

Central extensions of groups and adjoint groups of quandles

By

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Abstract

This paper develops an approach for describing centrally extended groups, as determining the adjoint groups associated with quandles. Furthermore, we explicitly describe such groups of some quandles. As a corollary, we determine some second quandle homology groups.

§ 1. Introduction

A quandle is a set with a binary operation $\triangleleft : X^2 \rightarrow X$ such that

1. The identity $a \triangleleft a = a$ holds for any $a \in X$.
2. The map $(\bullet \triangleleft a) : X \rightarrow X$ defined by $x \mapsto x \triangleleft a$ is bijective for any $a \in X$.
3. The identity $(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$ holds for any $a, b, c \in X$.

The axioms are partially motivated by knot theory and braidings. For example, any group is a quandle by the operation $a \triangleleft b := b^{-1}ab$; see §4 for other examples. Conversely, for a quandle X , we can define *the adjoint group* as the following group presentation:

$$\text{As}(X) = \langle e_x \ (x \in X) \mid e_{x \triangleleft y}^{-1} \cdot e_y^{-1} \cdot e_x \cdot e_y \ (x, y \in X) \rangle.$$

It is known that the correspondence $X \mapsto \text{As}(X)$ yields a functor from the category of quandles to that of groups with left-adjointness.

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Let us briefly explain some significance to determine $\text{As}(X)$. First, when the adjoint functor $F : \mathcal{C} \rightleftharpoons \mathcal{C}' : G$ is given, the difference between $\text{id}_{\mathcal{C}}$ and $G \circ F$ is important in some areas (see [Ger, N3] for the quantum representations, or see [AG, CJKLS, FRS, Kab, Joy, N1] and references therein for knot-invariants and pointed Hopf algebras), with a relation to centrally extended groups. Furthermore, as a result of Eisermann [Eis2] (see also Theorem 4.1), the second quandle homology $H_2^Q(X)$ can be computed from concrete expressions of $\text{As}(X)$. Moreover, the main theorem in [N2] shows that certain universal knot-invariants from quandles turn out to be characterized by the third group homology $H_3^{\text{gr}}(\text{As}(X))$. However, as we see the definition of $\text{As}(X)$ or some explicit computation [Cla2], it has been considered to be hard to deal with $\text{As}(X)$ concretely.

In this paper, we develop a method for formulating practically $\text{As}(X)$ in a purely algebraic way. This method is roughly summarized to ‘universal central extensions of groups modulo type-torsion’ (see §2–3); the main theorem 2.1 emphasizes importance of the concept of types. Furthermore, Section 4 demonstrates practical applications of the method. Actually, we succeed in determining $\text{As}(X)$ and the associated homology $H_2^Q(X)$ of some quandles X (up to torsion). As a special case, Subsection 4.5 compares the main theorem with Howlett’s theorem [How] concerning the Schur multipliers of Coxeter groups. Furthermore, in Section 5, we will see that the method is applicable to coverings in quandle theory.

Notation and convention. For a group G , we denote by $H_n^{\text{gr}}(G)$ the usual group homology in trivial integral coefficients. Moreover, a homomorphism $f : A \rightarrow B$ between abelian groups is said to be a $[1/N]$ -isomorphism and is denoted by $f : A \cong_{[1/N]} B$, if the localization of f at ℓ is an isomorphism for any prime ℓ that does not divide N . This paper does not need any basic knowledge in quandle theory, but assumes basic facts of group cohomology as in [Bro, Sections I, II and VII].

§ 2. Preliminaries and the main theorem

This section aims to state Theorem 2.1. We start by reviewing properties of quandles. A quandle X is said to be of type t_X , if $t_X > 0$ is the minimal N such that $x = x \triangleleft^N y$ for any $x, y \in X$, where we denote by $\bullet \triangleleft^N y$ the N -times on the right operation with y . Note that, if X is of finite order, it is of type t_X for some $t_X \in \mathbb{Z}$.

Next, let us study the adjoint group $\text{As}(X)$ in some details. Define a right action $\text{As}(X)$ on X by $x \cdot e_y := x \triangleleft y$ for $x, y \in X$. Notice the equality

$$(2.1) \quad e_{x \cdot g} = g^{-1} e_x g \in \text{As}(X) \quad (x \in X, \ g \in \text{As}(X)),$$

by definitions. The orbits of this action of $\text{As}(X)$ on X are called *connected components* of X , denoted by $O(X)$. For $i \in O(X)$, we let $X_i \subset X$ be the orbit with respect to

i . If the action is transitive (i.e., $O(X)$ is a singleton), X is said to be *connected*. Furthermore, with respect to $i \in O(X)$, define a homomorphism

$$(2.2) \quad \varepsilon_i : \text{As}(X) \longrightarrow \mathbb{Z} \quad \text{by} \quad \begin{cases} \varepsilon_i(e_x) = 1 \in \mathbb{Z}, & \text{if } x \in X_i, \\ \varepsilon_i(e_x) = 0 \in \mathbb{Z}, & \text{if } x \in X \setminus X_i. \end{cases}$$

Note that the direct sum $\oplus_{i \in O(X)} \varepsilon_i$ yields the abelianization $\text{As}(X)_{\text{ab}} \cong \mathbb{Z}^{\oplus O(X)}$ by (2.1), which means that the group $\text{As}(X)$ is of infinite order. Furthermore, if $O(X)$ is a singleton, we often omit writing the index i .

