

Some subordination criteria for analytic functions

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1 Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

For functions $f(z)$ and $g(z)$ in the class $\mathcal{A}$, we say that $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$ if there exists an analytic function $w(z)$ satisfying $w(0) = 0$, $|w(z)| < 1$ ($z \in \mathbb{U}$) and $f(z) = g(w(z))$ ($z \in \mathbb{U}$). We denote this subordination by $f(z) \prec g(z)$. In particular, if $g(z)$ is univalent in $\mathbb{U}$, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

We need the following lemma given by Miller and Mocanu [2] (see also [3, p. 132]).

Lemma 1.1 *Let the function $q(z)$ be analytic and univalent in $\mathbb{U}$. Also let $\phi(\omega)$ and $\psi(\omega)$ be analytic in a domain $\mathcal{C}$ containing $q(\mathbb{U})$, with*

$$\psi(\omega) \neq 0 \quad (\omega \in q(\mathbb{U}) \subset \mathcal{C}).$$

Set

$$Q(z) = zq'(z)\psi(q(z)) \quad \text{and} \quad h(z) = \phi(q(z)) + Q(z),$$

and suppose that

$$(i) \quad Q(z) \text{ is starlike and univalent in } \mathbb{U};$$

and

$$(ii) \quad \operatorname{Re} \left(\frac{zh'(z)}{Q(z)} \right) = \operatorname{Re} \left(\frac{\phi'(q(z))}{\psi(q(z))} + \frac{zQ'(z)}{Q(z)} \right) > 0 \quad (z \in \mathbb{U}).$$

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If $p(z)$ is analytic in $\mathbb{U}$, with

$$p(0) = q(0) \quad \text{and} \quad p(\mathbb{U}) \subset \mathcal{C},$$

and

$$\phi(p(z)) + zp'(z)\psi(p(z)) \prec \phi(q(z)) + zq'(z)\psi(q(z)) =: h(z) \quad (z \in \mathbb{U}),$$

then

$$p(z) \prec q(z) \quad (z \in \mathbb{U})$$

and $q(z)$ is the best dominant of this subordination.

By making use of Lemma 1.1, Kuroki, Owa and Srivastava [1] deduced each of the following lemmas.

Lemma 1.2 Let the function $f(z) \in \mathcal{A}$ be so chosen that

$$\frac{f(z)}{z} \neq 0 \quad (z \in \mathbb{U}).$$

Suppose also that the real parameters α ($\alpha \neq 0$) and β ($-1 \leq \beta \leq 1$), as well as the complex parameters A and B constrained by

$$|A| \leq 1, |B| < 1, A \neq B, \text{ and } \operatorname{Re}(1 - A\bar{B}) \geq |A - B|,$$

are so prescribed that

$$\frac{\beta(1 - \alpha)}{\alpha} + \frac{(1 + \beta)\{\operatorname{Re}(1 - A\bar{B}) - |A - B|\}}{1 - |B|^2} + \frac{1 - \beta}{1 + |A|} + \frac{1 + \beta}{1 + |B|} - 1 \geq 0.$$

If

$$\left(\frac{zf'(z)}{f(z)}\right)^\beta \left(1 + \alpha \frac{zf''(z)}{f'(z)}\right) \prec h(z) \quad (z \in \mathbb{U}),$$

where

$$h(z) = \left(\frac{1 + Az}{1 + Bz}\right)^{\beta-1} \left\{ (1 - \alpha) \frac{1 + Az}{1 + Bz} + \frac{\alpha(1 + Az)^2 + \alpha(A - B)z}{(1 + Bz)^2} \right\},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}).$$

Lemma 1.3 Let the function $f(z) \in \mathcal{A}$ be so chosen that

$$\frac{f(z)}{z} \neq 0 \quad (z \in \mathbb{U}).$$

Suppose also that the real parameters α ($\alpha \neq 0$) and β ($-1 \leq \beta \leq 1$), as well as the complex parameters A and B constrained by

$$|A| \leq 1, |B| = 1, A \neq B, \text{ and } 1 - A\bar{B} > 0,$$

are so prescribed that

$$\frac{\beta(1-\alpha)}{\alpha} + \frac{(1+\beta)(1-|A|^2)}{2(1-A\bar{B})} + \frac{(1-\beta)(1-|A|)}{2(1+|A|)} \geq 0.$$

