

Uniformly Finite-to-one and Onto Extensions  
of Homomorphisms between Directed Graphs

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Introduction

For two directed graphs  $G_1$  and  $G_2$ , a homomorphism  $h$  of  $G_1$  into  $G_2$  is, roughly speaking, a mapping of the set of arcs of  $G_1$  into the set of arcs of  $G_2$  that preserves the adjacency of arcs. The homomorphism  $h$  is naturally extended to a mapping  $h^*$  of the set of all paths in  $G_1$  into the set of all paths in  $G_2$ , which is called the extension of  $h$ .

The main object of this paper is to establish two properties of uniformly finite-to-one and onto extensions of homomorphisms between strongly connected directed graphs. One of them is shown to be also a property of those between directed graphs with no restriction, from which the following result is immediately obtained. For two directed graphs  $G_1$  and  $G_2$ , if there exists a homomorphism  $h$  of  $G_1$  into  $G_2$  such that the extension  $h^*$  of  $h$  is uniformly finite-to-one and onto, then the adjacency matrices  $M(G_1)$  of  $G_1$  and  $M(G_2)$  of  $G_2$  have the same maximal characteristic value but also the characteristic polynomial of  $M(G_1)$  is divided by that of  $M(G_2)$ . The other property is stated as follows. If  $G_1$  and  $G_2$  are two strongly connected directed graphs such that their adjacency matrices have the same maximal characteristic value, then for any homomorphism  $h$  of  $G_1$  into  $G_2$ ,  $h^*$  is uniformly finite-to-one if and only if  $h^*$  is onto.

## 1. Preliminaries

Let  $A$  be a finite nonempty set. A sequence of finite length of elements of  $A$  is called a string over  $A$ . The sequence of length 0 is also a string and is denoted by  $\Lambda$ . For a string  $x$ ,  $lg(x)$  denotes the length of  $x$ . The set of all strings over  $A$  is denoted by  $A^*$ . For a non-negative integer  $n$ ,  $A^n$  is the set of all strings of length  $n$  over  $A$ . For  $x, y \in A^*$ ,  $xy$  denotes the string obtained by concatenating the two strings  $x$  and  $y$ .

A graph (directed graph with labeled arcs and labeled points)  $G$  is defined to be a triple  $\langle P, A, \zeta \rangle$  where  $P$  is a finite set of elements called points,  $A$  is a finite set of elements called arcs and  $\zeta$  is a mapping of  $A$  into  $P \times P$ . If  $\zeta(a) = (u, v)$  for  $a \in A$  and  $u, v \in P$ , then  $u$  is called the initial endpoint of  $a$ ,  $v$  is called the terminal endpoint of  $a$ , and we say that  $a$  goes from  $u$  to  $v$ .

Let  $G = \langle P, A, \zeta \rangle$  be a graph. A string  $x = a_1 \cdots a_p$  ( $p \geq 1$ ) over  $A$  with  $a_i \in A$  ( $i = 1, \dots, p$ ) is called a path of length  $p$  in  $G$  if the terminal endpoint of  $a_i$  is the initial endpoint of  $a_{i+1}$  for  $i = 1, \dots, p-1$ . The initial endpoint  $u$  of  $a_1$  is called the initial endpoint of  $x$ , the terminal endpoint  $v$  of  $a_p$  is called the terminal endpoint of  $x$ , and we say that  $x$  goes from  $u$  to  $v$ . Each point  $u$  of  $G$  is a path of length 0 (going from  $u$  to itself). The set of all paths in  $G$  is denoted by  $\Pi(G)$ . The set of all paths of length  $p$  ( $p \geq 0$ ) in  $G$  is denoted by  $\Pi^{(p)}(G)$ . Note that  $\Pi^{(p)}(G) = A^p \cap \Pi(G)$  for  $p \geq 1$ .

A graph  $G = \langle P, A, \zeta \rangle$  is said to be strongly connected if  $p \neq \emptyset$  and for any  $u, v \in P$ , there exists a path from  $u$  to  $v$  in  $G$ . Of course, a graph consisting of exactly one point and no arc is strongly connected. But, for convenience, in what follows we assume, unless otherwise stated, that a strongly connected graph has at least one arc. However, we remark that all theorems, the proposition, and all lemmas concerning strongly connected graphs in this

paper trivially hold for strongly connected graphs with one point and no arc.

Let  $G_1 = \langle P, A, \zeta_1 \rangle$  and  $G_2 = \langle Q, B, \zeta_2 \rangle$  be two graphs. A homomorphism  $h$  of  $G_1$  into  $G_2$  is a pair  $(h, \phi)$  of a mapping  $h : A \rightarrow B$  and a mapping  $\phi : P \rightarrow Q$  such that for any  $a \in A$ , if  $\zeta_1(a) = (u, v)$  with  $u, v \in P$ , then  $\zeta_2(h(a)) = (\phi(u), \phi(v))$ .

If  $G_1$  is strongly connected, then a homomorphism  $h = (h, \phi)$  of  $G_1$  into  $G_2$  is uniquely determined by  $h$ . Therefore, when  $G_1$  is strongly connected, we say that  $h$  is a homomorphism of  $G_1$  into  $G_2$  and we denote by  $\phi_h$  the unique mapping  $\phi$  such that  $(h, \phi)$  is a homomorphism of  $G_1$  into  $G_2$ .

For a homomorphism  $h = (h, \phi)$  of a graph  $G_1 = \langle P, A, \zeta_1 \rangle$  into a graph  $G_2 = \langle Q, B, \zeta_2 \rangle$  and a subgraph  $G'_1 = \langle P', A', \zeta'_1 \rangle$  of  $G_1$ , we denote the subgraph  $\langle \phi(P'), h(A'), \zeta'_2 \rangle$  of  $G_2$  by  $h(G'_1)$ . (A graph  $G' = \langle P', A', \zeta' \rangle$  is a subgraph of a graph  $G = \langle P, A, \zeta \rangle$  if  $P' \subset P$ ,  $A' \subset A$ , and  $\zeta'(a) = \zeta(a)$  for all  $a' \in A'$ .) It is easy to see that if  $G'_1$  is strongly connected, then  $h(G'_1)$  is strongly connected. When  $G'_1$  is strongly connected, we often denote  $h(G'_1)$  by  $h(G'_1)$ .

Let  $G_1 = \langle P, A, \zeta_1 \rangle$  and  $G_2 = \langle Q, B, \zeta_2 \rangle$  be graphs. Let  $h = (h, \phi)$  be a homomorphism of  $G_1$  into  $G_2$ . We define the extension  $h^* : \Pi(G_1) \rightarrow \Pi(G_2)$  of  $h$  as follows. For each  $x \in \Pi(G_1)$ , if  $lg(x) = 0$ , i.e.,  $x$  is a point of  $G_1$ , then

$$h^*(x) = \phi(x)$$

and if  $x = a_1 \cdots a_p$  ( $p \geq 1$ ) with  $a_i \in A$  ( $i = 1, \dots, p$ ), then

$$h^*(x) = h(a_1) \cdots h(a_p).$$

When  $G_1$  is strongly connected, we often use  $h^*$  instead of  $h$  and say that  $h^*$  is the extension of the homomorphism  $h$ .

A mapping  $f : A \rightarrow B$  is said to be uniformly finite-to-one if there exists a positive number  $N$  such that  $|f^{-1}(y)| \leq N$  for all  $y \in B$ .

Let  $G_1$  and  $G_2$  be two graphs. Let  $h$  be a homomorphism of  $G_1$  into  $G_2$ . Two paths  $x$  and  $y$  in  $G_1$  are said to be indistinguishable by  $h$  if  $x$  and  $y$  have the same initial endpoint and the same terminal endpoint and  $h^*(x) = h^*(y)$ .

Proposition 1. Let  $G_1 = \langle P, A, \zeta_1 \rangle$  be a strongly connected graph and let  $G_2 = \langle Q, B, \zeta_2 \rangle$  be a graph. Let  $h : A \rightarrow B$  be a homomorphism of  $G_1$  into  $G_2$ . Then  $h^* : \Pi(G_1) \rightarrow \Pi(G_2)$  is uniformly finite-to-one if and only if no two distinct paths in  $G_1$  are indistinguishable by  $h$ .