In addition, we briefly review *the inner automorphism group*, $\text{Inn}(X)$, of a quandle X . Regard the action of $\text{As}(X)$ as a group homomorphism ψ_X from $\text{As}(X)$ to the symmetric group $\text{Bij}(X, X)$. The group $\text{Inn}(X)$ is defined as the image $\text{Im}(\psi_X) \subset \text{Bij}(X, X)$. Hence, we have a group extension

$$(2.3) \quad 0 \longrightarrow \text{Ker}(\psi_X) \longrightarrow \text{As}(X) \xrightarrow{\psi_X} \text{Inn}(X) \longrightarrow 0 \quad (\text{exact}).$$

By the equality (2.1), this kernel $\text{Ker}(\psi_X)$ is contained in the center. Therefore, it is natural to focus on their second group homology; we show a theorem on $H_2^{\text{gr}}(\text{As}(X))$ as a useful estimate:

Theorem 2.1. *For any connected quandle X of type t_X (possibly, X could be of infinite order), the second group homology $H_2^{\text{gr}}(\text{As}(X))$ is annihilated by t_X . Furthermore, the abelian kernel $\text{Ker}(\psi_X)$ in (2.3) is $[1/t_X]$ -isomorphic to $\mathbb{Z} \oplus H_2^{\text{gr}}(\text{Inn}(X))$.*

The proof will appear in §6. In conclusion, metaphorically speaking, $\text{As}(X)$ turns out to be the ‘universal central extension’ of $\text{Inn}(X)$ up to t_X -torsion; hence, this theorem emphasizes importance of the concept of types; so as to investigate $\text{As}(X)$, we shall study $\text{Inn}(X)$ and $H_2^{\text{gr}}(\text{Inn}(X))$.

§ 3. Methods on inner automorphism groups.

Following the preceding theorem to study the group $\text{As}(X)$, we shall develop a method for describing the inner automorphism group $\text{Inn}(X)$:

Theorem 3.1. *Let a group G act on a quandle X . Let a map $\kappa : X \rightarrow G$ satisfy the followings:*

1. *The identity $x \triangleleft y = x \cdot \kappa(y) \in X$ holds for any $x, y \in X$.*
2. *The image $\kappa(X) \subset G$ generates the group G , and the action $X \curvearrowright G$ is effective.*

Then, there is an isomorphism $\text{Inn}(X) \cong G$, and the action $X \curvearrowright G$ agrees with the natural action of $\text{Inn}(X)$.

Proof. Identify the action $X \curvearrowright G$ with a group homomorphism $f : G \rightarrow \text{Bij}(X, X)$. It follows from the first assumption that $f(\kappa(X)) \subset \text{Inn}(X)$ and $f(\kappa(X))$ generates $\text{Inn}(X)$; thus, f gives rise to an epimorphism $F : \langle \kappa(X) \rangle \rightarrow \text{Inn}(X)$, where $\langle \kappa(X) \rangle$ is the subgroup of G generated by $\kappa(X)$. Then, the second assumption ensures that $\langle \kappa(X) \rangle = G$, and the bijectivity of F , i.e., $\text{Inn}(X) \cong G$. Moreover, the agreement of the two actions follows by construction. $\square$

This theorem is applicable to many quandles, in practice. Actually, as seen in Section 4, we can determine $\text{Inn}(X)$ of many quandles X . However, we here explain that this theorem is inspired by the Cartan embeddings in symmetric space theory as follows:

Example 3.2. Let X be a symmetric space in differential geometry. Consider the group $\text{Inn}(X) \subset \text{Diff}(X)$ generated by the symmetries $\bullet \triangleleft y$ with compact-open topology. As is well known, $\text{Inn}(X)$ has a Lie group structure, and the map $X \rightarrow \text{Inn}(X)$ that sends y to $\bullet \triangleleft y$ is commonly called the Cartan embedding. As seen in textbooks on symmetric spaces, Theorem 3.1 had been used to determine $\text{Inn}(X)$ concretely.

Furthermore, we suggest another computation when $\text{Inn}(X)$ is perfect.