If

$$\left(\frac{zf'(z)}{f(z)}\right)^\beta \left(1 + \alpha \frac{zf''(z)}{f'(z)}\right) \prec h(z) \quad (z \in \mathbb{U}),$$

where

$$h(z) = \left(\frac{1+Az}{1+Bz}\right)^{\beta-1} \left\{ (1-\alpha) \frac{1+Az}{1+Bz} + \frac{\alpha(1+Az)^2 + \alpha(A-B)z}{(1+Bz)^2} \right\},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}).$$

2 Main results

In this section, we begin by starting and proving one of our main results.

Theorem 2.1 *Let the function $f(z) \in \mathcal{A}$ be so chosen that*

$$\frac{f(z)}{z} \neq 0 \quad (z \in \mathbb{U}).$$

Suppose also that the real parameters α ($\alpha \neq 0$), β ($-1 \leq \beta \leq 1$), and

$$\delta \left(1 \leq \delta \leq \frac{\pi}{2 \left(\left| \sin^{-1} \frac{\operatorname{Im}(1-A\bar{B})}{|1-A\bar{B}|} \right| + \cos^{-1} \frac{\sqrt{(1-|A|^2)(1-|B|^2)}}{|1-A\bar{B}|} \right)} \right),$$

as well as the complex parameters A and B constrained by

$$|A| \leq 1, |B| < 1, A \neq B, \text{ and } \operatorname{Re}(1-A\bar{B}) \geq |A-B|,$$

are so prescribed that

$$1 - |A||B| + \delta\beta|A| - \delta\beta|B| \geq 0,$$

and

$$\frac{\beta(1-\alpha)}{\alpha} + (1+\beta)m_0 + \frac{1-\delta\beta}{1+|A|} + \frac{1+\delta\beta}{1+|B|} \geq 0,$$

where

$$m_0 = \min_w \left\{ |w|^\delta \cos \left(\delta \left(\left| \sin^{-1} \frac{\operatorname{Im}(1-A\bar{B})}{|1-A\bar{B}|} \right| + \cos^{-1} \frac{1-|A|^2+|w|^2(1-|B|^2)}{2|w||1-A\bar{B}|} \right) \right) \right\} :$$

$$\left(\frac{|1 - A\bar{B} - |A - B||}{1 - |B|^2} \leq |w| \leq \frac{\sqrt{(1 - |A|^2)(1 - |B|^2)}}{1 - |B|^2} \right).$$

If

$$(1) \quad \left(\frac{zf'(z)}{f(z)} \right)^\beta \left(1 + \alpha \frac{zf''(z)}{f'(z)} \right) \prec h(z) \quad (z \in \mathbb{U}),$$

where

$$h(z) = \left(\frac{1 + Az}{1 + Bz} \right)^{\delta\beta-1} \left\{ (1 - \alpha) \frac{1 + Az}{1 + Bz} + \frac{\alpha(1 + Az)^{1+\delta}(1 + Bz)^{1-\delta} + \alpha\delta(A - B)z}{(1 + Bz)^2} \right\},$$

then

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1 + Az}{1 + Bz} \right)^\delta \quad (z \in \mathbb{U}).$$

Proof. If we define the function $p(z)$ by

$$p(z) = \frac{zf'(z)}{f(z)} \quad (z \in \mathbb{U}),$$

then $p(z)$ is analytic in $\mathbb{U}$ and the condition (1) can be written as follows:

$$\{p(z)\}^\beta \{(1 - \alpha) + \alpha p(z)\} + \alpha zp'(z) \{p(z)\}^{\beta-1} \prec h(z) \quad (z \in \mathbb{U}).$$

We also set

$$q(z) = \left(\frac{1 + Az}{1 + Bz} \right)^\delta, \quad \phi(z) = z^\beta(1 - \alpha + \alpha z), \quad \text{and} \quad \psi(z) = \alpha z^{\beta-1}$$

for $z \in \mathbb{U}$. Then, clearly, the function $q(z)$ is analytic and univalent in $\mathbb{U}$.

