

Entangled Quantum Markov Chain satisfyfying Entanglement Condition

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Abstract

The entropic criterion of entanglement is applied to prove that entangled Markov chain with unitarily implementable transition operator is indeed an entangled state on infinite multiple algebras.

1 Introduction and Preliminaries

Accardi and Fidaleo [2] proposed a construction to relate, based on classical Markov chain with discrete state space, to a quantum Markov chain (in the sense of [1]) on infinite tensor products of type I factors. They called *entangled Markov chain* (EMC) the special class of quantum Markov chains obtained in this way.

Using the PPT entanglement criterion [13, 8] (positivity of the partial transpose of the density matrix) Miyadera showed [9] that the finite volume restriction of a class of EMC on infinite tensor products of 2×2 matrix algebras is entangled.

In our previous paper [3], using the entropic type of entanglement criterion for pure states [11, 3], which is based on the notion of *degree of entanglement*, we proved that the vector states defining the EMC's on infinite tensor products of $d \times d$ matrix algebras ($d \in \mathbb{N}$) "generically" are entangled (see Definition (3) below).

Our result did not include Miyadear's one because, by restricting an EMC to some local algebra, one obtains a mixed state to which the above criterion for a pure state is not applicable. However our entanglement criterion gives the sufficient condition for entanglement in the case of mixtures (for pure states this condition is necessary and sufficient) [4]. Moreover our entanglement criterion, being based on a numerical inequality, is in many cases easier to verify than the positivity condition required by the PPT criterion.

In this note we will show some results obtained in [4] with proof for the reader's convenience. Our entanglement condition is applied to the restriction of EMC's, generated by a unitarily implementable $d \times d$ stochastic matrix, to algebras localized which is obtained as a mixed state. This allows to prove that the above EMC induce an entangled state on infinite tensor products of $d \times d$ matrix algebras for any $d \in \mathbb{N}$.

We consider a classical Markov chain (S_n) with state space $S = \{1, 2, \dots, d\}$, initial distribution $p = (p_j)$ and transition probability matrix $P = (p_{ij})$

$$p_{ij} \geq 0 \quad ; \quad \sum_j p_{ij} = 1$$

Let $\{e_i\}_{i \leq d}$ be an orthonormal basis (ONB) of $\mathbb{C}^{|S|}$. For a fixed vector e_0 in this basis, denote

$$\mathcal{H}_{\mathbb{N}} := \bigotimes_{\mathbb{N}}^{(e_0)} \mathbb{C}^{|S|} \quad (1)$$

the infinite tensor product of $\mathbb{N}$ -copies of the Hilbert space $\mathbb{C}^{|S|}$ with respect to the constant sequence (e_0) . An orthonormal basis of $\mathcal{H}_{\mathbb{N}}$ is given by the vectors

$$|e_{j_0}, \dots, e_{j_n}\rangle := \left(\bigotimes_{\alpha \in [0, n]} e_{j_\alpha} \right) \otimes \left(\bigotimes_{\alpha \in [0, n]^c} e_0 \right).$$

For any Hilbert space $\mathcal{H}$ we denote $\mathcal{H}^*$ its dual and $\xi \in \mathcal{H} \mapsto \xi^* \in \mathcal{H}^*$ the canonical embedding. Thus, if $\xi \in \mathcal{H}$ is a unit vector, $\xi\xi^*$ denotes the projection onto the subspace generated by ξ .

Let M_d denote the algebra of complex $d \times d$ matrices and let $\mathcal{A} := \bigotimes_{\mathbb{N}} M_d = M_d \otimes M_d \otimes \dots$ be the C^* -infinite tensor product of $\mathbb{N}$ -copies of M_d .