Proof. Suppose that  $x_1$  and  $x_2$  are two distinct paths in  $G_1$  such that they have the same initial endpoint, say  $u$ , and the same terminal endpoint, say  $v$ , and  $h^*(x_1) = h^*(x_2)$ . Since  $G_1$  is strongly connected, there exists a path  $z$  going from  $v$  to  $u$ . For any positive integer  $N$ , we have  $|(h^*)^{-1}((h^*(x_1)h^*(z))^N)| \geq 2^N$ . Hence  $h^*$  is not uniformly finite-to-one.

Suppose that  $h^*$  is not uniformly finite-to-one. Then there exists a path  $y \in \Pi(G_2)$  such that  $|(h^*)^{-1}(y)| > |P|^2$ . Since the number of paths  $x$ 's in  $(h^*)^{-1}(y)$  is greater than the number of all possible pairs of the initial endpoint and the terminal endpoint of a path in  $G_1$ , there exist two distinct paths with the same initial endpoint and the same terminal endpoint in  $(h^*)^{-1}(y)$ .  $\square$

For a graph  $G = \langle P, A, \zeta \rangle$  with  $P = \{u_1, \dots, u_n\}$ , the adjacency matrix  $M(G)$  is the square matrix  $(m_{ij})$  of order  $n$  such that  $m_{ij}$  is the number of arcs going from  $u_i$  to  $u_j$  ( $1 \leq i, j \leq n$ ).

A matrix  $M$  with real elements is said to be non-negative if all elements of  $M$  are non-negative. By the Frobenius Theorem (cf. Gantmacher [2] or Nikaido [7]), any non-negative square matrix  $M$  has a non-negative real characteristic value which the moduli of all the other characteristic values of  $M$  do not exceed. We call that maximum real characteristic value the maximal characteristic value of  $M$ . For a graph  $G$ , we denote the maximal characteristic value of  $M(G)$  by  $r(G)$ .

A square matrix  $M$  is said to be irreducible if there is no permutation

matrix  $H$  such that  $H^{-1}MH$  has the form

$$\begin{pmatrix} M_1 & 0 \\ M_2 & M_3 \end{pmatrix} \quad (1.1)$$

where  $M_1$  and  $M_3$  are square matrices and  $0$  is a zero matrix. For a graph  $G$ ,  $G$  is strongly connected if and only if  $M(G)$  is irreducible.

Let  $M$  be an irreducible non-negative square matrix of order  $n$ . Let  $r$  be the maximal characteristic value of  $M$ . By the Perron-Frobenius Theorem (cf. Gantmacher [2] or Nikaido [7]),  $r > 0$  and to the maximal characteristic value  $r$  there corresponds a characteristic vector  $w = (w_1, \dots, w_n)$  with  $w_i > 0$  for  $i = 1, \dots, n$ . Let  $D = (d_{ij})$  be the diagonal matrix of order  $n$  such that  $d_{ii} = w_i$  ( $i = 1, \dots, n$ ). Then the sum of all the coordinates of each column vector of  $DMD^{-1}$  is equal to  $r$ . (Cf. [2]). For any matrix  $K$ , let us denote the sum of all the elements of  $K$  by  $S(K)$ . Then, since for each non-negative integer  $p$ ,  $M^p = D^{-1}(DMD^{-1})^p D$  and  $S((DMD^{-1})^p) = nr^p$ , we have

$$\alpha r^p \leq S(M^p) \leq \beta r^p \quad (p = 0, 1, \dots) \quad (1.2)$$

where  $\alpha = n \min_{1 \leq i, j \leq n} (w_i/w_j)$  and  $\beta = n \max_{1 \leq i, j \leq n} (w_i/w_j)$ .

Lemma 1. let  $G_1$  and  $G_2$  be two strongly connected graphs. Let  $h$  be a homomorphism of  $G_1$  into  $G_2$ . Then the following two statements are valid.

- (1) If  $h^*$  is uniformly finite-to-one, then  $r(G_1) \leq r(G_2)$ .
- (2) If  $h^*$  is onto, then  $r(G_1) \geq r(G_2)$ .

Proof. Assume that  $h^*$  is uniformly finite-to-one. Then there exists a positive number  $N$  such that  $|(h^*)^{-1}(y)| \leq N$  for all  $y \in \Pi(G_2)$ . Thus since for each non-negative integer  $p$ ,  $|\Pi^{(p)}(G_1)| = \sum_{y \in \Pi^{(p)}(G_2)} |(h^*)^{-1}(y)|$ , it follows that

$$|\Pi^{(p)}(G_1)| \leq N |\Pi^{(p)}(G_2)| \quad (p = 0, 1, \dots) \quad (1.3).$$

Since for  $i = 1, 2$ ,

$$|\Pi^{(p)}(G_i)| = S((M(G_i))^p) \quad (p = 0, 1, \dots),$$

using (1.2) and (1.3) we have  $r(G_1) \leq r(G_2)$ .

Assume that  $h^*$  is onto. Then it follows that

$$|\Pi^{(p)}(G_1)| \geq |\Pi^{(p)}(G_2)| \quad (p = 0, 1, \dots).$$

Hence by the same argument as above, we have  $r(G_1) \geq r(G_2)$ .  $\square$

## 2. Uniformly finite-to-one and onto extensions.

Let  $G = \langle P, A, \zeta \rangle$  be a graph with  $A = \{a_1, \dots, a_\ell\}$ . Let  $Z$  be the ring of integers. We consider the polynomial ring  $Z[a_1, \dots, a_\ell]$  in indeterminates  $a_1, \dots, a_\ell$  over  $Z$ . Let  $P = \{u_1, \dots, u_n\}$ . Let  $\hat{M} = (\hat{m}_{ij})$  be the matrix of order  $n$  with elements in  $Z[a_1, \dots, a_\ell]$  such that  $\hat{m}_{ij} = a_{p_1} + \dots + a_{p_k}$  if  $a_{p_1}, \dots, a_{p_k}$  are all arcs from  $u_i$  to  $u_j$  in  $G$ , and  $\hat{m}_{ij} = 0$  if there exists no arc from  $u_i$  to  $u_j$  ( $1 \leq i, j \leq n$ ). Then the matrix  $\hat{M}$  is called the representation matrix of  $G$  and is denoted by  $\hat{M}(G)$ . Let  $X$  be an indeterminate not contained in  $A$ . Let  $\hat{f}_G$  be the polynomial in  $Z[a_1, \dots, a_\ell, X]$  which is equal to the characteristic polynomial of  $\hat{M}(G)$ , i.e., let  $\hat{f}_G$  be the polynomial defined by

$$\hat{f}_G(a_1, \dots, a_\ell, X) = \det (XI_n - \hat{M}(G))$$

where  $I_n$  is the identity matrix of order  $n$ . Then  $\hat{f}_G(a_1, \dots, a_\ell, X)$  is homogeneous of degree  $n$ . Let  $f_G(X)$  be the characteristic polynomial of the adjacency matrix  $M(G)$  of  $G$ . Then clearly

$$f_G(X) = \hat{f}_G(1, \dots, 1, X).$$

In this section we shall prove the following theorem.

**Theorem 1.** Let  $G_1 = \langle P, A, \zeta_1 \rangle$  and  $G_2 = \langle Q, B, \zeta_2 \rangle$  be two strongly connected graphs with  $A = \{a_1, \dots, a_\ell\}$  and  $B = \{b_1, \dots, b_m\}$ . Let  $h : A \rightarrow B$  be a homomorphism of  $G_1$  into  $G_2$ . Let  $g$  be the polynomial in  $Z[b_1, \dots, b_m, X]$  obtained from  $\hat{f}_{G_1}(a_1, \dots, a_\ell, X)$  by substituting  $h(a_i)$  for  $a_i$  in  $\hat{f}_{G_1}$  for  $i = 1, \dots, \ell$ . Then, if  $h^*$  is uniformly finite-to-one and onto, then  $r(G_1) = r(G_2)$  and  $\hat{f}_{G_2}(b_1, \dots, b_m, X)$  divides  $g(b_1, \dots, b_m, X)$  in  $Z[b_1, \dots, b_m, X]$ .

Theorem 1 can be generalized to graphs with no restriction. Using Theorem 1, we can prove the following theorem. But the proof is omitted in this paper. (It is found in [5].)