Now, for $q_1(z) = \frac{1 + Az}{1 + Bz}$, it is clear that $q_1(z)$ is univalent in $\mathbb{U}$ and $q_1(\mathbb{U})$ is the open disk given by

$$\left| q_1 - \frac{1 - A\bar{B}}{1 - |B|^2} \right| < \frac{|A - B|}{1 - |B|^2},$$

which shows that

$$\operatorname{Re}(q_1(z)) > \frac{\operatorname{Re}(1 - A\bar{B}) - |A - B|}{1 - |B|^2} \geq 0 \quad (z \in \mathbb{U}).$$

Here, we define that

$$W := \left\{ w : w \in \mathbb{C} \quad \text{and} \quad \left| w - \frac{1 - A\bar{B}}{1 - |B|^2} \right| = \frac{|A - B|}{1 - |B|^2} \right\}.$$

Then, for $w \in W$, $\arg(w)$ satisfy the following condition:

$$\begin{aligned} \sin^{-1} \frac{\operatorname{Im}(1 - A\bar{B})}{|1 - A\bar{B}|} - \cos^{-1} \frac{\sqrt{(1 - |A|^2)(1 - |B|^2)}}{|1 - A\bar{B}|} &\leq \arg(w) \\ &\leq \sin^{-1} \frac{\operatorname{Im}(1 - A\bar{B})}{|1 - A\bar{B}|} + \cos^{-1} \frac{\sqrt{(1 - |A|^2)(1 - |B|^2)}}{|1 - A\bar{B}|}. \end{aligned}$$

From this condition, for

$$1 \leq \delta \leq \frac{\pi}{2 \left(\left| \sin^{-1} \frac{\operatorname{Im}(1 - A\bar{B})}{|1 - A\bar{B}|} \right| + \cos^{-1} \frac{\sqrt{(1 - |A|^2)(1 - |B|^2)}}{|1 - A\bar{B}|} \right)},$$

we obtain

$$|\delta \arg(w)| \leq \frac{\pi}{2}.$$

Thus we see that

$$\operatorname{Re}(w^\delta) = |w|^\delta \cos(\delta \arg(w)) \geq 0.$$

Furthermore, we define that

$$W_1 := \left\{ w_1 : w_1 \in W \text{ and } \arg(w_1) \geq \arg\left(\frac{1 - A\bar{B}}{1 - |B|^2}\right) \right\},$$

and

$$W_2 := \left\{ w_2 : w_2 \in W \text{ and } \arg(w_2) < \arg\left(\frac{1 - A\bar{B}}{1 - |B|^2}\right) \right\}.$$

Then, $\arg(w_1)$ and $\arg(w_2)$ can write to as follows:

$$\begin{cases} \arg(w_1) = \sin^{-1} \frac{\operatorname{Im}(1 - A\bar{B})}{|1 - A\bar{B}|} + \cos^{-1} \frac{1 - |A|^2 + |w_1|^2(1 - |B|^2)}{2|w_1||1 - A\bar{B}|}, \\ \arg(w_2) = \sin^{-1} \frac{\operatorname{Im}(1 - A\bar{B})}{|1 - A\bar{B}|} - \cos^{-1} \frac{1 - |A|^2 + |w_2|^2(1 - |B|^2)}{2|w_2||1 - A\bar{B}|}. \end{cases}$$

Here, the following two things can be said about the minimum value of $\operatorname{Re}(w^\delta)$ ($w \in W$).

(i) When $\operatorname{Im}(1 - A\bar{B}) \geq 0$, for $w_1 \in W_1$,

$$\min_{w_1} \{ \operatorname{Re}(w_1^\delta) \} : \left(\frac{|1 - A\bar{B} - |A - B||}{1 - |B|^2} \leq |w_1| \leq \frac{\sqrt{(1 - |A|^2)(1 - |B|^2)}}{1 - |B|^2} \right)$$

is minimum value of $\operatorname{Re}(w^\delta)$ ($w \in W$).

