepsilon_1(t) \neq 0$.

Definition 3.1. Let $\gamma : I \longrightarrow \mathbb{R}_1^3$ be a regular curve ($\varepsilon_1(t) \neq 0$) with the lightcone semi-polar coordinates (α, β, θ) , then we define an evolute $Ev(\gamma) : I \rightarrow \mathbb{R}_1^3$ of γ by

$$\begin{aligned} Ev(\gamma)(t) = & \gamma(t) \\ & + 4\left(\frac{2\alpha\beta}{\varepsilon_1}(\alpha\dot{\beta} - \beta\dot{\alpha}) + \dot{\theta}|\alpha\beta|^{\frac{1}{2}}(\beta - \alpha)\left(\frac{\alpha\beta\varepsilon_1}{\varepsilon_1\varepsilon_2} - 3\frac{\alpha\dot{\beta} + \beta\dot{\alpha}}{\varepsilon_2}\right)\right)(t)(\alpha(t)L_{\theta(t)}^+ - \beta(t)L_{\theta(t)}^-) \\ & + 8\left(\frac{2\alpha\beta}{\varepsilon_1}\dot{\theta}\alpha\beta(\alpha - \beta) + |\alpha\beta|^{\frac{1}{2}}(\alpha\dot{\beta} - \beta\dot{\alpha})\left(\frac{\alpha\beta\varepsilon_1}{\varepsilon_1\varepsilon_2} - 3\frac{\alpha\dot{\beta} + \beta\dot{\alpha}}{\varepsilon_2}\right)\right)(t)M_{\theta(t)}. \end{aligned}$$

Proposition 3.2. *If $\gamma : I \rightarrow \mathbb{R}_1^3$ is a non-lightlike curve ($\varepsilon_1(t) \neq 0$) with the lightcone semi-polar coordinates (α, β, θ) , then*

$$\begin{aligned} Ev(\gamma)(t) &= \gamma(t) \\ &+ 4\left(\frac{2\alpha\beta}{\varepsilon_1}(\alpha\dot{\beta} - \beta\dot{\alpha}) + \dot{\theta}|\alpha\beta|^{\frac{1}{2}}(\beta - \alpha)\left(\frac{\alpha\beta\varepsilon_1}{\varepsilon_1\varepsilon_2} - 3\frac{\alpha\dot{\beta} + \beta\dot{\alpha}}{\varepsilon_2}\right)\right)(t)(\alpha(t)L_{\theta(t)}^+ - \beta(t)L_{\theta(t)}^-) \\ &+ 8\left(\frac{2\alpha\beta}{\varepsilon_1}\dot{\theta}\alpha\beta(\alpha - \beta) + |\alpha\beta|^{\frac{1}{2}}(\alpha\dot{\beta} - \beta\dot{\alpha})\left(\frac{\alpha\beta\varepsilon_1}{\varepsilon_1\varepsilon_2} - 3\frac{\alpha\dot{\beta} + \beta\dot{\alpha}}{\varepsilon_2}\right)\right)(t)M_{\theta(t)} \\ &= \gamma(t) + \delta_1\delta_2\frac{1}{k}\mathbf{N}(t) + \delta_1\delta_2\frac{\dot{k}}{\kappa^2\tau}\mathbf{B}(t). \end{aligned}$$

Remark 3.3. *If $\gamma(t_0)$ is a lightlike point of $\gamma(t)$ ($\varepsilon_1(t) \neq 0$), we have*

$$\alpha(t_0) = 0, \quad \beta(t_0) \neq 0$$

or

$$\alpha(t_0) \neq 0, \quad \beta(t_0) = 0.$$

So

$$Ev(\gamma)(t_0) = \gamma(t_0).$$

In appropriate conditions, we also define an evolute of a mixed type curve with $\varepsilon_1(t_0) = 0$.

Definition 3.4. *The evolute $Ev(\gamma) : I \rightarrow \mathbb{R}_1^3$ of γ is given by*

$$\begin{aligned} Ev(\gamma)(t) &= \gamma(t) \\ &+ 2(2\lambda(\alpha\dot{\beta} - \beta\dot{\alpha}) + \dot{\theta}|\alpha\beta|^{\frac{1}{2}}(\beta - \alpha)\left(\lambda\frac{\varepsilon_1}{\varepsilon_2} - 6\frac{\alpha\dot{\beta} + \beta\dot{\alpha}}{\varepsilon_2}\right))(t)(\alpha(t)L_{\theta(t)}^+ - \beta(t)L_{\theta(t)}^-) \\ &+ 4(2\lambda\dot{\theta}\alpha\beta(\alpha - \beta) + |\alpha\beta|^{\frac{1}{2}}(\alpha\dot{\beta} - \beta\dot{\alpha})\left(\lambda\frac{\varepsilon_1}{\varepsilon_2} - 6\frac{\alpha\dot{\beta} + \beta\dot{\alpha}}{\varepsilon_2}\right))(t)M_{\theta(t)}, \end{aligned}$$

if there exists a unique smooth function $\lambda : I \rightarrow \mathbb{R}$ such that

$$2\alpha(t)\beta(t) = \lambda(t)\varepsilon_1(t).$$

4 Examples

In this section we give some examples.