An element $A_\Lambda \in \mathcal{A}$ (observable) will be said to be *localized* in a finite region $\Lambda \subset \mathbb{N}$ if there exists an operator $\bar{A}_\Lambda \in \bigotimes_{\Lambda} M_d$ such that

$$A_\Lambda = \bar{A}_\Lambda \otimes 1_{\Lambda^c}$$

In the following we will identify $A_\Lambda = \bar{A}_\Lambda$ and we denote $\mathcal{A}_\Lambda$ the local algebra at Λ . Let $\sqrt{p_i}$ (resp. $\sqrt{p_{ij}}$) be any complex square root of p_i (resp. p_{ij}) (i.e. $|\sqrt{p_i}|^2 = p_i$ (resp. $|\sqrt{p_{ij}}|^2 = p_{ij}$)) and define the vector

$$\Psi_n = \sum_{j_0, \dots, j_n} \sqrt{p_{j_0}} \prod_{\alpha=0}^{n-1} \sqrt{p_{j_\alpha j_{\alpha+1}}} |e_{j_0}, \dots, e_{j_n}\rangle \quad (2)$$

Although the limit $\lim_{n \rightarrow \infty} \Psi_n$ will not exist, the basic property of Ψ_n is the following [2].

Proposition 1 *There exists a unique quantum Markov chain ψ on $\mathcal{A}$ such that, for every $k \in \mathbb{N}$ and for every $A \in \mathcal{A}_{[0, k]}$, one has*

$$\langle \Psi_{k+1}, A \Psi_{k+1} \rangle = \lim_{n \rightarrow \infty} \langle \Psi_n, A \Psi_n \rangle =: \psi(A) \quad (3)$$

Moreover ψ is stationary if and only if the associated classical Markov chain $\{p := (p_i), P = (p_{ij})\}$ is stationary, i.e.

$$\sum_i p_i p_{ij} = p_j \quad ; \quad \forall j \quad (4)$$

2 Notions of entanglement and degree of entanglement

Definition 2 Let $\mathcal{A}_j$ ($j \in \{1, 2, \dots, n\}$) with $n < \infty$ be C^* -algebras and let $\mathcal{A} = \bigotimes_{j=1}^n \mathcal{A}_j$ be a tensor product of C^* -algebras. A state $\omega \in \mathcal{S}\left(\bigotimes_{j=1}^n \mathcal{A}_j\right)$ is called separable if

$$\omega \in \overline{\text{Conv}} \left\{ \bigotimes_{j=1}^n \omega_j ; \omega_j \in \mathcal{S}(\mathcal{A}_j), j \in \{1, 2, \dots, n\} \right\}$$

where $\overline{\text{Conv}}$ denotes norm closure of the convex hull.

A nonseparable state is called entangled.

Notice that the notion of separability may depend on the choice of the tensor product of C^* -algebras. Unless otherwise specified, one realizes the C^* -algebras on Hilbert spaces and one considers the induced tensor product. In any case a separable pure state must be a product of pure states.

Definition 3 [3] In the notations of Definition (2) a state $\omega \in \mathcal{S}(\mathcal{A})$ is called 2-separable if

$$\omega \in \overline{\text{Conv}} \{ \omega_k \otimes \omega_{(k)} : \omega_k \in \mathcal{S}(\mathcal{A}_k), \omega_{(k)} \in \mathcal{S}(\mathcal{A}_{(k)}), \forall k \in \{1, 2, \dots, n\} \}$$

where $\mathcal{A} = \mathcal{A}_k \otimes \mathcal{A}_{(k)} := \mathcal{A}_{[1,k]} \otimes \mathcal{A}_{(k,n]}$.

A non-2-separable state is called 2-entangled.

Remark Notice that, for $n = 2$, 2-entanglement is equivalent to usual entanglement. For $n > 2$, 2-entanglement is a strictly stronger property than usual entanglement.

Definition 4 Let $\mathcal{H}_1, \mathcal{H}_2$ be separable Hilbert spaces and let θ be density matrices in $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ with its marginal densities denoted by ρ and σ in $\mathcal{B}(\mathcal{H}_1), \mathcal{B}(\mathcal{H}_2)$ respectively.