Then, we obtain

$$\operatorname{Re}(w_1^\delta) = |w_1|^\delta \cos \left(\delta \left(\sin^{-1} \frac{\operatorname{Im}(1 - A\bar{B})}{|1 - A\bar{B}|} + \cos^{-1} \frac{1 - |A|^2 + |w_1|^2(1 - |B|^2)}{2|w_1||1 - A\bar{B}|} \right) \right).$$

(ii) When $\text{Im}(1 - A\bar{B}) < 0$, for $w_2 \in W_2$,

$$\min_{w_2} \{\text{Re}(w_2^\delta)\} : \left(\frac{|1 - A\bar{B} - |A - B||}{1 - |B|^2} \leq |w_2| \leq \frac{\sqrt{(1 - |A|^2)(1 - |B|^2)}}{1 - |B|^2} \right)$$

is minimum value of $\text{Re}(w^\delta)$ ($w \in W$).

Then, we can write

$$\begin{aligned} \text{Re}(w_2^\delta) &= |w_2|^\delta \cos \left(\delta \left(\sin^{-1} \frac{\text{Im}(1 - A\bar{B})}{|1 - A\bar{B}|} - \cos^{-1} \frac{1 - |A|^2 + |w_2|^2(1 - |B|^2)}{2|w_2||1 - A\bar{B}|} \right) \right) \\ &= |w_2|^\delta \cos \left(\delta \left(\left| \sin^{-1} \frac{\text{Im}(1 - A\bar{B})}{|1 - A\bar{B}|} \right| + \cos^{-1} \frac{1 - |A|^2 + |w_2|^2(1 - |B|^2)}{2|w_2||1 - A\bar{B}|} \right) \right). \end{aligned}$$

From the above-mentioned, for $w \in W$, we see that

$$\begin{aligned} m_0 &:= \min_w \left\{ |w|^\delta \cos \left(\delta \left(\left| \sin^{-1} \frac{\text{Im}(1 - A\bar{B})}{|1 - A\bar{B}|} \right| + \cos^{-1} \frac{1 - |A|^2 + |w|^2(1 - |B|^2)}{2|w||1 - A\bar{B}|} \right) \right) \right\} : \\ &\quad \left(\frac{|1 - A\bar{B} - |A - B||}{1 - |B|^2} \leq |w| \leq \frac{\sqrt{(1 - |A|^2)(1 - |B|^2)}}{1 - |B|^2} \right) \end{aligned}$$

is minimum value of $\text{Re}(w^\delta)$ ($w \in W$).

Thus, we obtain

$$\text{Re}(q(z)) > m_0 \geq 0 \quad (z \in \mathbb{U}).$$

Therefore, ϕ and ψ are analytic in a domain $\mathcal{C}$ containing $q(\mathbb{U})$, with

$$\psi(\omega) \neq 0 \quad (\omega \in q(\mathbb{U}) \subset \mathcal{C}).$$

The function $Q(z)$ given by

$$Q(z) = zq'(z)\psi(q(z)) = \frac{\alpha(A - B)z(1 + Az)^{\beta-1}}{(1 + Bz)^{\beta+1}}$$

is univalent and starlike in $\mathbb{U}$, because

$$\begin{aligned} \text{Re} \left(\frac{zQ'(z)}{Q(z)} \right) &= (1 - \delta\beta) \text{Re} \left(\frac{1}{1 + Az} \right) + (1 + \delta\beta) \text{Re} \left(\frac{1}{1 + Bz} \right) - 1 \\ &> \frac{1 - \delta\beta}{1 + |A|} + \frac{1 + \delta\beta}{1 + |B|} - 1 = \frac{1 - |A||B| + \delta\beta|A| - \delta\beta|B|}{(1 + |A|)(1 + |B|)} \geq 0 \quad (z \in \mathbb{U}). \end{aligned}$$