**Theorem 1'.** Let  $G_1 = \langle P, A, \zeta_1 \rangle$  and  $G_2 = \langle Q, B, \zeta_2 \rangle$  be two graphs with  $A = \{a_1, \dots, a_\ell\}$  and  $B = \{b_1, \dots, b_m\}$ . Let  $h = (h, \phi)$  be a homomorphism of  $G_1$  into  $G_2$ . Let  $g$  be the polynomial in  $Z[b_1, \dots, b_m, X]$  obtained from  $\hat{f}_{G_1}(a_1, \dots, a_\ell, X)$  by substituting  $h(a_i)$  for  $a_i$  in  $\hat{f}_{G_1}$  for  $i = 1, \dots, \ell$ . Then if  $h^*$  is uniformly finite-to-one and onto, then  $r(G_1) = r(G_2)$  and  $\hat{f}_{G_2}(b_1, \dots, b_m, X)$  divides  $g(b_1, \dots, b_m, X)$  in  $Z[b_1, \dots, b_m, X]$ .

As a direct consequence of Theorem 1', we have the following result.

**Corollary 1.** Let  $G_1$  and  $G_2$  be two graphs. If there exists a homomorphism of  $G_1$  into  $G_2$  such that the extension  $h^*$  of  $h$  is uniformly finite-to-one and onto, then not only  $r(G_1) = r(G_2)$  but also  $f_{G_1}(X)$  is divided by  $f_{G_2}(X)$ .

To prove Theorem 1, we use Lemma 1 and furthermore four lemmas.

**Lemma 2.** Let  $G_1 = \langle P, A, \zeta_1 \rangle$  and  $G_2 = \langle Q, B, \zeta_2 \rangle$  be two strongly connected graphs with  $A = \{a_1, \dots, a_\ell\}$  and  $B = \{b_1, \dots, b_m\}$ . Let  $h : A \rightarrow B$  be a homomorphism of  $G_1$  into  $G_2$ . Write  $\hat{f}_{G_1} = \hat{f}_{G_1}(a_1, \dots, a_\ell, X)$  and  $\hat{f}_{G_2} = \hat{f}_{G_2}(b_1, \dots, b_m, X)$ . Let  $g$  be the polynomial in  $Z[b_1, \dots, b_m, X]$  defined by

$$g(b_1, \dots, b_m, X) = \hat{f}_{G_1}(h(a_1), \dots, h(a_\ell), X)$$

Then if  $h^*$  is uniformly finite-to-one and onto, then for any  $m$  positive integers  $p_1, \dots, p_m$ , there exists a real number  $r$  such that

$$g(p_1, \dots, p_m, r) = \hat{f}_{G_2}(p_1, \dots, p_m, r) = 0.$$

Proof. Assume that  $h^* : \Pi(G_1) \rightarrow \Pi(G_2)$  is uniformly finite-to-one and onto. Let  $p_1, \dots, p_m$  be any  $m$  positive integers. We construct two graphs  $G'_1$  and  $G'_2$  as follows. For each  $i = 1, \dots, \ell$ , let  $j(i)$  be the number such that  $h(a_i) = b_{j(i)}$  and let  $A_i = \{a_{i,1}, a_{i,2}, \dots, a_{i,p_{j(i)}}\}$  where  $a_{i,v}$ ,  $v = 1, \dots, p_{j(i)}$ , are new distinct elements for every  $i$ . The graph  $G'_1$  is obtained from  $G_1$  by replacing each arc  $a_i$  with the arcs consisting of the elements of  $A_i$ . That is,  $G'_1 = \langle P, A', \zeta'_1 \rangle$  where  $A' = \bigcup_{i=1}^{\ell} A_i$  and  $\zeta'_1 : A' \rightarrow P \times P$  is defined by

$$\zeta'_1(a_{i,v}) = \zeta_1(a_i) \quad (v = 1, \dots, p_{j(i)}, i = 1, \dots, \ell).$$

For each  $j = 1, \dots, m$ , let  $B_j = \{b_{j,1}, \dots, b_{j,p_j}\}$  where  $b_{j,v}$ ,  $v = 1, \dots, p_j$ , are new distinct elements for every  $j$ . The graph  $G'_2$  is obtained from  $G_2$  by replacing each arc  $b_j$  with the arcs  $b_{j,1}, \dots, b_{j,p_j}$ . That is,  $G'_2 = \langle Q, B', \zeta'_2 \rangle$  where  $B' = \bigcup_{j=1}^m B_j$  and  $\zeta'_2 : B' \rightarrow Q \times Q$  is defined by

$$\zeta'_2(b_{j,v}) = \zeta_2(b_j) \quad (v = 1, \dots, p_j, j = 1, \dots, m).$$

Let  $h' : A' \rightarrow B'$  be the mapping defined by

$$h'(a_{i,v}) = b_{j(i),v} \quad (v = 1, \dots, p_{j(i)}, i = 1, \dots, \ell).$$

Then clearly,  $G'_1$  and  $G'_2$  are strongly connected and  $h'$  is a homomorphism of  $G'_1$  into  $G'_2$ . Furthermore, for any path  $b_{j_1,v_1} \dots b_{j_s,v_s}$  in  $G'_2$ ,  $b_{j_1} \dots b_{j_s}$  is a path of  $G_2$  and it follows that

$$|((h')^*)^{-1}(b_{j_1,v_1} \dots b_{j_s,v_s})| = |(h^*)^{-1}(b_{j_1} \dots b_{j_s})|$$

Therefore, since  $h^*$  is uniformly finite-to-one and onto, so is  $(h')^*$ . Hence, by Lemma 1,  $r(G'_1) = r(G'_2)$ . Let  $r = r(G'_1)$ . Then the characteristic polynomials  $f_{G'_1}(X)$  of  $M_{G'_1}$  and  $f_{G'_2}(X)$  of  $M_{G'_2}$  have  $r$  as their common root.

Write  $A' = \{a'_1, \dots, a'_\ell\}$  and  $B' = \{b'_1, \dots, b'_m\}$ . Write  $\hat{f}_{G'_1} = \hat{f}_{G'_1}(a'_1, \dots, a'_\ell, X)$  and  $\hat{f}_{G'_2} = \hat{f}_{G'_2}(b'_1, \dots, b'_m, X)$ . Then since for each arc  $b_j$  in  $G_2$ , if  $\zeta_2(b_j) = (u_j, v_j)$  ( $u_j, v_j \in Q$ ), then there exist  $p_j$  arcs going from  $u_j$  to  $v_j$  in  $G'_2$ , we have

$$\hat{f}_{G'_2}(p_1, \dots, p_m, X) = \hat{f}_{G'_2}(1, \dots, 1, X).$$

Also, since for each arc  $a_i$  in  $G_1$ , if  $h(a_i) = b_j$  and  $\zeta_1(a_i) = (s_i, t_i)$  ( $s_i, t_i \in P$ ), then there exist  $p_j$  arcs going from  $s_i$  to  $t_i$  in  $G'_1$ , it follows that

$$g(p_1, \dots, p_m, X) = \hat{f}_{G'_1}(1, \dots, 1, X).$$

Since  $\hat{f}_{G'_1}(1, \dots, 1, X) = f_{G'_1}(X)$ ,  $\hat{f}_{G'_2}(1, \dots, 1, X) = f_{G'_2}(X)$  and  $f_{G'_1}(r) = f_{G'_2}(r) = 0$ , we have

$$g(p_1, \dots, p_m, r) = \hat{f}_{G'_2}(p_1, \dots, p_m, r) = 0. \quad \square$$

An arc  $a$  of a graph  $G$  is called a loop of  $G$  if the initial and terminal endpoints of  $a$  are the same.

Lemma 3. Let  $G_1 = \langle P, A, \zeta_1 \rangle$  and  $G_2 = \langle Q, B, \zeta_2 \rangle$  be two strongly connected graphs. Let  $h : A \rightarrow B$  be a homomorphism of  $G_1$  into  $G_2$ . Let  $v \in Q$ . Let  $G'_2$  be a graph obtained from  $G_2$  by adding a loop  $b_v (\in B)$  from  $v$  to itself. Let  $G'_1$  be a graph obtained from  $G_1$  by adding a loop  $a_u (\in A)$  from  $u$  to itself for every point  $u$  in  $\phi_h^{-1}(v)$ . ( $\phi_h$  was defined in the preceding section.) Let  $h'$  be the mapping of  $A \cup \{a_u \mid u \in \phi_h^{-1}(v)\}$  into  $B \cup \{b_v\}$  defined by

$$h'(a) = \begin{cases} h(a) & \text{if } a \in A \\ b_v & \text{if } a = a_u \text{ with } u \in \phi_h^{-1}(v). \end{cases}$$

Then  $h'$  is a homomorphism of  $G'_1$  into  $G'_2$  and if  $h^*$  is uniformly finite-to-one and onto, then  $(h')^*$  is also uniformly finite-to-one and onto.