The quasi mutual entropy of ρ and σ w.r.t θ is defined by [10]

$$I_\theta(\rho, \sigma) \equiv \text{tr} \theta (\log \theta - \log \rho \otimes \sigma) \quad (5)$$

The degree of entanglement of θ , denoted by $D_{EN}(\theta)$, is defined by [11]

$$D_{EN}(\theta) \equiv \frac{1}{2} \{ S(\rho) + S(\sigma) \} - I_\theta(\rho, \sigma) \quad (6)$$

where $S(\cdot)$ is the von-Neumann entropy.

In the following we identify normal states on $\mathcal{B}(\mathcal{H})$ ($\mathcal{H}$ some separable Hilbert space) with their density matrices and, if θ is such a state, we will use indifferently the notations

$$\theta(x) = \text{tr}(\theta x) \quad ; \quad x \in \mathcal{B}(\mathcal{H}) \quad (7)$$

Recalling that, for density operators θ, δ in $\mathcal{B}(\mathcal{H})$, the relative entropy of δ with respect to θ is defined by:

$$R(\theta|\delta) := \text{tr} \theta (\log \theta - \log \delta) \quad (8)$$

(see [5, 12] for a more general discussion) we see that $I_\theta(\rho, \sigma)$ is the relative entropy of the tensor product of its marginal densities with respect to θ itself. Since it is known that the relative entropy is a kind of distance between states, it is clear why the degree of entanglement of θ by (6) is a measure of how far θ is from being a product state. Moreover we see also that D_{EN} is a kind of symmetrized quantum conditional entropy. In the classical case the conditional entropy always takes non-negative value, however our new criterion can be negative according to the strength of quantum correlation between ρ and σ [4].

Theorem 5 *A necessary condition for a (normal) state θ on $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ to be separable is that*

$$D_{EN}(\theta) \geq 0 \quad (9)$$

Equivalently: a sufficient condition for θ to be entangled is that

$$D_{EN}(\theta) < 0. \quad (10)$$

Proof. Let θ be a state on $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. If θ is separable, there exist density matrices ρ_n, σ_n respectively in $\mathcal{B}(\mathcal{H}_1), \mathcal{B}(\mathcal{H}_2)$ such that

$$\theta = \sum_n p_n \rho_n \otimes \sigma_n$$

with

$$p_n \geq 0, \forall n \quad ; \quad \sum_n p_n = 1$$

Let $\{x_n\}$ be an ONB in $\mathcal{H}_1$ and define the completely positive unital (CP1) map $\Lambda_0 : \mathcal{B}(\mathcal{H}_1) \rightarrow \mathcal{B}(\mathcal{H}_1)$ by

$$\Lambda_0(A) = \sum_n \text{tr}(A \rho_n) x_n x_n^* \quad ; \quad A \in \mathcal{B}(\mathcal{H}_1) \quad (11)$$

Then its dual is

$$\Lambda_0^*(\delta) = \sum_n \langle x_n, \delta x_n \rangle \rho_n \quad ; \quad \delta \in \mathcal{B}(\mathcal{H}_1)_* \quad (12)$$

so that defining the CP1 map

$$\Lambda := \Lambda_0 \otimes \text{id} : \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2) \rightarrow \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$$

and the density matrix

$$\theta_d := \sum_n p_n x_n x_n^* \otimes \sigma_n$$

one easily verifies that

$$\Lambda^*(\theta_d) = \theta$$

Moreover, denoting

$$\rho = \sum_n p_n \rho_n \quad \text{and} \quad \sigma = \sum_n p_n \sigma_n$$

the marginal densities of θ and $\rho_d = \sum_n p_n |x_n\rangle \langle x_n|$ the first marginal density of θ_d , one has:

$$\Lambda^*(\rho_d \otimes \sigma) = \rho \otimes \sigma$$

Recall now that the monotonicity property of the relative entropy (see [12] for proof and history) that for any pair of von Neumann algebras $\mathcal{M}$, $\mathcal{M}^0$, for any normal CP1 map $\Lambda : \mathcal{M} \rightarrow \mathcal{M}^0$ and for any pair of normal states ω_0, φ_0 on $\mathcal{M}^0$ one has

$$R(\Lambda^*(\omega_0) | \Lambda^*(\varphi_0)) \leq R(\omega_0 | \varphi_0) \quad (13)$$