Furthermore, we have

$$\begin{aligned} h(z) &= \phi(q(z)) + Q(z) \\ &= \left(\frac{1 + Az}{1 + Bz} \right)^{\delta\beta} \left\{ 1 - \alpha + \alpha \left(\frac{1 + Az}{1 + Bz} \right)^\delta \right\} + \frac{\alpha\delta(A - B)z(1 + Az)^{\delta\beta-1}}{(1 + Bz)^{\delta\beta+1}}, \end{aligned}$$

and

$$\begin{aligned} \operatorname{Re} \left(\frac{zh'(z)}{Q(z)} \right) &= \frac{\beta(1-\alpha)}{\alpha} + (1+\beta)\operatorname{Re}(q(z)) + \operatorname{Re} \left(\frac{zQ'(z)}{Q(z)} \right) \\ &> \frac{\beta(1-\alpha)}{\alpha} + (1+\beta)m_0 + \frac{1-\delta\beta}{1+|A|} + \frac{1+\delta\beta}{1+|B|} - 1 \geq 0 \quad (z \in \mathbb{U}). \end{aligned}$$

Since all conditions of Lemma 1.1 are satisfied, we conclude that

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+Az}{1+Bz} \right)^\delta \quad (z \in \mathbb{U}).$$

This completes the proof of Theorem 2.1. $\square$

Remark 2.1 Letting $\delta = 1$ in Theorem 2.1, we obtain Lemma 1.2 proven by Kuroki, Owa and Srivastava [1, Theorem 2].

Also, taking $A, B \in \mathbb{R}$ ($-1 < B < A \leq 1$ or $-1 \leq A < B < 1$) in Theorem 2.1, we get the following Corollary 2.1 and Corollary 2.2.

Corollary 2.1 Let the function $f(z) \in \mathcal{A}$ be so chosen that

$$\frac{f(z)}{z} \neq 0 \quad (z \in \mathbb{U}).$$

Suppose also that the parameters

$$\alpha (\alpha \neq 0), \beta (-1 \leq \beta \leq 1), A, B (-1 < B < A \leq 1),$$

and

$$\delta \left(1 \leq \delta \leq \frac{\pi}{2 \cos^{-1} \frac{\sqrt{(1-A^2)(1-B^2)}}{1-AB}} \right)$$

are so prescribed that

$$1 - |A||B| + \delta\beta|A| - \delta\beta|B| \geq 0,$$

and

$$\frac{\beta(1-\alpha)}{\alpha} + (1+\beta)m_0 + \frac{1-\delta\beta}{1+|A|} + \frac{1+\delta\beta}{1+|B|} \geq 0,$$

where

$$m_0 = \min_w \left\{ |w|^\delta \cos \left(\delta \cos^{-1} \frac{1-A^2+|w|^2(1-B^2)}{2|w|(1-AB)} \right) \right\} : \left(\frac{1-A}{1-B} \leq |w| \leq \sqrt{\frac{1-A^2}{1-B^2}} \right).$$

If

$$\left(\frac{zf'(z)}{f(z)} \right)^\beta \left(1 + \alpha \frac{zf''(z)}{f'(z)} \right) \prec h(z) \quad (z \in \mathbb{U}),$$

where

$$h(z) = \left(\frac{1+Az}{1+Bz} \right)^{\delta\beta-1} \left\{ (1-\alpha) \frac{1+Az}{1+Bz} + \frac{\alpha(1+Az)^{1+\delta}(1+Bz)^{1-\delta} + \alpha\delta(A-B)z}{(1+Bz)^2} \right\},$$

then

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+Az}{1+Bz} \right)^{\delta} \quad (z \in \mathbb{U}).$$

Corollary 2.2 Let the function $f(z) \in \mathcal{A}$ be so chosen that

$$\frac{f(z)}{z} \neq 0 \quad (z \in \mathbb{U}).$$

Suppose also that the parameters

$$\alpha \ (\alpha \neq 0), \ \beta \ (-1 \leq \beta \leq 1), \ A, B \ (-1 \leq A < B < 1),$$

and

$$\delta \left(1 \leq \delta \leq \frac{\pi}{2 \cos^{-1} \frac{\sqrt{(1-A^2)(1-B^2)}}{1-AB}} \right)$$

are so prescribed that

$$1 - |A||B| + \delta\beta|A| - \delta\beta|B| \geq 0,$$

and

$$\frac{\beta(1-\alpha)}{\alpha} + (1+\beta)m_0 + \frac{1-\delta\beta}{1+|A|} + \frac{1+\delta\beta}{1+|B|} \geq 0,$$

where

$$m_0 = \min_w \left\{ |w|^{\delta} \cos \left(\delta \cos^{-1} \frac{1-A^2+|w|^2(1-B^2)}{2|w|(1-AB)} \right) \right\} : \left(\frac{1+A}{1+B} \leq |w| \leq \sqrt{\frac{1-A^2}{1-B^2}} \right).$$