Proof. Clearly  $h'$  is a homomorphism of  $G'_1$  into  $G'_2$ . Any path  $z$  in  $\Pi(G'_2)$  is written as  $z = x_1 y_1 x_2 y_2 \cdots x_\ell y_\ell$  where  $x_i \in \Pi(G_2)$  ( $i = 1, \dots, \ell$ ),  $x_1 x_2 \cdots x_\ell \in \Pi(G_2)$ , and  $y_i \in \{b_v\}^*$  for  $i = 1, \dots, \ell$ . It is easily seen that  $|((h')^*)^{-1}(z)| = |(h^*)^{-1}(x_1 \cdots x_\ell)|$ . Hence if  $h^*$  is uniformly finite-to-one and onto, then  $(h')^*$  is also uniformly finite-to-one and onto.  $\square$

Lemma 4. Let  $G = \langle P, A, \zeta \rangle$  with  $A = \{a_1, \dots, a_m\}$  be a graph such that for every  $u \in P$ , there exists at least one loop going from  $u$  to itself. Then

(1)  $\det \hat{M}(G)$  is an irreducible polynomial in  $\mathbb{Z}[a_1, \dots, a_m]$  if (and only if)  $G$  is strongly connected, and

(2)  $\hat{f}_G(a_1, \dots, a_m, X)$  is an irreducible polynomial in  $\mathbb{Z}[a_1, \dots, a_m, X]$  if (and only if)  $G$  is strongly connected.

Proof. We first note that for any homogeneous polynomial  $f(X_1, \dots, X_\ell)$  in indeterminates  $X_1, \dots, X_\ell$  over an integral domain  $k$  such that  $f(0, X_2, \dots, X_\ell) \neq 0$ , if  $f(0, X_2, \dots, X_\ell)$  is irreducible, then  $f(X_1, \dots, X_\ell)$  is irreducible. (Since  $f(X_1, \dots, X_\ell)$  is homogeneous and  $f(0, X_2, \dots, X_\ell) \neq 0$ ,  $\deg f(0, X_2, \dots, X_\ell) = \deg f(X_1, \dots, X_\ell)$ . Assume that  $f(0, X_2, \dots, X_\ell)$  is irreducible but  $f(X_1, \dots, X_\ell)$  is reducible. Then there exist polynomials  $g_1$  and  $g_2$  such that  $f(X_1, \dots, X_\ell) = g_1(X_1, \dots, X_\ell)g_2(X_1, \dots, X_\ell)$  and  $1 \leq \deg g_1 < \deg f$ . Hence  $f(0, X_2, \dots, X_\ell) = g_1(0, X_2, \dots, X_\ell)g_2(0, X_2, \dots, X_\ell)$ . Since  $f(0, X_2, \dots, X_\ell) \neq 0$ ,  $g_1(0, X_2, \dots, X_\ell) \neq 0$ . Since  $f(X_1, \dots, X_\ell)$  is homogeneous,  $g_1(X_1, \dots, X_\ell)$  is homogeneous. (Cf. van der Waerden [9], § 23.) Hence  $\deg g_1(0, X_2, \dots, X_\ell) = \deg g_1(X_1, \dots, X_\ell)$ . Thus  $1 \leq \deg g_1(0, X_2, \dots, X_\ell) < \deg f(0, X_2, \dots, X_\ell)$ . Therefore,  $f(0, X_2, \dots, X_\ell)$  is reducible, which contradicts the assumption.) Therefore, since  $\hat{f}_G(a_1, \dots, a_m, 0) = \det(-\hat{M}(G))$ , the if part of (2) follows from that of (1).

Moreover, to prove the if part of (1), it suffices to show the following :

(3)  $\det \hat{M}(G)$  is an irreducible polynomial if  $G = \langle P, A, \zeta \rangle$  is a graph such that (i)  $G$  is strongly connected, (ii) for any  $u \in P$ , there exists at least one loop from  $u$  to itself in  $G$ , and (iii) any graph obtained from  $G$  by deleting an arc of  $G$  does not satisfy both (i) and (ii).

Let  $G = \langle P, A, \zeta \rangle$  be any graph. A path  $z = a_1 \cdots a_\ell$  with  $a_i \in A$  ( $i = 1, \dots, \ell$ ) is called a circuit of length  $\ell$  if  $a_i \neq a_j$  for any  $i, j, 1 \leq i < j \leq \ell$ , and the initial and terminal endpoints of  $z$  are the same. A circuit  $z = a_1 \cdots a_\ell$  is said to be elementary if  $a_i$  and  $a_j$  have distinct initial endpoints for any  $i, j, 1 \leq i < j \leq \ell$ . (Each loop is an elementary circuit of length 1.) A set  $E$  of elementary circuits in  $G$  is called a circuit-cover of  $G$  if each point of  $G$  is on exactly one circuit in  $E$ .

Let  $E_G$  be the set of all circuit covers of  $G$ . Write  $\hat{M}(G) = (a_{ij})$ . Then  $\det \hat{M}(G)$  is written as

$$\det \hat{M}(G) = \sum_{\sigma} \varepsilon(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

where  $n = |P|$ ,  $\sigma$  is a permutation on  $\{1, \dots, n\}$ , and  $\varepsilon(\sigma) = 1$  or  $-1$  if permutation  $\sigma$  is even or odd, respectively. For every permutation  $\sigma$  such that  $a_{1\sigma(1)} \cdots a_{n\sigma(n)} \neq 0$ , the arcs (indeterminates)  $a_{i\sigma(i)}$ ,  $i = 1, \dots, n$ , constitute all elementary circuits in a circuit-cover of  $G$ . Conversely, for any circuit-cover  $E$  of  $G$ , the product of all arcs (indeterminates) that are on the circuits in  $E$  is equal to a term in  $\det \hat{M}(G)$  up to the sign. Precisely, we can write

$$\det \hat{M}(G) = \sum_{E \in E(G)} t_E$$

where for any circuit cover  $E \in E(G)$ ,  $t_E$  is the monomial  $\prod_{z \in E} t_z$  where for any elementary circuit  $z = a_1 \cdots a_\ell$  ( $a_i \in A$ ) in  $G$ ,  $t_z$  is the monomial defined by  $t_z = (-1)^{\ell+1} a_1 \cdots a_\ell$ . Therefore, we shall prove the following by induction on the number  $n_G$  of the points of  $G$ .

(4)  $p_G = \sum_{E \in \tilde{E}(G)} t_E$  is an irreducible polynomial if  $G$  is a graph satisfying the conditions (i), (ii), and (iii).

If  $n_G = 1$ , then clearly (4) holds. Let  $n \geq 2$  and assume that (4) holds when  $n_G \leq n-1$ . Let  $G = \langle P, A, \zeta \rangle$  be a graph with  $n_G = n$  satisfying (i), (ii), and (iii). Since  $n_G > 1$  and  $G$  is strongly connected, there exists an elementary circuit  $z = a_1 \cdots a_\ell$  of length  $\ell > 1$  in  $G$  where  $a_i \in A$  ( $i = 1, \dots, \ell$ ). Let  $u_1, \dots, u_\ell$  be the points on the circuit  $z$ . Since  $G$  satisfies the conditions (iii) and (i), if we delete all the arcs  $a_1, \dots, a_\ell$  from  $G$ , we obtain pairwise disconnected  $\ell$  strongly connected subgraphs  $G_i$ ,  $i = 1, \dots, \ell$ , of  $G$  such that  $u_i$  is a point in  $G_i$  and each point of  $G$  is a point of exactly one of  $G_i$ 's. Clearly each  $G_i$  is a graph satisfying (i), (ii), and (iii). For  $i = 1, \dots, \ell$ , let  $H_i$  be the subgraph of  $G_i$  obtained by deleting the point  $u_i$  from  $G_i$ . Then we can write

$$p_G = (-1)^{\ell+1} a_1 \cdots a_\ell p_{H_1} \cdots p_{H_\ell} + p_{G_1} \cdots p_{G_\ell}$$

where if  $G_i$  is a graph consisting of the point  $u_i$  and a loop from  $u_i$  to itself, then we set  $p_{H_i} = 1$ . Clearly  $p_G$  is a polynomial of degree 1 with respect to  $a_1$ . By the induction hypothesis,  $p_{G_i}$  is an irreducible polynomial for  $i = 1, \dots, \ell$ . For  $i = 1, \dots, \ell$ , any indeterminate in  $p_{H_i}$  appears only in  $p_{G_i}$ . Hence any factor of the  $a_2 \cdots a_\ell p_{H_1} p_{H_2} \cdots p_{H_\ell}$  can not divide  $p_{G_1} \cdots p_{G_\ell}$ . Thus we conclude that  $p_G$  is an irreducible polynomial.