Using this property one finds:

$$I_\theta(\rho, \sigma) = R(\theta | \rho \otimes \sigma) = R(\Lambda^*(\theta_d) | \Lambda^*(\rho_d \otimes \sigma)) \leq R(\theta_d | \rho_d \otimes \sigma) = I_{\theta_d}(\rho_d, \sigma)$$

so that

$$S(\sigma) - I_\theta(\rho, \sigma) \geq S(\sigma) - I_{\theta_d}(\rho_d, \sigma) = - \sum_n p_n \text{tr}(\sigma_n \log \sigma_n) \geq 0 \quad (14)$$

Introducing the density operator

$$\hat{\theta}_d = \sum_n p_n \rho_n \otimes y_n y_n^*$$

where $\{y_n\}$ is an ONB in $\mathcal{H}_2$, and using a variant of the above argument (in which the density θ_d is replaced by $\hat{\theta}_d$) one proves the analogue inequality

$$S(\rho) - I_\theta(\rho, \sigma) \geq S(\rho) - I_{\hat{\theta}_d}(\rho, \sigma_d) = - \sum_n p_n \text{tr}(\rho_n \log \rho_n) \geq 0 \quad (15)$$

Combining (14) and (15) one obtains:

$$\begin{aligned} D_{EN}(\theta; \rho, \sigma) &= \frac{1}{2} ((S(\sigma) - I_\theta(\rho, \sigma)) + (S(\rho) - I_\theta(\rho, \sigma))) \geq \\ &\geq \frac{1}{2} \left(- \sum_n p_n \text{tr}(\rho_n \log \rho_n) - \sum_n p_n \text{tr}(\sigma_n \log \sigma_n) \right) \geq 0 \end{aligned} \quad (16)$$

which is (9). ■

Remark For pure (normal) states θ on $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ condition (10) is also necessary for entanglement (see [11, 3]).

3 The localized EMC and its marginal states

We discuss the entanglement of the finite volume restrictions of a class of EMC on infinite tensor products of $d \times d$ matrix algebras. By restricting an EMC to some local algebra one obtains a mixed state to which our entanglement criterion D_{EN} is applicable because of theorem 5. In the following arguments we will denote $u_{ij} = \sqrt{p_{ij}}$ any (fixed) complex square root of p_{ij} so that

$$|u_{ij}|^2 = p_{ij} \quad ; \quad \forall i, j$$

and we assume that $U = (u_{ij})$ is a unitary matrix.

Let denote the unitarily implementable EMC state restricted to a finite region $[0, \nu]$ by $\rho_{[0, \nu]}$, then for every local observable $A \in \mathcal{A}_{[0, \nu]}$ one has $\rho_{[0, \nu]}(A) = \langle \Psi_{\nu+1}, (A \otimes I) \Psi_{\nu+1} \rangle$. Hence the density operator $\rho_{[0, \nu]}$ is given by:

$$\begin{aligned} \rho_{[0, \nu]} &= \text{tr}_{\mathcal{H}_{\nu+1}} |\Psi_{\nu+1}\rangle \langle \Psi_{\nu+1}| \\ &= \sum_{i_0, \dots, i_{\nu+1}, j_0, \dots, j_{\nu+1}, l_{\nu+1}} \sqrt{p_{i_0}}^* \sqrt{p_{j_0}} \prod_{\alpha=0}^{\nu} u_{i_{\alpha} i_{\alpha+1}}^* u_{j_{\alpha} j_{\alpha+1}} \\ &\quad \langle e_{l_{\nu+1}}, e_{j_{\nu+1}} \rangle \langle e_{i_{\nu+1}}, e_{l_{\nu+1}} \rangle |e_{j_0}, \dots, e_{j_{\nu}}\rangle \langle e_{i_0}, \dots, e_{i_{\nu}}| \\ &= \sum_{j_0, \dots, j_{\nu}, l, i_0, \dots, i_{\nu}} \sqrt{p_{i_0}}^* \sqrt{p_{j_0}} \prod_{\alpha=0}^{\nu-1} u_{i_{\alpha} i_{\alpha+1}}^* u_{j_{\alpha} j_{\alpha+1}} \\ &\quad u_{i_{\nu} l}^* u_{j_{\nu} l} |e_{j_0}, \dots, e_{j_{\nu}}\rangle \langle e_{i_0}, \dots, e_{i_{\nu}}| \end{aligned}$$

From the unitarity of $U = (u_{ij})$ one has $\sum_l u_{i_{\nu} l}^* u_{j_{\nu} l} = (UU^*)_{j_{\nu} i_{\nu}} = \delta_{i_{\nu}, j_{\nu}}$. Using this unitarity one has

$$\begin{aligned} \rho_{[0, \nu]} &= \sum_{j_0, j_1, \dots, j_{\nu}, i_0, i_1, \dots, i_{\nu}, k} \sqrt{p_{i_0}}^* \sqrt{p_{j_0}} \prod_{\alpha=0}^{\nu-2} u_{i_{\alpha} i_{\alpha+1}}^* u_{j_{\alpha} j_{\alpha+1}} \\ &\quad u_{i_{\nu-1} k}^* u_{j_{\nu-1} k} |e_{j_0}, e_{j_1}, \dots, e_{j_{\nu-1}}, e_{\nu}(k)\rangle \langle e_{i_0}, e_{i_1}, \dots, e_{i_{\nu-1}}, e_{\nu}(k)| \\ &= \sum_k p_k e_{[0, \nu]}(k) e_{[0, \nu]}(k)^*, \end{aligned} \tag{17}$$

where

$$e_{[0, \nu]}(k) := \frac{1}{\sqrt{p_k}} \sum_{j_0, \dots, j_{\nu-1}} \sqrt{p_{j_0}} \left(\prod_{\alpha=0}^{\nu-2} u_{j_{\alpha} j_{\alpha+1}} \right) u_{j_{\nu-1} k} |e_{j_0}, \dots, e_{j_{\nu-1}}, e_{\nu}(k)\rangle.$$

The vectors $\{e_{[0,\nu]}(k)\}_k$ are normalized and orthogonal each other. In fact

$$\begin{aligned} \|e_{[0,\nu]}(k)\|^2 &= \frac{1}{p_k} \sum_{j_{\mu+1}, j_1, \dots, j_{\nu-1}} p_{j_{\mu+1}} \prod_{\alpha=\mu+1}^{\nu-2} p_{j_{\alpha} j_{\alpha+1}} p_{j_{\nu-1} k} \\ &= \frac{1}{p_k} \sum_{j_{\nu-1}} p_{j_{\nu-1}} p_{j_{\nu-1} k} = \frac{p_k}{p_k} = 1, \end{aligned}$$

and the orthogonality of $\{e_{[0,\nu]}(k)\}_k$ is clear because of the orthogonality of $\{e_{\nu}(k)\}_k$. We see that the decomposition (17) gives a Schatten decomposition.