Proposition 3.3. *Let X be a quandle, and $O(X)$ be the set of orbits of the action $X \curvearrowright \text{As}(X)$. Set the epimorphism $\varepsilon_i : \text{As}(X) \rightarrow \mathbb{Z}$ associated with $i \in O(X)$ defined in (2.2). If the group $\text{Inn}(X)$ is perfect, i.e., $H_1^{\text{gr}}(\text{Inn}(X)) = 0$, then we have an isomorphism*

$$(3.1) \quad \text{As}(X) \cong \text{Ker}(\oplus_{i \in O(X)} \varepsilon_i) \times \mathbb{Z}^{\oplus O(X)},$$

and this $\text{Ker}(\oplus_{i \in O(X)} \varepsilon_i)$ is a central extension of $\text{Inn}(X)$ and is perfect. In particular, if X is connected and the group homology $H_2^{\text{gr}}(\text{Inn}(X))$ vanishes, then $\text{As}(X) \cong \text{Inn}(X) \times \mathbb{Z}$.

Proof. We will show the isomorphism (3.1). By the assumption $H_1^{\text{gr}}(\text{Inn}(X)) = 0$, the composite $\text{Ker}(\psi_X) \hookrightarrow \text{As}(X) \xrightarrow{\text{proj.}} H_1^{\text{gr}}(\text{As}(X)) = \mathbb{Z}^{\oplus O(X)}$ obtained from (2.3) is surjective. Since $\mathbb{Z}^{\oplus O(X)}$ is free, we can choose a section $\mathfrak{s} : \mathbb{Z}^{\oplus O(X)} \rightarrow \text{Ker}(\psi_X)$ of the composite. Hence, by the equality (2.1) and the inclusion $\text{Ker}(\psi_X) \subset \text{As}(X)$, the semi-direct product $\text{As}(X) \cong \text{Ker}(\oplus_{i \in O(X)} \varepsilon_i) \rtimes \mathbb{Z}^{\oplus O(X)}$ is trivial, leading to (3.1) as desired. Furthermore the kernel $\text{Ker}(\oplus_{i \in O(X)} \varepsilon_i)$ is a central extension of $\text{Inn}(X)$ by construction, and is perfect by the Kunneth theorem and $\text{As}(X)_{\text{ab}} \cong \mathbb{Z}^{\oplus O(X)}$. Hence we complete the proof. $\square$

Remark. In general, the kernel $\text{Ker}(\oplus_{i \in O(X)} \varepsilon_i)$ is not always the universal central extension of the perfect group $\text{Inn}(X)$; see [N3, Theorem 4] with $g = 3$ as a counterexample such that the extension $\text{Ker}(\oplus_{i \in O(X)} \varepsilon_i) \rightarrow \text{Inn}(X)$ is not universal.

Finally, we conclude this section by giving two lemmas on $\text{As}(X)$, which are used later.

Lemma 3.4. *Let X be a connected quandle of type $t < \infty$. Then, for any $x, y \in X$, we have the identity $(e_x)^t = (e_y)^t$ in the center of $\text{As}(X)$.*

Proof. For every $b \in X$, note the equalities $(e_x)^{-t} e_b e_x^t = e_{(\dots(b \triangleleft x) \dots) \triangleleft x} = e_b$ in $\text{As}(X)$. Namely $(e_x)^t$ lies in the center. Furthermore the connectivity admits $g \in \text{As}(X)$ such that $x \cdot g = y$. Hence, it follows from (2.1) that $(e_x)^t = g^{-1}(e_x)^t g = (e_{x \cdot g})^t = (e_y)^t$ as desired. $\square$

Lemma 3.5. *Let X be a connected quandle of finite order. Then $|\text{Inn}(X)|/|X|$ is divisible by its type t_X .*

Proof. For $x, y \in X$, we define $m_{x,y}$ as the minimal n satisfying $x \triangleleft^n y = x$. Note that $(\bullet \triangleleft^{m_{x,y}} y)$ lies in the stabilizer $\text{Stab}(x)$. Since $|\text{Stab}(x)| = |\text{Inn}(X)|/|X|$ by connectivity, any $m_{x,y}$ divides $|\text{Inn}(X)|/|X|$; hence so does the type t_X . $\square$

Furthermore, in some cases, we can calculate some torsion parts of their group homology:

Lemma 3.6. *Let X be a connected quandle of type t_X . If $H_2^{\text{gr}}(\text{Inn}(X))$ is annihilated by $t_X < \infty$, then there is a $[1/t_X]$ -isomorphism $H_3^{\text{gr}}(\text{Adj}(X)) \cong H_3^{\text{gr}}(\text{Inn}(X))$.*

Proof. Consider the Lyndon-Hochschild spectral sequence of (2.3). It is sufficient for the proof to show that the differential

$$d_2 : E_{3,0}^2 = H_3(\text{Inn}(X); H_0(\text{Ker}(\psi_X))) \longrightarrow E_{1,1}^2 = H_1^{\text{gr}}(\text{Inn}(X); H_1^{\text{gr}}(\text{Ker}(\psi_X)))$$

is trivial modulo t_X . To show this, the inflation-restriction sequence of (2.3)

$$\text{Ker}(\psi_X) \longrightarrow H_1^{\text{gr}}(\text{As}(X); \mathbb{Z}) \longrightarrow H_1^{\text{gr}}(\text{Inn}(X); \mathbb{Z}) \longrightarrow 0$$

and Lemma 3.4 imply that $H_1^{\text{gr}}(\text{Inn}(X); \mathbb{Z})$ is annihilated by t_X . Furthermore, we obtain the $[1/t_X]$ -isomorphism $\text{Ker}(\psi_X) \cong_{[1/t_X]} \mathbb{Z}$ from Theorem 2.1. Hence, the image of d_2 is trivial as required. $\square$