Thus we have proved the if part of the theorem. (It is easy to see that the only if part of the theorem holds.)  $\square$

Lemma 5.<sup>†</sup> Let  $f$  and  $g$  be two polynomials in  $Z[a_1, \dots, a_m, X]$ . If  $f$  is irreducible in  $Z[a_1, \dots, a_m, X]$  and if for any  $m$  positive integers  $p_1, \dots,$

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<sup>†</sup> See Acknowledgement.

$p_m$ , there exists a real number  $r$  such that

$$f(p_1, \dots, p_m, r) = g(p_1, \dots, p_m, r) = 0,$$

then  $f$  divides  $g$  in  $Z[a_1, \dots, a_m, X]$ .

Proof. Assume that  $f$  is irreducible in  $Z[a_1, \dots, a_m, X]$  and assume that for any  $m$  positive integers  $p_1, \dots, p_m$ , there exists a real number  $r$  such that  $f(p_1, \dots, p_m, r) = g(p_1, \dots, p_m, r) = 0$ . Suppose that  $g$  is not divided by  $f$  in  $Z[a_1, \dots, a_m, X]$ . Let  $K$  be the quotient field of  $Z[a_1, \dots, a_m]$ . We can consider  $f(X) = f(a_1, \dots, a_m, X)$  and  $g(X) = g(a_1, \dots, a_m, X)$  as polynomials in  $K[X]$ . Since  $f(X)$  is irreducible and  $g(X)$  is not divided by  $f(X)$  in  $Z[a_1, \dots, a_m][X]$ ,  $f(X)$  is irreducible and  $g(X)$  is not divided by  $f(X)$  in  $K[X]$ . (Cf. van der Waerden [9], §23.) Therefore, there exist  $s(X)$  and  $t(X)$  in  $K[X]$  such that

$$f(X)s(X) + g(X)t(X) = 1 \quad (3.1).$$

Since the coefficients of  $s(X)$  and  $t(X)$  are rational functions in indeterminates  $a_1, \dots, a_m$  over  $Z$ , there exists a nonzero polynomial  $u$  in  $Z[a_1, \dots, a_m]$  such that both  $us$  and  $ut$  are in  $Z[a_1, \dots, a_m, X]$ . Let  $\hat{s} = us$  and  $\hat{t} = ut$ . Then by equation (3.1) we have

$$f(a_1, \dots, a_m, X)\hat{s}(a_1, \dots, a_m, X) + g(a_1, \dots, a_m, X)\hat{t}(a_1, \dots, a_m, X) = u(a_1, \dots, a_m). \quad (3.2)$$

Since  $u$  is a nonzero polynomial, there exist  $m$  positive integers  $p_1, \dots, p_m$  such that  $u(p_1, \dots, p_m) \neq 0$ . (Cf. van der Waerden [9], §21.) By hypothesis, there exists a real number  $r$  such that  $f(p_1, \dots, p_m, r) = g(p_1, \dots, p_m, r) = 0$ . Substituting  $(p_1, \dots, p_m, r)$  for  $(a_1, \dots, a_m, X)$  in the polynomials in equation (3.2), we are lead to a contradiction. Thus the lemma is proved.  $\square$

Now we are ready to prove Theorem 1.

Proof of Theorem 1. Assume that  $h^*$  is uniformly finite-to-one and onto.

By Lemma 1, we have  $r(G_1) = r(G_2)$ .

It follows from Lemma 3 that by adding new loops  $a'_1, \dots, a'_p$  to  $G_1$  and new loops  $b'_1, \dots, b'_q$  to  $G_2$  if necessary, we can obtain two strongly connected graphs  $G'_1$  and  $G'_2$  and a homomorphism  $h'$  of  $G'_1$  into  $G'_2$  satisfying the following conditions, from  $G_1, G_2$ , and  $h$ .

- (1) For each point  $u$  of  $G'_2$ , there exists at least one loop from  $u$  to  $u$ .
- (2)  $h'(a) = h(a)$  for all  $a \in A$  and  $h'(\{a'_1, \dots, a'_p\}) = \{b'_1, \dots, b'_q\}$ .
- (3)  $(h')^*$  is uniformly finite-to-one and onto.

Let  $\hat{f}_{G'_1}(a_1, \dots, a_\ell, a'_1, \dots, a'_p, X)$  and  $\hat{f}_{G'_2}(b_1, \dots, b_m, b'_1, \dots, b'_q, X)$  be the polynomials which are equal to the characteristic polynomials of  $\hat{M}(G'_1)$  and  $\hat{M}(G'_2)$ , respectively. Let  $g'$  be the polynomial in  $Z[b_1, \dots, b_m, b'_1, \dots, b'_q]$  defined by

$$\begin{aligned} & g'(b_1, \dots, b_m, b'_1, \dots, b'_q, X) \\ &= \hat{f}_{G'_1}(h'(a_1), \dots, h'(a_\ell), h'(a'_1), \dots, h'(a'_p), X). \end{aligned}$$

Then, since  $G'_2$  is a strongly connected graph and condition (1) is satisfied, it follows from Lemma 4 that  $\hat{f}_{G'_2}$  is irreducible in  $Z[b_1, \dots, b_m, b'_1, \dots, b'_q, X]$ . Since  $(h')^*$  is uniformly finite-to-one and onto, it follows from Lemma 2 that for any  $m + q$  positive integers  $p_1, \dots, p_m, p'_1, \dots, p'_q$ , there exists a real number  $r$  such that

$$\begin{aligned} & g'(p_1, \dots, p_m, p'_1, \dots, p'_q, r) \\ &= \hat{f}_{G'_2}(p_1, \dots, p_m, p'_1, \dots, p'_q, r) = 0. \end{aligned}$$

Hence it follows from Lemma 5 that  $\hat{f}_{G'_2}$  divides  $g'$  in  $Z[b_1, \dots, b_m, b'_1, \dots, b'_q, X]$ . Obviously,

$$\hat{f}_{G'_2}(b_1, \dots, b_m, X) = \hat{f}_{G'_2}(b_1, \dots, b_m, 0, \dots, 0, X)$$

and by condition (2),

$$g(b_1, \dots, b_m, X) = g'(b_1, \dots, b_m, 0, \dots, 0, X).$$

Thus we conclude that  $\hat{f}_{G_2}$  divides  $g$  in  $Z[b_1, \dots, b_m, X]$ .  $\square$

**Example 1.** Let  $G = \langle P, A, \zeta \rangle$  be a graph. For any non-negative integer  $p$ , we define a graph  $L^{(p)}(G)$  as follows.  $L^{(0)}(G) = G$ . For  $p > 1$ ,  $L^{(p)}(G) = \langle \Pi^{(p)}(G), \Pi^{(p+1)}(G), \zeta^{(p)} \rangle$  where  $\zeta^{(p)}(a_1 \dots a_{p+1}) = (a_1 \dots a_p, a_2 \dots a_{p+1})$  for  $a_1 \dots a_{p+1} \in \Pi^{(p+1)}(G)$  with  $a_i \in A$  ( $i = 1, \dots, p+1$ ). (Recall that  $\Pi^{(p)}(G) = A^p \cap \Pi(G)$  for  $p \geq 1$ .) We call  $L^{(p)}(G)$  the path graph of length  $p$  of  $G$ .