Example 4.1. Let $\gamma : \mathbb{R} \rightarrow \mathbb{R}_1^3$ be a regular curve defined by

$$\gamma(t) = \left(\frac{2}{3}t^3 + t, \sin t, -\cos t\right).$$

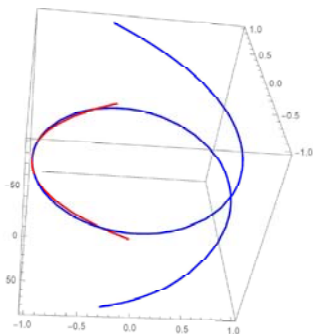
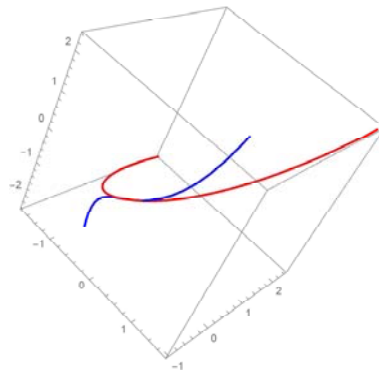
We can calculate

$$2\alpha(t)\beta(t) = 2t^2(1 + t^2), \quad \varepsilon_1(t) = 4t^2(t^2 + 5),$$

so

$$\lambda(t) = \frac{2\alpha(t)\beta(t)}{\varepsilon_1(t)} = \frac{1 + t^2}{2(t^2 + 5)}.$$

The expression of $Ev(\gamma)(t)$ (the evolute of $\gamma(t)$) is too long and complicated, so we do not write it here and we show it in the following figures (see Figure 3 and Figure 4).

Figure 3: $\gamma(t)$ (blue) and $Ev(\gamma)(t)$ (red)Figure 4: $\gamma(t)$ (blue) and $Ev(\gamma)(t)$ (red) around the lightlike point

Example 4.2. Let $\gamma : [0, 4\pi) \rightarrow \mathbb{R}_1^3$ be a regular curve defined by

$$\gamma(t) = (\tfrac{1}{2} \sin 2t, -2(-\cos \tfrac{1}{2}t - \tfrac{1}{3} \cos \tfrac{3}{2}t), -2(\sin \tfrac{1}{2}t - \tfrac{1}{3} \sin \tfrac{3}{2}t)).$$

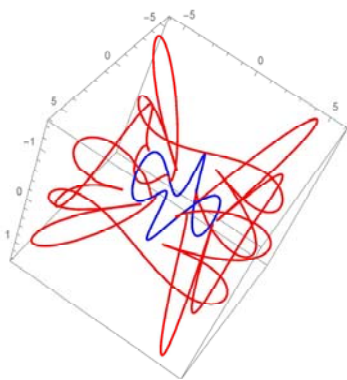
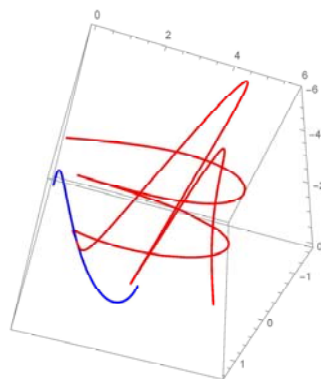
We can calculate

$$2\alpha(t)\beta(t) = 2\cos^2 t(4\cos^2 t - 3), \quad \varepsilon_1(t) = 4\cos^2 t(12\sin^4 t + 17\sin^2 t + 4),$$

so

$$\lambda(t) = \frac{2\alpha(t)\beta(t)}{\varepsilon_1(t)} = \frac{4\cos^2 t - 3}{2(12\sin^4 t + 17\sin^2 t + 4)}.$$

The expression of $Ev(\gamma)(t)$ (the evolute of $\gamma(t)$) is too long and complicated, so we do not write it here and we show it in the following figures (see Figure 5 and Figure 6).

Figure 5: $\gamma(t)$ (blue) and $Ev(\gamma)(t)$ (red)Figure 6: $\gamma(t)$ (blue) and $Ev(\gamma)(t)$ (red) around the lightlike point

References

- [1] A. Honda. Fundamental theorem of spacelike curves in Lorentz-Minkowski space. arXiv:1905.03367 [math.DG].
- [2] D. Fuchs. Evolutes and involutes of spatial curves. Amer. Math. Monthly, 2013, 120(3): 217–231.
- [3] J.W. Bruce, P.J. Giblin. Curves and Singularities: a geometrical introduction to singularity theory. Cambridge university press, Cambridge, 1992.
- [4] R. Lopez. Differential geometry of curves and surfaces in Lorentz-Minkowski space. Int. Electron. J. Geom., 2014, 7(1): 44–107.
- [5] S. Izumiya, A. Takiyama. A time-like surface in Minkowski 3-space which contains pseudo-circles. Proc. Edinburgh Math. Soc. (2), 1997, 40(1): 127–136.
- [6] S. Izumiya, M.C. Romero Fuster, M. Takahashi. Evolutes of curves in the Lorentz-Minkowski plane. Singularities in Generic Geometry. Adv. Stud. Pure Math., 78, Math. Soc. Japan, Tokyo, 2018: 313–330.