§ 4. Six examples of $\text{As}(X)$ and second quandle homology

Based on the preceding results on $\text{As}(X)$, this section calculates $\text{Inn}(X)$ and $\text{As}(X)$ for six kinds of connected quandles X : Alexander, symplectic, spherical, Dehn, Coxeter and core quandles. These quandles are dealt with in six subsections in turn.

Furthermore, to determine the second quandle homology $H_2^Q(X)$ in trivial $\mathbb{Z}$ -coefficients (see §6 for the definition), we will employ the following computation of Eisermann:

Theorem 4.1 ([Eis2, Theorem 1.15]). *Let X be a connected quandle. Fix an element $x_0 \in X$. Let $\text{Stab}(x_0) \subset \text{As}(X)$ be the stabilizer of x_0 , and $\varepsilon : \text{As}(X) \rightarrow \mathbb{Z}$ be the abelianization mentioned in (2.2). Then, $H_2^Q(X)$ is isomorphic to the abelianization of $\text{Stab}(x_0) \cap \text{Ker}(\varepsilon)$.*

§ 4.1. Alexander quandles

We start by discussing the class of Alexander quandles. Every $\mathbb{Z}[T^{\pm 1}]$ -module X has a quandle structure with the operation $x \triangleleft y = y + T(x - y)$ for $x, y \in X$, and is called *the Alexander quandle*. This operation $\bullet \triangleleft y$ can be geometrically compared to the T -multiple with center y . The type is the minimal N such that $T^N = \text{id}_X$ since $x \triangleleft^n y = y + T^n(x - y)$. Furthermore, it can be easily verified that an Alexander quandle X is connected if and only if $(1 - T)X = X$.

Let us review the concrete presentation of $\text{As}(X)$, which is due to Clauwens [Cla2]. When X is connected, set up the homomorphism $\mu_X : X \otimes X \rightarrow X \otimes X$ defined by $\mu_X(x \otimes y) = x \otimes y - Ty \otimes x$. Further, he defined a group operation on $\mathbb{Z} \times X \times \text{Coker}(\mu_X)$ by setting

$$(n, x, \alpha) \cdot (m, y, \beta) = (n + m, T^m x + y, \alpha + \beta + [T^m x \otimes y]),$$

and constructed a group isomorphism $\text{As}(X) \rightarrow \mathbb{Z} \times X \times \text{Coker}(\mu_X)$, which takes e_x to $(1, x, 0)$. As a result, the kernel of $\psi_X : \text{As}(X) \rightarrow \text{Inn}(X)$ equals $t_X \mathbb{Z} \times \text{Coker}(\mu_X)$.

Thanks to his presentation of $\text{As}(X)$, we can easily show a result of Clauwens that determines the homology $H_2^Q(X)$ of a connected Alexander quandle X . To be precise,

Proposition 4.2 (Clauwens [Cla2]). *Let X be a connected Alexander quandle. The homology $H_2^Q(X)$ is isomorphic to the quotient module $\text{Coker}(\mu_X) = X \otimes_{\mathbb{Z}} X / (x \otimes y - Ty \otimes x)_{x, y \in X}$.*

Proof. By definition we can see that the $\text{Ker}(\varepsilon) \cap \text{Stab}(0)$ is the cokernel $\text{Coker}(\mu_X)$. □

§ 4.2. Symplectic quandles

Let K be a commutative field, and let Σ_g be the closed surface of genus g . Consider the multiplicative group $K^\times$, and the quotient $K^\times / (K^\times)^2$ modulo 2. For $[r] \in K^\times / (K^\times)^2$, we fix a representative $r \in K^\times$, and consider the copy of $H^1(\Sigma_g; K) \setminus \{0\} =$

$K^{2g} \setminus \{0\}$, denoted by X_r . Let X be the union $\cup_{r \in K^\times / (K^\times)^2} X_r$ (here, we should notice that $X = X_r$ if K is an algebraically closed field.). Using the standard symplectic 2-form $\langle, \rangle : H^1(\Sigma_g; K) \times H^1(\Sigma_g; K) \rightarrow K$, the set X is made into a quandle by the operation $x \triangleleft y := r\langle x, y \rangle y + x \in X$ for $x \in X_r$ and $y \in X$, and is called a *symplectic quandle (over K)*. The operation $\bullet \triangleleft y : X \rightarrow X$ is commonly called *the transvection of y* . Note that the type of the quandle X is the characteristic of K since $x \triangleleft^N y = Nr\langle x, y \rangle y + x$.