Especially,  $L^{(1)}(G)$  is called the line graph of  $G$  and is denoted by  $L(G)$ .

(This is the same as the line digraph of  $G$  in Hemminger and Beinke [4] and the adjoint of  $G$  in Berge [1].) Clearly, if  $G$  is strongly connected, then  $L^{(p)}(G)$  is strongly connected for all  $p \geq 0$ . Let  $p$  be any non-negative integer. We define mappings  $h : \Pi^{(p+1)}(G) \rightarrow A$  and  $\phi : \Pi^{(p)}(G) \rightarrow P$  as follows. For any  $a_1 \dots a_{p+1} \in \Pi^{(p+1)}(G)$  with  $a_i \in A$  ( $i = 1, \dots, p+1$ ),  $h(a_1 \dots a_{p+1}) = a_{p+1}$  and for any  $x \in \Pi^{(p)}(G)$ ,  $\phi(x)$  is the terminal endpoint of  $x$ . Then clearly  $h = (h, \phi)$  is a homomorphism of  $L^{(p)}(G)$  into  $G$  and  $h^*$  is uniformly finite-to-one. Clearly if  $G$  is strongly connected, then  $h^*$  is uniformly finite-to-one and onto. Let  $g$  be the polynomial obtained from  $\hat{f}_{L^{(p)}(G)}$  by substituting  $h(y)$  for  $y$  for all indeterminates  $y \in \Pi^{(p+1)}(G)$  in  $\hat{f}_{L^{(p)}(G)}$ . Then, whether  $G$  is strongly connected or not, we can show that

$$g = X^m \hat{f}_G, \quad (3.3)$$

and hence we have

$$f_{L^{(p)}(G)}(X) = X^m f_G(X),$$

where  $m = |\Pi^{(p)}(G)| - |P|$  and we assume that  $\hat{f}_\phi = f_\phi = 1$  for the graph  $\phi$  with no point. Note that  $m$  may be negative. Particularly we have

$$r(L^{(p)}(G)) = r(G).$$

Proof of Equation (3.3). It suffices to show the result for  $p = 1$  because  $L^{(p)}(G)$  is isomorphic to  $L^p(G)$  for any  $p \geq 0$ <sup>†</sup> and hence the result for general  $p (\geq 0)$  is straightforwardly proved by induction. Thus we assume that  $p = 1$ .

Assume that  $G$  is strongly connected. Let  $P = \{u_1, \dots, u_n\}$  and let  $A = \{a_1, \dots, a_\ell\}$ . Then  $h : \Pi^{(2)}(G) \rightarrow A$  is a homomorphism of the strongly connected graph  $L(G)$  into the strongly connected graph  $G$ . Since  $h^*$  is uniformly finite-to-one and onto, it follows from Theorem 1 that there exists  $\alpha \in Z[a_1, \dots, a_\ell, X]$  such that

$$g(a_1, \dots, a_\ell, X) = \alpha(a_1, \dots, a_\ell, X) \hat{f}_G(a_1, \dots, a_\ell, X). \quad (3.4)$$

Let  $\tilde{M}$  be the square matrix  $(m_{ij})$  of order  $\ell$  in which  $m_{ij} = a_j$  if the terminal endpoint of  $a_i$  is the initial endpoint of  $a_j$  and  $m_{ij} = 0$  otherwise. We can consider  $g(X) = g(a_1, \dots, a_\ell, X)$  as a polynomial in  $K[X]$  where  $K$  is the quotient field of  $Z[a_1, \dots, a_\ell]$ . Then  $g(X)$  is the characteristic polynomial of  $\tilde{M}$ . By the construction of  $\tilde{M}$ , if  $a_i$  and  $a_j$  have the same terminal endpoint, then the  $i$ th and  $j$ th rows of  $\tilde{M}$  are the same. Therefore, for each point  $u_k$  of  $G$ , if the number of arcs going to  $u_k$  is  $d_k$ , then  $d_k$  rows of  $\tilde{M}$  are the same. Let  $m = |A| - |P|$ . Then it is easily shown that there exist  $m$  linearly independent row vectors  $V$ 's such that  $\tilde{V}\tilde{M} = 0$  where  $0$  is the zero vector. Thus  $0$  is a characteristic value of  $\tilde{M}$  with at least  $m$  linearly independent characteristic vectors corresponding to it. Hence  $0$  is a root of  $g(X)$  of multiplicity at least  $m$ . Thus  $g(X)$  is divided by  $X^m$ .

We assume without loss of generality that (\*) for each point  $u$  of  $G$

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<sup>†</sup> We define  $L^p(G)$  by  $L^0(G) = G$  and  $L^p(G) = L(L^{p-1}(G))$  ( $p > 0$ ).

there exists at least one loop going from  $u$  to itself. (For assume that  $a_1$  is a loop. Let  $G'$  be the graph obtained from  $G$  by deleting the loop  $a_1$ . Let  $g'$  be the polynomial in  $\mathbb{Z}[a_2, \dots, a_\ell, X]$  defined for  $G'$  in the same way as  $g$  for  $G$ . Then it is easily checked that  $g(0, a_2, \dots, a_\ell, X) = Xg'(a_2, \dots, a_\ell, X)$  and  $\hat{f}_G(0, a_2, \dots, a_\ell, X) = \hat{f}_{G'}(a_2, \dots, a_\ell, X)$ . Hence it follows that if (3.3) holds in case  $G$  satisfies  $(*)$ , then it also holds in case  $G$  does not satisfy  $(*)$ .) Thus since  $G$  is strongly connected, it follows from Lemma 4 that  $\hat{f}_G$  is irreducible in  $\mathbb{Z}[a_1, \dots, a_\ell, X]$ . Therefore, since  $g(a_1, \dots, a_\ell, X)$  is divided by  $X^m$ , so is  $\alpha(a_1, \dots, a_\ell, X)$  in (3.4). Since  $\deg \alpha = m$ , we conclude that (3.3) holds.

For the general case where  $G$  is not necessarily strongly connected, the result is straightforwardly proved by induction on the number of the maximal strongly connected subgraphs<sup>†</sup> of  $G$ . Therefore, the remainder of the proof is omitted.  $\square$

It was mentioned in Hemminger and Beinke [4], p.298 that A.J. Hoffman had asked whether one can determine  $f_{L(G)}(X)$  in terms of  $f_G(X)$ . The question has been solved by a result in Example 1.

3. The extension of a homomorphism between two strongly connected graphs  $G_1$  and  $G_2$  with  $r(G_1) = r(G_2)$ .

In this section, we prove that if  $G_1$  and  $G_2$  are strongly connected graphs such that their adjacency matrices have the same maximal characteristic value, then for any homomorphism  $h$  of  $G_1$  into  $G_2$ ,  $h^*$  is onto if and only if  $h^*$  is

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<sup>†</sup> Here, "strongly connected graph" is used in the usual sense, that is, it includes "strongly connected graph with one point and no arc".

uniformly finite-to-one, and hence the surjectivity of  $h^*$  is equivalent to the nonexistence of two distinct paths which are indistinguishable by  $h$ , in  $G_1$ .

Let  $G = \langle P, A, \zeta \rangle$  be a strongly connected graph. Let  $p$  be a non-negative integer. We define a mapping  $\theta_{G,p} : \Pi(L^{(p)}(G)) \rightarrow \bigcup_{i=p}^{\infty} \Pi^{(i)}(G)$  as follows. (Cf. Example 1.) If  $z$  is a path of length 0 in  $L^{(p)}(G)$ , then  $z \in \Pi^{(p)}(G)$ . For this case, we define

$$\theta_{G,p}(z) = z.$$

If  $z$  is a path of length  $\ell (\geq 1)$  in  $L^{(p)}(G)$ , then  $z$  is of the form

$$z = (a_1 \cdots a_{p+1})(a_2 \cdots a_{p+1}) \cdots (a_\ell \cdots a_{\ell+p})$$

with  $a_1, \dots, a_{\ell+p} \in A$  such that  $a_1 \cdots a_{\ell+p} \in \Pi^{(\ell+p)}(G)$ . For this case, we define

$$\theta_{G,p}(z) = a_1 \cdots a_{\ell+p}.$$

Clearly,  $\theta_{G,p}$  is one-to-one and onto.