Let us consider the marginal states of density $\rho_{[0,\nu]}$ for each $\mu \in [0, \nu-1]$ given by

$$\rho_{[\mu]} \equiv \text{tr}_{H_{(\mu,\nu)}} \rho_{[0,\nu]}, \quad \rho_{(\mu)} \equiv \text{tr}_{H_{[0,\mu]}} \rho_{[0,\nu]} \quad (18)$$

Since, by Proposition (1), the family $(\rho_{[0,\nu]})_{\nu}$ is projective, for each $\mu \in [0, \nu-1]$ the restriction of $\rho_{[0,\nu]}$ to the algebra localized on $[0, \mu]$ is $\rho_{[0,\mu]}$. This implies

$$\rho_{[\mu]} \equiv \text{tr}_{H_{(\mu,\nu)}} \rho_{[0,\nu]} = \rho_{[0,\mu]}. \quad (19)$$

On the other hand the marginal state $\rho_{(\mu)}$ is given by

$$\begin{aligned} \rho_{(\mu)} &= \text{tr}_{\mathcal{H}_{[0,\mu]}} \rho_{[0,\nu]} \\ &= \sum_{j_0, \dots, j_{\mu}, j_{\mu+1}, \dots, j_{\nu-1}, i_{\mu+1}, \dots, i_{\nu-1}, k} p_{j_0} \left(\prod_{\alpha=0}^{\mu-1} p_{j_{\alpha} j_{\alpha+1}} \right) u_{j_{\mu} i_{\mu+1}}^* u_{j_{\mu} j_{\mu+1}} \\ &\quad \left(\prod_{\alpha=\mu+1}^{\nu-2} u_{i_{\alpha} i_{\alpha+1}}^* u_{j_{\alpha} j_{\alpha+1}} \right) u_{i_{\nu-1} k}^* u_{j_{\nu-1} k} \\ &\quad |e_{j_{\mu+1}}, \dots, e_{j_{\nu-1}}, e_{\nu}(k)\rangle \langle e_{i_{\mu+1}}, \dots, e_{i_{\nu-1}}, e_{\nu}(k)| \\ &= \sum_{n, j_{\mu+1}, \dots, j_{\nu-1}, i_{\mu+1}, \dots, i_{\nu-1}, k} p_n u_{n i_{\mu+1}}^* u_{n j_{\mu+1}} \prod_{\alpha=\mu+1}^{\nu-2} u_{i_{\alpha} i_{\alpha+1}}^* u_{j_{\alpha} j_{\alpha+1}} \\ &\quad u_{i_{\nu-1} k}^* u_{j_{\nu-1} k} |e_{j_{\mu+1}}, \dots, e_{j_{\nu-1}}, e_{\nu}(k)\rangle \langle e_{i_{\mu+1}}, \dots, e_{i_{\nu-1}}, e_{\nu}(k)| \\ &= \sum_{n, k} p_n e_{(\mu,\nu]}^n(k) e_{(\mu,\nu]}^n(k)^* \end{aligned} \quad (20)$$

where

$$e_{(\mu,\nu]}^n(k) = \sum_{j_{\mu+1}, \dots, j_{\nu-1}} u_{n j_{\mu+1}} \left(\prod_{\alpha=\mu+1}^{\nu-2} u_{j_{\alpha} j_{\alpha+1}} \right) u_{j_{\nu-1} k} |e_{j_{\mu+1}}, \dots, e_{j_{\nu-1}}, e_{\nu}(k)\rangle.$$

Remark If we put

$$\rho_{(\mu,\nu]}(n) := \sum_k e_{(\mu,\nu]}^n(k) e_{(\mu,\nu]}^n(k)^*, \quad (21)$$

then it is shown that (21) can be recognized as an orthogonal decompositions of a density operator. In fact we can show the following properties of $\rho_{(\mu,\nu]}(n)$.