We will determine $\text{Inn}(X)$ and $\text{As}(X)$ associated with the symplectic quandle X over K .

Lemma 4.3. *$\text{Inn}(X)$ is isomorphic to the symplectic group $Sp(2g; K)$.*

Proof. Recall from the Cartan-Dieudonné theorem that the classical group $Sp(2g; K)$ is generated by transvections $(\bullet \triangleleft y)$.

We will show the desired isomorphism. For any $y \in X$, the map $(\bullet \triangleleft y) : X \rightarrow X$ is a restriction of a linear map $K^{2g} \rightarrow K^{2g}$. It thus yields a map $\kappa : X \rightarrow GL(2g; K)$, which factors through $Sp(2g; K)$ and satisfies the conditions in Theorem 3.1. Indeed, the condition (2) follows from the classical theorem and the effectivity of the standard action $K^{2g} \curvearrowright Sp(2g; K)$. Therefore $\text{Inn}(X) \cong Sp(2g; K)$ as desired. $\square$

Proposition 4.4. *Take a field K of positive characteristic p and with $|K| > 10$. Assume the connectivity, that is, every $x \in K$ admits a square $\sqrt{x}$ in K . Let $X = K^2 \setminus \{0, 0\}$ be the symplectic quandle over K , and $\widetilde{Sp}(2g; K)$ be the universal central extension of $Sp(2g; K)$. Then $\text{As}(X) \cong \mathbb{Z} \times \widetilde{Sp}(2g; K)$.*

Proof. Since X is connected and $\text{Inn}(X) \cong Sp(2g; K)$ by Lemma 4.3, Proposition 3.3 implies $\text{As}(X) \cong \text{Ker}(\varepsilon) \times \mathbb{Z}$. Further, it follows from Theorem 2.1 that $H_2^{\text{gr}}(\text{As}(X))$ is annihilated by p . Hence, following the fact [Sus] that $H_2^{\text{gr}}(Sp(2g; K))$ has no p -torsion, the kernel $\text{Ker}(\varepsilon)$ must be the universal central extension of $Sp(2g; K)$, which completes the proof. $\square$

Remark. This proposition holds even if the characteristic of K is zero and X is not connected; see [N2] for the proof. Furthermore, the paper [N2] also determines $H_2^Q(X)$ in the case where K is of infinite order.

Accordingly, hereafter, we will focus on finite fields $K = \mathbb{F}_q$ with $q > 10$:

Proposition 4.5. *Let X be the symplectic quandle over $\mathbb{F}_q$. If $q > 10$, then $\text{As}(X) \cong \mathbb{Z}^{O(X)} \times Sp(2g; \mathbb{F}_q)$. Furthermore, $H_3^{\text{gr}}(\text{As}(X)) \cong \mathbb{Z}/(q^2 - 1)$.*

Proof. Since $(\mathbb{F}_q)^\times$ is cyclic, we first should notice that, if q is even $|O(X)| = 1$, and that, if q is odd, $|O(X)| = 2$ or 1 according to $q = 4r + 1$ or $q = 4r + 3$ for some $r \in \mathbb{Z}$.

Since $q > 10$, the first and second homology groups of $\text{Inn}(X) \cong Sp(2g; \mathbb{F}_q)$ are known to be zero (see [FP, Fri]). Thus, $\widetilde{Sp}(2g; \mathbb{F}_q) = Sp(2g; \mathbb{F}_q)$, leading to $\text{As}(X) \cong \mathbb{Z}^{O(X)} \times Sp(2g; \mathbb{F}_q)$ as stated. Furthermore, the latter part follows from the result $H_3^{\text{gr}}(Sp(2g; \mathbb{F}_q)) \cong \mathbb{Z}/(q^2 - 1)$ in [FP, Fri]. $\square$

As a result, we will determine the second homology $H_2^Q(X)$.

Proposition 4.6. *Let $q > 10$, and X be as above. If $g \geq 2$, the homology $H_2^Q(X)$ vanishes. If $g = 1$, then $H_2^Q(X) \cong (\mathbb{Z}/p)^{d|O(X)|}$, where $q = p^d$.*

Proof. Recall $\text{As}(X) \cong \mathbb{Z}^{O(X)} \times Sp(2g; \mathbb{F}_q)$. Considering the standard action $X \curvearrowright Sp(2g; \mathbb{F}_q)$, denote by G_X the stabilizer of $(1, 0, \dots, 0) \in (\mathbb{F}_q)^{2g}$. Since Theorem 4.1 immediately means $H_2^Q(X) \cong H_1^{\text{gr}}(G_X)^{|O(X)|}$, we will calculate $H_1^{\text{gr}}(G_X)$ as follows. First, for $g = 1$, it can be verified that the stabilizer G_X is exactly the product $(\mathbb{Z}/p)^d$ as an abelian group; hence $H_2^Q(X) \cong (\mathbb{Z}/p)^{d|O(X)|}$ in the sequel. Next, for $g \geq 2$, the vanishing $H_2^Q(X) = H_1^{\text{gr}}(G_X) = 0$ immediately follows from Lemma 4.7 below. $\square$