**Theorem 2.** Let  $G_1 = \langle P, A, \zeta_1 \rangle$  and  $G_2 = \langle Q, B, \zeta_2 \rangle$  be two strongly connected graphs with  $r(G_1) = r(G_2)$ . Then for any homomorphism  $h : A \rightarrow B$  of  $G_1$  into  $G_2$ ,  $h^*$  is uniformly finite-to-one if and only if  $h^*$  is onto.

**Proof.** Let  $h : A \rightarrow B$  be a homomorphism of  $G_1$  into  $G_2$ . Let  $p$  be any non-negative integer. We define a mapping  $h^{(p)} : \Pi^{(p+1)}(G_1) \rightarrow \Pi^{(p+1)}(G_2)$  by

$$h^{(p)}(x) = h^*(x) \quad (x \in \Pi^{(p+1)}(G_1)).$$

Then  $h^{(p)}$  is a homomorphism of the strongly connected graph  $L^{(p)}(G_1)$  into the strongly connected graph  $L^{(p)}(G_2)$ . Moreover, we have, for each  $z \in \Pi(L^{(p)}(G_1))$ ,

$$h^*(\theta_{G_1,p}(z)) = \theta_{G_2,p}((h^{(p)})^*(z)).$$

Therefore, since  $\theta_{G_1,p}$  and  $\theta_{G_2,p}$  are one-to-one and onto,  $h^*$  is uniformly

finite-to-one [onto] if and only if  $(h^{(p)})^*$  is uniformly finite-to-one [onto].

Assume that  $h^*$  is uniformly finite-to-one but not onto. Then there exists  $y \in \Pi(G_2)$  with  $lg(y) \geq 1$  such that  $(h^*)^{-1}(y) = \emptyset$ . Let  $p = lg(y) - 1$ . We consider the homomorphism  $h^{(p)}$  of  $L^{(p)}(G_1)$  into  $L^{(p)}(G_2)$ . Since  $h^*$  is uniformly finite-to-one,  $(h^{(p)})^*$  is uniformly finite-to-one. Moreover,  $y$  is an arc of  $L^{(p)}(G_2)$  such that  $(h^{(p)})^{-1}(y) = \emptyset$ . Let  $H = h^{(p)}(L^{(p)}(G_1))$  (cf. Section 1 for this notation.). Then  $H$  is a strongly connected subgraph of  $L^{(p)}(G_2)$ . We define a mapping  $(h^{(p)})' : \Pi^{(p+1)}(G_1) \rightarrow h^{(p)}(\Pi^{(p+1)}(G_2))$  by

$$(h^{(p)})'(x) = h^{(p)}(x) \quad (x \in \Pi^{(p+1)}(G_1)).$$

Then clearly  $(h^{(p)})'$  is a homomorphism of  $L^{(p)}(G_1)$  into  $H$ . Since  $(h^{(p)})^*$  is uniformly finite-to-one,  $((h^{(p)})')^*$  is uniformly finite-to-one. Thus, by Lemma 1, we have

$$r(L^{(p)}(G_1)) \leq r(H).$$

On the other hand, the following result is known (e.g., Nikaido [13]).

(i) For any two distinct non-negative square matrices  $M_1$  and  $M_2$  of the same order, if  $M_1$  is irreducible and  $M_1 - M_2$  is non-negative, then the maximal characteristic value of  $M_1$  is greater than that of  $M_2$ .

Since  $h^{(p)}(\Pi^{(p+1)}(G_1)) \subset \Pi^{(p+1)}(G_2) - \{y\}$ , it follows from (i) that the maximal characteristic value of  $M(L^{(p)}(G_2))$  is greater than that of  $M(H)$ .

Hence we have

$$r(L^{(p)}(G_2)) > r(H).$$

By Example 1 and hypothesis

$$r(L^{(p)}(G_1)) = r(G_1) = r(G_2) = r(L^{(p)}(G_2)).$$

Therefore, we have

$$r(L^{(p)}(G_1)) > r(H),$$

which is a contradiction. Thus we have proved that if  $h^*$  is uniformly finite-to-one, then  $h^*$  is onto.

To show the converse, we assume that  $h^*$  is onto but not uniformly finite-to-one. We construct two graphs  $G'_1$  and  $G'_2$  from  $G_1 = \langle P, A, \zeta_1 \rangle$  and  $G_2 = \langle Q, B, \zeta_2 \rangle$ , respectively, as follows. Let  $\bar{A} = \{\bar{a} | a \in A\}$  and let  $\bar{B} = \{\bar{b} | b \in B\}$ . The graph  $G'_1$  is obtained from  $G_1$  by adding a new arc  $\bar{a}$  having the same initial and terminal endpoints as  $a$  for every arc  $a$  of  $G_1$ . That is,  $G'_1 = \langle P, A', \zeta'_1 \rangle$  where  $A' = A \cup \bar{A}$  and  $\zeta'_1$  is defined by  $\zeta'_1(a) = \zeta'_1(\bar{a}) = \zeta_1(a)$  ( $a \in A$ ). In the same way,  $G'_2$  is obtained from  $G_2$ . That is,  $G'_2 = \langle Q, B', \zeta'_2 \rangle$  where  $B' = B \cup \bar{B}$  and  $\zeta'_2$  is defined by  $\zeta'_2(b) = \zeta'_2(\bar{b}) = \zeta_2(b)$  ( $b \in B$ ). Clearly,  $G'_1$  and  $G'_2$  are strongly connected and  $M(G'_i) = 2M(G_i)$  for  $i = 1, 2$  so that  $r(G'_i) = 2r(G_i)$  for  $i = 1, 2$ . Since, by hypothesis,  $r(G_1) = r(G_2)$ ,  $G'_1$  and  $G'_2$  are two strongly connected graphs with  $r(G'_1) = r(G'_2)$ . Define a mapping  $h' : A' \rightarrow B'$  as follows. For each  $a \in A$ ,  $h'(a) = b$  and  $h'(\bar{a}) = \bar{b}$  where  $b = h(a)$ . Then, since  $h$  is a homomorphism of  $G_1$  into  $G_2$ ,  $h'$  is a homomorphism of  $G'_1$  into  $G'_2$ . Let  $d_1 \cdots d_\ell$  be any path of length  $\geq 1$  in  $G'_2$  with  $d_i \in B'$  ( $i = 1, \dots, \ell$ ). For  $i = 1, \dots, \ell$ , let  $b_i$  be the element of  $B$  such that  $b_i = d_i$  or  $\bar{b}_i = d_i$ . Then  $b_1 \cdots b_\ell$  is a path in  $G_2$  and

$$|((h')^*)^{-1}(d_1 \cdots d_\ell)| = |(h^*)^{-1}(b_1 \cdots b_\ell)|.$$

Therefore, since  $h^*$  is onto,  $(h')^*$  is onto.

Since  $h^*$  is not uniformly finite-to-one, it follows from Proposition 1 that there exist two distinct paths  $x_1$  and  $x_2$  in  $G_1$  which are indistinguishable by  $h$ . Let  $p = \lg(x_1) - 1$ . For  $i = 1, 2$ , we write  $x_i = a_{i1} \cdots a_{i(p+1)}$  with  $a_{ij} \in A$  ( $j = 1, \dots, p+1$ ). For  $i = 1, 2$ , let  $x'_i = \bar{a}_{i1} a_{i2} \cdots a_{i(p+1)}$ . Then  $x'_1$  and  $x'_2$  are two distinct paths in  $G'_1$  which are indistinguishable by  $h'$ .

Put  $H_1 = L^{(p)}(G'_1)$ ,  $H_2 = L^{(p)}(G'_2)$ , and  $g = (h')^{(p)}$ . Then  $H_1$  and  $H_2$  are strongly connected graphs and  $g$  is a homomorphism of  $H_1$  into  $H_2$ . Moreover,

$x'_1$  and  $x'_2$  are distinct arcs of  $H_1$ . Let  $\tilde{H}_1 = \langle R, E, \zeta \rangle$  be the maximal strongly connected subgraph of  $H_1$  having the arc  $x'_1$  but not having the arc  $x'_2$ . Now we shall prove the following.

(\*)  $\tilde{H}_1$  exists and for any  $z \in \Pi(H_1) - \Pi(\tilde{H}_1)$ , there exists  $\tilde{z} \in \Pi(\tilde{H}_1)$  such that  $g^*(\tilde{z}) = g^*(z)$ .