If

$$\left(\frac{zf'(z)}{f(z)} \right)^{\beta} \left(1 + \alpha \frac{zf''(z)}{f'(z)} \right) \prec h(z) \quad (z \in \mathbb{U}),$$

where

$$h(z) = \left(\frac{1+Az}{1+Bz} \right)^{\delta\beta-1} \left\{ (1-\alpha) \frac{1+Az}{1+Bz} + \frac{\alpha(1+Az)^{1+\delta}(1+Bz)^{1-\delta} + \alpha\delta(A-B)z}{(1+Bz)^2} \right\},$$

then

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+Az}{1+Bz} \right)^{\delta} \quad (z \in \mathbb{U}).$$

Next, we derive our second main result contained in Theorem 2.2 below.

Theorem 2.2 *Let the function $f(z) \in \mathcal{A}$ be so chosen that*

$$\frac{f(z)}{z} \neq 0 \quad (z \in \mathbb{U}).$$

Suppose also that real parameters α ($\alpha \neq 0$), β ($-1 \leq \beta \leq 1$), and

$$\varepsilon \left(0 \leq \varepsilon \leq \frac{1 - |A|^2}{2(1 - A\bar{B})} \right),$$

as well as the complex parameters A and B constrained by

$$|A| \leq 1, |B| = 1, A \neq B, \text{ and } 1 - A\bar{B} > 0,$$

are so prescribed that

$$\frac{\beta(1 - \alpha)}{\alpha} + \frac{(1 + \beta)\{1 - |A|^2 - 2\varepsilon(1 - A\bar{B})\}}{2(1 - \varepsilon)(1 - A\bar{B})} + \frac{(1 - \beta)\{1 - \varepsilon - |A - \varepsilon B|\}}{2\{1 - \varepsilon + |A - \varepsilon B|\}} \geq 0.$$

If

$$(2) \quad \left(\frac{zf'(z)}{f(z)} \right)^\beta \left(1 + \alpha \frac{zf''(z)}{f'(z)} \right) \prec h(z) \quad (z \in \mathbb{U}),$$

where

$$h(z) = \left\{ \frac{1 - \varepsilon + (A - \varepsilon B)z}{(1 - \varepsilon)(1 + Bz)} \right\}^{\beta-1} \left\{ (1 - \alpha) \frac{1 - \varepsilon + (A - \varepsilon B)z}{(1 - \varepsilon)(1 + Bz)} + \frac{\alpha\{1 - \varepsilon + (A - \varepsilon B)z\}^2 + \alpha(1 - \varepsilon)(A - B)z}{(1 - \varepsilon)^2(1 + Bz)^2} \right\},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1 - \varepsilon + (A - \varepsilon B)z}{(1 - \varepsilon)(1 + Bz)} \left(= \frac{\frac{1+A}{1+B} - \varepsilon}{1 - \varepsilon} \right) \quad (z \in \mathbb{U}).$$

Proof. Let us define the function $p(z)$, $q(z)$, $\phi(z)$, and $\psi(z)$ by

$$p(z) = \frac{zf'(z)}{f(z)}, \quad q(z) = \frac{1 - \varepsilon + (A - \varepsilon B)z}{(1 - \varepsilon)(1 + Bz)}, \quad \phi(z) = z^\beta(1 - \alpha + \alpha z), \quad \text{and} \quad \psi(z) = \alpha z^{\beta-1}$$

for $z \in \mathbb{U}$. Then, clearly, the function $q(z)$ is analytic and univalent in $\mathbb{U}$.