(i) Orthogonality:

$$\langle e_{(\mu,\nu]}^n(k), e_{(\mu,\nu]}^n(l) \rangle = \delta_{k,l},$$

(ii) Density:

$$\begin{aligned} \|e_{(\mu,\nu]}^n(k)\|^2 &= \sum_{j_{\mu+1}, \dots, j_{\nu-1}} p_{nj_{\mu+1}} \left(\prod_{\alpha=\mu+2}^{\nu-2} p_{j_{\alpha}j_{\alpha+1}} \right) p_{j_{\nu-1}k} \\ &\equiv \left(P^{\nu-(\mu+1)} \right)_{nk}. \end{aligned}$$

This matrix $(P^{\nu-(\mu+1)})$ can be recognized as a transition probability matrix generated by $P = (p_{ij})$ (i.e. a classical ergodic Markov chain). This implies

$$\text{tr} \rho_{(\mu,\nu]}(n) = \sum_k \left(P^{\nu-(\mu+1)} \right)_{nk} = 1.$$

Let denote $\widehat{e}_{(\mu,\nu]}^n(k)$ the normalized vector i.e.

$$\widehat{e}_{(\mu,\nu]}^n(k) = \frac{1}{\sqrt{\left(P^{\nu-(\mu+1)} \right)_{nk}}} e_{(\mu,\nu]}^n(k).$$

Then $\rho_{(\mu,\nu]}(n)$ is represented by

$$\rho_{(\mu,\nu]}(n) = \sum_k \left(P^{\nu-(\mu+1)} \right)_{nk} \widehat{e}_{(\mu,\nu]}^n(k) \widehat{e}_{(\mu,\nu]}^n(k)^* \quad (22)$$

which is a Schatten decomposition.

4 The DEN of EMC generated by unitarily implementable channel

We can define the entanglement criterion of EMC φ via the DEN of a localized EMC $\rho_{[0,\nu]}$. According to the definition of DEN one can compute the DEN of $\rho_{[0,\nu]}$ as follows:

$$\begin{aligned} D_{EN}(\rho_{[0,\nu]}; \rho_{[\mu]}, \rho_{(\mu)}) &= \frac{1}{2} \left\{ S(\rho_{[\mu]}) + S(\rho_{(\mu)}) \right\} - I_{\rho_{[0,\nu]}}(\rho_{[\mu]}, \rho_{(\mu)}) \\ &= \frac{1}{2} \left\{ S(\rho_{[\mu]}) + S(\rho_{(\mu)}) \right\} - \left\{ S(\rho_{[\mu]}) + S(\rho_{(\mu)}) - S(\rho_{[0,\nu]}) \right\} \\ &= S(\rho_{[0,\nu]}) - \frac{1}{2} \left\{ S(\rho_{[\mu]}) + S(\rho_{(\mu)}) \right\}. \end{aligned}$$

Definition 6 For a fixed $\mu \in \mathbb{N}$ we define the 2-entangled DEN of EMC φ by

$$D_{EN}(\varphi; \rho_{[\mu]}, \rho_{(\mu)}) \equiv \lim_{\nu \rightarrow \infty} D_{EN}(\rho_{[0, \nu]}; \rho_{[\mu]}, \rho_{(\mu)}), \quad (23)$$

where $\nu > \mu$. The D_{EN} of EMC φ is defined by the infimum of the 2-entangled DEN.

$$D_{EN}(\varphi) \equiv \inf_{\mu \in \mathbb{N}} D_{EN}(\varphi; \rho_{[\mu]}, \rho_{(\mu)}). \quad (24)$$

Then we have the following result [4].

Theorem 7

$$D_{EN}(\varphi) = -\frac{1}{2} H(P) < 0 \quad (25)$$

where $H(P)$ is a Shannon entropy of a initial distribution of P .

Proof. The localized state $\rho_{[0, \nu]}$ is decomposed to (17) and its marginal state $\rho_{[\mu]}$ has a similar decomposition because of (19) which implies

$$S(\rho_{[0, \nu]}) = S(\rho_{[\mu]}) = -\sum_{n=1}^d p_n \log p_n = H(P). \quad (26)$$