Let  $z \in \Pi(H_1) - \Pi(\tilde{H}_1)$ . Since  $H_1$  is strongly connected, there exists a circuit  $C$  in  $H_1$  such that the arc  $x'_2$  is on  $C$  and  $z$  is a subpath of  $C$ . (For two paths  $z_1$  and  $z_2$ ,  $z_1$  is a subpath of  $z_2$  if there exists paths  $w_1$  and  $w_2$  such that  $z_2 = w_1 z_1 w_2$ .<sup>†</sup>) Let  $D = \theta_{G'_1, p}(C)$ . Then  $D$  is a path in  $G'_1$  and  $x'_2$  appears in  $D$  at least once as a subpath of  $D$ . Hence we can write  $D = w_1 x'_2 w_2$  with  $w_1, w_2 \in \Pi(G'_1)$ . Let  $D_1 = w_1 x'_1 w_2$ . Then since  $x'_1$  and  $x'_2$  have the same initial endpoint and the same terminal endpoint and  $(h')^*(x'_1) = (h')^*(x'_2)$ ,  $D_1$  is a path in  $G'_1$  and  $(h')^*(D_1) = (h')^*(D)$ . Since  $x'_1 = \bar{a}_{11} a_{12} \cdots a_{1p}$  and  $x'_2 = \bar{a}_{21} a_{22} \cdots a_{2p}$ ,  $x'_1$  and  $x'_2$  do not intersect, that is, there exist no paths  $t_1, t_2$ , and  $s$  of length  $> 0$  in  $G'_1$  such that  $x'_1 = t_1 s$  and  $x'_2 = s t_2$  or  $x'_1 = s t_1$  and  $x'_2 = t_2 s$ . Hence replacing any subpath  $x'_2$  in  $D$  by  $x'_1$  does not generate a new subpath  $x'_2$  in  $D_1$ . Therefore, by replacing every subpath  $x'_2$  in  $D$  by  $x'_1$ , we can obtain a path  $\tilde{D}$  in  $G'_1$  such that  $x'_2$  is not a subpath of  $\tilde{D}$ ,  $x'_1$  is a subpath of  $\tilde{D}$ , and

$$(h')^*(\tilde{D}) = (h')^*(D).$$

$\tilde{D}$  has the form  $\tilde{D} = \tilde{d}_1 \cdots \tilde{d}_\ell \tilde{d}_1 \cdots \tilde{d}_p$  where  $\tilde{d}_i \in A'$  ( $i = 1, \dots, \ell$ ), by additional replacements if necessary. ( $D$  is of the form  $D = d_1 \cdots d_\ell d_1 \cdots d_p$  where  $d_i \in A'$  ( $i = 1, \dots, \ell$ )). If a part of one of the initial and terminal subpaths  $d_1 \cdots d_p$  of  $D$  is replaced by a subpath of  $x'_1$  in the above replacements, the corresponding part of the other subpath  $d_1 \cdots d_p$  of  $D$  must be replaced by the same subpath of  $x'_1$ .) Let  $\tilde{C} = (\theta_{G'_1, p})^{-1}(\tilde{D})$ . Then  $\tilde{C}$  is a circuit in  $H_1$ .

<sup>†</sup> We assume that  $uy = yv = y$  for paths  $u$  and  $v$  of length 0, i.e. points  $u$  and  $v$ , and a path  $y$  going from  $u$  to  $v$ .

Moreover,  $\tilde{C}$  passes through the arc  $x_1'$  but does not pass through the arc  $x_2'$ .

Hence,  $\tilde{H}_1$  exists and  $\tilde{C}$  is a circuit in  $\tilde{H}_1$ . Furthermore, we have

$$g^*(\tilde{C}) = (\theta_{G_2', P})^{-1}((h')^*(\tilde{D})) = (\theta_{G_2', P})^{-1}((h')^*(D)) = g^*(C).$$

Since  $z$  is a subpath of  $C$ , there exists a subpath  $\tilde{z}$  of  $\tilde{C}$  such that  $g^*(z) = g^*(\tilde{z})$ . Of course,  $\tilde{z}$  is a path in  $\tilde{H}_1$ . Thus we have proved (\*).

Let  $\tilde{g} : E \rightarrow \Pi^{(p+1)}(G_2)$  be the restriction of  $g$ . Then  $\tilde{g}$  is a homomorphism of  $\tilde{H}_1$  into  $H_2$ . Since  $(h')^*$  is onto,  $g^* = ((h')^{(p)})^*$  is onto. Therefore, it follows from (\*) that  $(\tilde{g})^*$  is onto. Thus it follows from Lemma 1 that

$$r(\tilde{H}_1) \geq r(H_2).$$

However,  $H_1$  is a strongly connected graph and  $\tilde{H}_1$  is a subgraph of  $H_1$  which has not the arc  $x_2'$  of  $H_1$ . Hence it follows from (i) that the maximal characteristic value of  $M(H_1)$  is greater than that of  $M(\tilde{H}_1)$ . Thus we have

$$r(\tilde{H}_1) < r(H_1).$$

From example 3,  $r(G_1') = r(L^{(p)}(G_1')) = r(H_1)$  for  $i = 1, 2$ . Therefore since  $r(G_1') = r(G_2')$ ,  $r(H_1) = r(H_2)$ . Thus we have

$$r(\tilde{H}_1) < r(H_2).$$

which is a contradiction. Thus we have proved that if  $h^*$  is onto, then  $h^*$  is uniformly finite-to-one. The proof of the theorem is completed.  $\square$

Corollary 2. Let  $G_1 = \langle P, A, \zeta_1 \rangle$  and  $G_2 = \langle Q, B, \zeta_2 \rangle$  be two strongly connected graphs with  $r(G_1) = r(G_2)$ . Let  $h : A \rightarrow B$  be a homomorphism of  $G_1$  into  $G_2$ . Then  $h^*$  is onto if and only if no two distinct paths in  $G_1$  are indistinguishable by  $h$ .

Proof. This follows from Theorem 2 and Proposition 1.  $\square$

We remark that Theorem 2 and Corollary 2 are no longer true if either

$G_1$  or  $G_2$  in them is not strongly connected. This is shown by the following examples.

Define graphs  $G_1$ ,  $G_2$  and  $G_3$  as follows.  $G_1 = \langle \{u_1, u_2\}, \{a_1, a_2, a_3\}, \zeta_1 \rangle$  where  $\zeta_1(a_1) = (u_1, u_1)$ ,  $\zeta_1(a_2) = (u_1, u_2)$ , and  $\zeta_1(a_3) = (u_2, u_2)$ ;  $G_2 = \langle \{v\}, \{b\}, \zeta_2 \rangle$  where  $\zeta_2(b) = (v, v)$ ;  $G_3 = \langle \{w_1, w_2\}, \{c_1, c_2\}, \zeta_3 \rangle$  where  $\zeta_3(c_1) = (w_1, w_1)$  and  $\zeta_3(c_2) = (w_1, w_2)$ . Then  $G_2$  is strongly connected but  $G_1$  and  $G_3$  are not strongly connected. Clearly  $r(G_1) = r(G_2) = r(G_3)$ . Let  $h_1 = (h_1, \phi_1)$  be the homomorphism of  $G_1$  into  $G_2$  defined by  $h_1(a_1) = h_1(a_2) = h_1(a_3) = b$  and  $\phi_1(u_1) = \phi_1(u_2) = v$ . Let  $h_2 = (h_2, \phi_2)$  be the homomorphism of  $G_2$  into  $G_3$  defined by  $h_2(b) = c_1$  and  $\phi_2(v) = w_1$ . Then  $h_1^*$  is onto but not uniformly finite-to-one because, for each  $n \geq 1$ ,  $h_1^*(a_1^i a_2 a_3^{n-i-1}) = b^n$  for any  $i$  with  $0 \leq i \leq n-1$  so that  $|(h_1^*)^{-1}(b^n)| \geq n$ . Note that  $a_1 a_2$  and  $a_2 a_3$  are distinct and indistinguishable by  $h_1$ . Clearly  $h_2^*$  is one-to-one but not onto.

#### Acknowledgements

The author would like to thank Professor Kôji Uchida for the proof of Lemma 5. He is also grateful to Dr. Akira Maruoka for helpfull discussions.

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