Lemma 4.7. *Let $g \geq 2$ and $q > 10$. Let G_X denote the stabilizer of the action $X \curvearrowright Sp(2g; \mathbb{F}_q)$ mentioned above. Then the homology groups $H_1^{\text{gr}}(G_X)$ and $H_2^{\text{gr}}(G_X)$ vanish.*

Proof. Since $q > 10$, recall from [FP, II. §6.3] the order of $Sp(2g; \mathbb{F}_q)$ as

$$|Sp(2g; \mathbb{F}_q)| = q^{g^2}(q^{2g} - 1)(q^{2g-2} - 1) \cdots (q^2 - 1).$$

Since $|X| = q^{2g} - 1$, the order of G_X is equal to $q^{g^2} \cdot |Sp(2g-2; \mathbb{F}_q)|$. Thereby $H_1^{\text{gr}}(G_X)$ and $H_2^{\text{gr}}(G_X)$ are zero up to p -torsion, because of the inclusion $Sp(2g-2; \mathbb{F}_q) \subset G_X$ by definitions and the vanishing $H_1^{\text{gr}} \oplus H_2^{\text{gr}}(Sp(2g-2; \mathbb{F}_q)) \cong 0$ up to p torsion.

Finally, we may focus on the p -torsion of $H_1^{\text{gr}} \oplus H_2^{\text{gr}}(G_X)$. Following the proof of [Fri, Proposition 4.4], there is a certain subgroup “ $\Delta(Sp(2g; \mathbb{F}_q))$ ” of G_X which contains a p -sylog group of $Sp(2g; \mathbb{F}_q)$ and this $\mathbb{Z}/p$ -homology vanishes. Hence, $H_1^{\text{gr}} \oplus H_2^{\text{gr}}(G_X) = 0$ as required. $\square$

§ 4.3. Spherical quandles

Let K be a field of characteristic not equal to 2, and fix $n \geq 2$ in this subsection. Take the standard symmetric bilinear form $\langle \cdot, \cdot \rangle : K^{n+1} \otimes K^{n+1} \rightarrow K$. Consider a set of the form

$$S_K^n := \{ x \in K^{n+1} \mid \langle x, x \rangle = 1 \}.$$

We define the operation $x \triangleleft y$ to be $2\langle x, y \rangle y - x \in S_K^n$. The pair $(S_K^n, \triangleleft)$ is a quandle of type 2, and is referred to as a *spherical quandle* (over K). This operation $\bullet \triangleleft y$ can

be interpreted as a linear transformation which identically acts on y and $-\text{Id}$ on the the subspace orthogonal to y .

Then, similar to the proof of Lemma 4.3, one can readily determine $\text{Inn}(X)$ as follows:

Lemma 4.8. *If n is odd, then $\text{Inn}(S_K^n)$ is isomorphic to the orthogonal group $O(n+1; K)$. If n is even, $\text{Inn}(S_K^n)$ is isomorphic to $SO(n+1; K)$.*

Next, we will focus on second homology group and $H_3^{\text{gr}}(\text{As}(X))$ of spherical quandles over $\mathbb{F}_q$. Here, the results are up to 2-torsion, whereas the 2-torsion part is the future problem.

Proposition 4.9. *Let X be a spherical quandle over $\mathbb{F}_q$. Let $q > 10$. For $n \geq 3$, the second homology $H_2^Q(X)$ is annihilated by 2. If $n = 1$, then the homology $H_2^Q(X)$ is $[1/2]$ -isomorphic to the cyclic group $\mathbb{Z}/(q - \delta_q)$, where $\delta_q = \pm 1$ is according to $q \equiv \pm 1 \pmod{4}$.*

Proof. Assume n is odd. Under the standard action $X \curvearrowright O(n+1; \mathbb{F}_q)$, the stabilizer of $(1, 0, \dots, 0) \in X$ is $O(n; \mathbb{F}_q)$. By a similar discussion to the proof of Proposition 4.6, $H_2^Q(X) \cong H_1^{\text{gr}}(O(n; \mathbb{F}_q))$ modulo 2-torsion. For $n \geq 3$, the abelianization of $O(n; \mathbb{F}_q)$ is $(\mathbb{Z}/2)^2$; see [FP, II. §3]; hence the $H_2^Q(X)$ is annihilated by 2 as required. The same discussion in the even case of n works well, since the inclusion $SO(n) \rightarrow O(n)$ induces $H_*(SO(n; \mathbb{F}_q)) \cong_{[1/2]} H_*(O(n; \mathbb{F}_q))$ modulo 2-torsion.