Now, for $q_1(z) = \frac{1 + Az}{1 + Bz}$, it is clear that $q_1(z)$ is univalent in $\mathbb{U}$ and $q_1(\mathbb{U})$ is the right half plane satisfying

$$\operatorname{Re}(q_1(z)) > \frac{1 - |A|^2}{2(1 - A\bar{B})} \geq 0.$$

Thus we see that

$$\begin{aligned} \operatorname{Re}(q(z)) &= \operatorname{Re} \left(\frac{q_1(z) - \varepsilon}{1 - \varepsilon} \right) = \frac{\operatorname{Re}(q_1(z)) - \varepsilon}{1 - \varepsilon} \\ &> \frac{\frac{1-|A|^2}{2(1-A\bar{B})} - \varepsilon}{1 - \varepsilon} = \frac{1 - |A|^2 - 2\varepsilon(1 - A\bar{B})}{2(1 - \varepsilon)(1 - A\bar{B})} \geq 0 \quad (z \in \mathbb{U}). \end{aligned}$$

Also, the functions ϕ and ψ satisfy the conditions required by Lemma 1.1. The function $Q(z)$ given by

$$Q(z) = zq'(z)\psi(q(z)) = \frac{\alpha(A - B)z\{1 - \varepsilon + (A - \varepsilon B)z\}^{\beta-1}}{(1 - \varepsilon)^\beta(1 + Bz)^{\beta+1}}$$

is univalent and starlike in $\mathbb{U}$, because

$$\begin{aligned} \operatorname{Re} \left(\frac{zQ'(z)}{Q(z)} \right) &= (1 - \beta)(1 - \varepsilon)\operatorname{Re} \left(\frac{1}{1 - \varepsilon + (A - \varepsilon B)z} \right) + (1 + \beta)\operatorname{Re} \left(\frac{1}{1 + Bz} \right) - 1 \\ &> (1 - \beta)(1 - \varepsilon)\frac{1}{1 - \varepsilon + |A - \varepsilon B|} + \frac{1}{2}(1 + \beta) - 1 \\ &= \frac{(1 - \beta)\{1 - |A|^2 - 2\varepsilon(1 - A\bar{B})\}}{2\{1 - \varepsilon + |A - \varepsilon B|\}^2} \geq 0 \quad (z \in \mathbb{U}). \end{aligned}$$

Furthermore, we have

$$\begin{aligned} h(z) &= \phi(q(z)) + Q(z) \\ &= \left\{ \frac{(1 - \varepsilon) + (A - \varepsilon B)z}{(1 - \varepsilon)(1 + Bz)} \right\}^\beta \left\{ 1 - \alpha + \alpha \frac{(1 - \varepsilon) + (A - \varepsilon B)z}{(1 - \varepsilon)(1 + Bz)} \right\} \\ &\quad + \frac{\alpha(A - B)z\{(1 - \varepsilon) + (A - \varepsilon B)z\}^{\beta-1}}{(1 - \varepsilon)^\beta(1 + Bz)^{\beta+1}}, \end{aligned}$$

and

$$\begin{aligned} \operatorname{Re} \left(\frac{zh'(z)}{Q(z)} \right) &= \frac{\beta(1 - \alpha)}{\alpha} + (1 + \beta)\operatorname{Re}(q(z)) + \operatorname{Re} \left(\frac{zQ'(z)}{Q(z)} \right) \\ &> \frac{\beta(1 - \alpha)}{\alpha} + \frac{(1 + \beta)\{1 - |A|^2 - 2\varepsilon(1 - A\bar{B})\}}{2(1 - \varepsilon)(1 - A\bar{B})} \\ &\quad + \frac{(1 - \beta)\{(1 - \varepsilon) - |A - \varepsilon B|\}}{2\{(1 - \varepsilon) + |A - \varepsilon B|\}} \geq 0 \quad (z \in \mathbb{U}). \end{aligned}$$

Similarly, the other conditions of Lemma 1.1 are also seen to be satisfied. Therefore, we conclude that

$$\frac{zf'(z)}{f(z)} \prec \frac{1 - \varepsilon + (A - \varepsilon B)z}{(1 - \varepsilon)(1 + Bz)} \quad (z \in \mathbb{U}),$$

which completes the proof of Theorem 2.2. $\square$

Remark 2.2 Setting $\varepsilon = 0$ in Theorem 2.2, we obtain Lemma 1.3 proven by Kuroki, Owa and Srivastava [1, Theorem 1].