On the other hand the another marginal state $\rho_{(\mu)}$ is decomposed to (20) which can not be recognized as a orthogonal decomposition in general. However we can estimate the orthogonality of the vectors $e_{(\mu, \nu]}^n(k)$ and $e_{(\mu, \nu]}^m(k)$ ($n \neq m$) asymptotically as follows:

$$\begin{aligned} \langle e_{(\mu, \nu]}^n(k), e_{(\mu, \nu]}^m(k) \rangle &= \sum_{j_{\mu+1}, \dots, j_{\nu-1}} u_{nj_{\mu+1}}^* u_{mj_{\mu+1}} \left(\prod_{\alpha=\mu+1}^{\nu-2} p_{j_{\alpha} j_{\alpha+1}} \right) p_{j_{\nu-1} k} \\ &= \sum_{j_{\mu+1}} u_{nj_{\mu+1}}^* u_{mj_{\mu+1}} \sum_{j_{\mu+2}, \dots, j_{\nu-1}} p_{j_{\mu+1} j_{\mu+2}} \left(\prod_{\alpha=\mu+2}^{\nu-2} p_{j_{\alpha} j_{\alpha+1}} \right) p_{j_{\nu-1} k} \\ &= \sum_{j_{\mu+1}} u_{nj_{\mu+1}}^* u_{mj_{\mu+1}} (P^{\nu-\mu-2})_{j_{\mu+1} k} \end{aligned}$$

From the ergodic property of $(P^{\nu-\mu-2})$ we have

$$\lim_{\nu \rightarrow \infty} (P^{\nu-\mu-2})_{j_{\mu+1} k} = p_k$$

Therefore

$$\begin{aligned} \lim_{\nu \rightarrow \infty} \langle e_{(\mu, \nu]}^n(k), e_{(\mu, \nu]}^m(k) \rangle &= p_k \sum_{j_{\mu+1}} u_{nj_{\mu+1}}^* u_{mj_{\mu+1}} \\ &= p_k \delta_{n, m}. \end{aligned} \quad (27)$$

In large $\nu \gg 0$ we can estimate the orthogonality of $\{\rho_{(\mu,\nu]}(n)\}_n$ approximately

$$\rho_{(\mu,\nu]}(n) \rho_{(\mu,\nu]}(m) \simeq 0 \quad (n \neq m). \quad (28)$$

It is known (see [12]) that, if a density operator ρ is a convex combination of densities ρ_n ,

$$\rho = \sum_n \lambda_n \rho_n \quad , \quad \lambda_n \geq 0 \quad , \quad \sum_n \lambda_n = 1$$

then the following inequality holds:

$$S(\rho) \leq \sum_n \lambda_n S(\rho_n) - \sum_n \lambda_n \log \lambda_n \quad (29)$$

and the equality holds if $\rho_n \perp \rho_m$ for $n \neq m$. Thanks to (28) we can apply the equality of (29) to $\rho_{(\mu)} = \sum_n p_n \rho_{(\mu,\nu]}(n)$.

$$\begin{aligned} \lim_{\nu \rightarrow \infty} S(\rho_{(\mu)}) &= \lim_{\nu \rightarrow \infty} S\left(\sum_n p_n \rho_{(\mu,\nu]}(n)\right) \\ &= -\sum_{n=1}^d p_n \sum_{k=1}^d p_k \log p_k - \sum_{n=1}^d p_n \log p_n \\ &= 2H(P). \end{aligned} \quad (30)$$

From the above arguments we have

$$\begin{aligned} \lim_{\nu \rightarrow \infty} D_{EN}(\rho_{[0,\nu]}; \rho_{(\mu)}, \rho_{(\mu)}) &= H(P) - \frac{1}{2} \{H(P) + 2H(P)\} \\ &= -\frac{1}{2} H(P). \end{aligned} \quad (31)$$

It is clear that the equation (31) holds for any $\mu \in \mathbb{N}$. This fact shows that the equation (25) holds. ■

This theorem says that the unitary implementable EMC is entangled state in the sense of definition 6. On the base of theorem 7 we can compute another entropic criteria, introduced in [6, 7], for EMC. As a result of such computations we can conclude that EMC gives an example of maximal entangled state on infinite multiple algebras. The detailed discussion will appear in a forthcoming paper [4].

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