Finally, when $n = 1$, the group $O(2; \mathbb{F}_q)$ is cyclic and of order $q - \delta_q$. Hence $H_2^Q(X) \cong H_1^{\text{gr}}(O(2; \mathbb{F}_q)) \cong_{[1/2]} \mathbb{Z}/(q - \delta_q)$. $\square$

Proposition 4.10. *Let $q > 10$. Then $H_3^{\text{gr}}(\text{As}(X)) \cong_{[1/2]} H_3^{\text{gr}}(O(n+1; \mathbb{F}_q))$ up to 2-torsion.*

Proof. Since $q > 10$, $H_1^{\text{gr}} \oplus H_2^{\text{gr}}(O(n+1; \mathbb{F}_q))$ is known to be annihilated by 2; see [Fri, FP]. Hence, the conclusion readily results from Lemma 3.6. $\square$

§ 4.4. Dehn quandle

Changing the subject, we now review Dehn quandle [Yet]. Denote by $\mathcal{M}_g$ the mapping class group of Σ_g , and consider the set, $\mathcal{D}_g$, defined by

$$\mathcal{D}_g := \{ \text{isotopy classes of (unoriented) non-separating simple closed curves } \gamma \text{ in } \Sigma_g \}.$$

For $\alpha, \beta \in \mathcal{D}_g$, we define $\alpha \triangleleft \beta \in \mathcal{D}_g$ by $\tau_\beta(\alpha)$, where $\tau_\beta \in \mathcal{M}_g$ is the positive Dehn twist along β . The pair $(\mathcal{D}_g, \triangleleft)$ is a quandle, and called (*non-separating*) *Dehn quandle*. As is

well-known, any two non-separating simple closed curves are conjugate by the product of some Dehn twists. Hence, the quandle $\mathcal{D}_g$ is connected, and is not of any type t . The Dehn quandle $\mathcal{D}_g$ is applicable to study 4-dimensional Lefschetz fibrations (see, e.g., [Yet, Zab, N3]). The natural inclusion $\kappa : \mathcal{D}_g \rightarrow \mathcal{M}_g$ implies $\text{Inn}(\mathcal{D}_g) \cong \mathcal{M}_g$ by Theorem 3.1. Furthermore, if $g \geq 4$, there is an isomorphism $\text{As}(\mathcal{D}_g) \cong \mathbb{Z} \times \mathcal{T}_g$ shown by [Ger], where $\mathcal{T}_g$ is the universal central extension of $\mathcal{M}_g$ associated with $H_2^{\text{gr}}(\mathcal{M}_g) \cong \mathbb{Z}$.

The result of this subsection is the following:

Proposition 4.11. *If $g \geq 5$, then $H_2^Q(\mathcal{D}_g) \cong \mathbb{Z}/2$.*

Proof. We will use the facts that an epimorphism $G \rightarrow H$ between groups induces an epimorphism $G_{\text{ab}} \rightarrow H_{\text{ab}}$, and that $\mathcal{M}_{g,r}$ is perfect.

Fixing $\alpha \in \mathcal{D}_g$, we begin by observing the stabilizer $\text{Stab}(\alpha) \subset \text{As}(\mathcal{D}_g)$. Note that the map $\mathcal{D}_g \rightarrow \mathcal{M}_g$ sending β to τ_β yields a group epimorphism $\pi : \text{As}(\mathcal{D}_g) \rightarrow \mathcal{M}_g$. Furthermore, by Proposition 3.3, the restriction of π to $\text{Ker}(\varepsilon) \cong \mathcal{T}_g$ coincides with the projection $\mathcal{T}_g \rightarrow \mathcal{M}_g$. In particular, we thus have $\pi(\text{Stab}(\alpha)) = \pi(\text{Stab}(\alpha) \cap \text{Ker}(\varepsilon)) \subset \mathcal{M}_g$.

We will construct a surjection $H_2^Q(\mathcal{D}_g) \rightarrow \mathbb{Z}/2$. By the virtue of Theorem 4.1, it is enough to construct a surjection from the preceding $\pi(\text{Stab}(\alpha) \cap \text{Ker}(\varepsilon))$ to $\mathbb{Z}/2$ for $g \geq 2$. As is shown [PR, Proposition 7.4], we have the following exact sequence:

$$(4.1) \quad 0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{M}_{g-1,2} \xrightarrow{\xi} \pi(\text{Stab}(\alpha)) \xrightarrow{\lambda} \mathbb{Z}/2 \quad (\text{exact}).$$

Here ξ is the homomorphism induced from the gluing $(\Sigma_{g-1,2}, \partial(\Sigma_{g-1,2})) \rightarrow (\Sigma_g, \alpha)$, and λ is defined by the transposition of the connected components of boundaries of $\Sigma_g \setminus \alpha$. By considering a hyper-elliptic involution preserving the above α , the map λ is surjective. Hence $\pi(\text{Stab}(\alpha) \cap \text{Ker}(\varepsilon))$ surjects onto $\mathbb{Z}/2$ as desired.