Also, making $A, B \in \mathbb{R}$ ($B = -1$; $-1 < A \leq 1$ or $B = 1$; $-1 \leq A < 1$) in Theorem 2.2, we find the following Corollary 2.3 and Corollary 2.4.

Corollary 2.3 Let the function $f(z) \in \mathcal{A}$ be so chosen that

$$\frac{f(z)}{z} \neq 0 \quad (z \in \mathbb{U}).$$

Suppose also that the parameters

$$\alpha \ (\alpha \neq 0), \ \beta \ (-1 \leq \beta \leq 1), \ A \ (-1 < A \leq 1),$$

and

$$\varepsilon \left(0 \leq \varepsilon \leq \frac{1}{2}(1 - A) \right)$$

are so prescribed that

$$\frac{\beta(1 - \alpha)}{\alpha} + \frac{(1 + \beta)(1 - A - 2\varepsilon)}{2(1 - \varepsilon)} + \frac{(1 - \beta)\{1 - \varepsilon - |A + \varepsilon|\}}{2\{1 - \varepsilon + |A + \varepsilon|\}} \geq 0.$$

If

$$\left(\frac{zf'(z)}{f(z)} \right)^\beta \left(1 + \alpha \frac{zf''(z)}{f'(z)} \right) \prec h(z) \quad (z \in \mathbb{U}),$$

where

$$h(z) = \left\{ \frac{1 - \varepsilon + (A + \varepsilon)z}{(1 - \varepsilon)(1 - z)} \right\}^{\beta-1} \left\{ (1 - \alpha) \frac{1 - \varepsilon + (A + \varepsilon)z}{(1 - \varepsilon)(1 - z)} + \frac{\alpha \{1 - \varepsilon + (A + \varepsilon)z\}^2 + \alpha(1 - \varepsilon)(A + 1)z}{(1 - \varepsilon)^2(1 - z)^2} \right\},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1 - \varepsilon + (A + \varepsilon)z}{(1 - \varepsilon)(1 - z)} \quad (z \in \mathbb{U}).$$

Corollary 2.4 Let the function $f(z) \in \mathcal{A}$ be so chosen that

$$\frac{f(z)}{z} \neq 0 \quad (z \in \mathbb{U}).$$

Suppose also that the parameters

$$\alpha \ (\alpha \neq 0), \ \beta \ (-1 \leq \beta \leq 1), \ A \ (-1 \leq A < 1),$$

and

$$\varepsilon \left(0 \leq \varepsilon \leq \frac{1}{2}(1 + A) \right)$$

are so prescribed that

$$\frac{\beta(1-\alpha)}{\alpha} + \frac{(1+\beta)(1+A-2\varepsilon)}{2(1-\varepsilon)} + \frac{(1-\beta)\{1-\varepsilon-|A-\varepsilon|\}}{2\{1-\varepsilon+|A-\varepsilon|\}} \geq 0.$$

If

$$\left(\frac{zf'(z)}{f(z)}\right)^\beta \left(1 + \alpha \frac{zf''(z)}{f'(z)}\right) \prec h(z) \quad (z \in \mathbb{U}),$$

where

$$h(z) = \left\{ \frac{1-\varepsilon+(A-\varepsilon)z}{(1-\varepsilon)(1+z)} \right\}^{\beta-1} \left\{ (1-\alpha) \frac{1-\varepsilon+(A-\varepsilon)z}{(1-\varepsilon)(1+z)} + \frac{\alpha\{1-\varepsilon+(A-\varepsilon)z\}^2 + \alpha(1-\varepsilon)(A-1)z}{(1-\varepsilon)^2(1+z)^2} \right\},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1-\varepsilon+(A-\varepsilon)z}{(1-\varepsilon)(1+z)} \quad (z \in \mathbb{U}).$$

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