Finally, we will complete the proof. By Theorem 4.1 again, recall that $(\text{Stab}(\alpha) \cap \text{Ker}(\varepsilon))_{\text{ab}} \cong H_2^Q(\mathcal{D}_g)$. To compute this, put the inclusion $\iota : \pi(\text{Stab}(\alpha)) \rightarrow \mathcal{M}_g$. By the Harer-Ivanov stability theorem (see [Iva]), the composition $\iota \circ \xi : \mathcal{M}_{g-1,2} \rightarrow \mathcal{M}_g$ induces an epimorphism

$$(4.2) \quad (\iota \circ \xi)_* : H_2^{\text{gr}}(\mathcal{M}_{g-1,2}; \mathbb{Z}) \longrightarrow H_2^{\text{gr}}(\mathcal{M}_g; \mathbb{Z}) \quad \text{for } g \geq 5.$$

Since $H_2^{\text{gr}}(\mathcal{M}_{g-1,2}; \mathbb{Z}) \cong H_2^{\text{gr}}(\mathcal{M}_g; \mathbb{Z}) \cong \mathbb{Z}$ is known (see, e.g., [FM]), the epimorphism (4.2) is isomorphic. Let $(\iota \circ \xi)^*(\mathcal{T}_g)$ denote the central extension of $\mathcal{M}_{g-1,2}$ obtained by $\iota \circ \xi$. Since $\mathcal{M}_g$ and $\mathcal{M}_{g-1,2}$ are perfect, the group $(\iota \circ \xi)^*(\mathcal{T}_g)$ is also perfect by the isomorphism (4.2). Note that the group $\text{Stab}(\alpha) \cap \text{Ker}(\varepsilon)$ is isomorphic to $\iota^*(\mathcal{T}_g)$. Hence the abelianization $(\text{Stab}(\alpha) \cap \text{Ker}(\varepsilon))_{\text{ab}}$ never be bigger than $\mathbb{Z}/2$. In conclusion, we arrive at the conclusion. $\square$

§ 4.5. Coxeter quandles

We will focus on Coxeter quandles, and study the associated groups, and show Theorem 4.12.

This subsection assumes basic knowledge of Coxeter groups, as explained in [Aki1, How]. Given a Coxeter graph Γ , we can set the Coxeter group W . Let X_Γ be the set of the reflections in W , that is, the set of elements conjugate to the generators of W . Equipping X_Γ with conjugacy operation, X_Γ is made into a quandle of type 2. Denote the inclusion $X_\Gamma \hookrightarrow W$ by κ . Since W subject to the center Z_W effectivity acts on X_Γ , we have $\text{Inn}(X_\Gamma) \cong W/Z_W$. Moreover, W is, by definition, isomorphic to the quotient of $\text{As}(X_\Gamma)$ subject to the squared relations $(e_x)^2 = 1$ for any $x \in X_\Gamma$.

In this situation, we now give another easy proof of a part of the theorem shown by Howlett:

Theorem 4.12 (A connected result in [How, §2–4]). *Assume that the Coxeter quandle X_Γ is connected. Then, the second group homology $H_2^{\text{gr}}(W)$ is annihilated by 2.*

Proof. Recall from Theorem 2.1 that $H_1^{\text{gr}}(\text{As}(X_\Gamma)) \cong \mathbb{Z}$ and $H_2^{\text{gr}}(\text{As}(X_\Gamma))$ is annihilated by 2. Therefore, the inflation-restriction exact sequence from the central extension $\text{As}(X_\Gamma) \rightarrow W$ implies the desired 2-vanishing of $H_2^{\text{gr}}(W)$. $\square$

Finally, we will end this subsection by giving some comments. Recently, Akita [Aki2] determined the associated group $\text{As}(X_\Gamma)$ as a $\mathbb{Z}^N$ -central extended group of W . Furthermore, concerning the third homology $H_3(\text{As}(X_\Gamma))$ in the case where X_Γ is connected, we obtain $H_3(\text{As}(X_\Gamma)) \cong H_3(W)$ up to 2-torsion from Lemma 3.6. The odd torsion of $H_3^{\text{gr}}(W)$ in a certain stable range is studied by Akita [Aki1].

§ 4.6. Core quandles

Given a group G , we let $X = G$ equipped with a quandle operation $g \triangleleft h := hg^{-1}h$. This quandle is called *core quandle* [Joy] and is of type 2. This last subsection will deal with core quandles, and show Proposition 4.13.

Let us give some terminologies to state the proposition. Let $\mathbb{Z}/2$ be $\{\pm 1\}$. Take the wreath product $(G \times G) \rtimes \mathbb{Z}/2$, and the commutator subgroup $[G, G]$. Consider the epimorphism $(G \times G) \rtimes \mathbb{Z}/2 \rightarrow G/[G, G]$ which sends (g, h, σ) to $[gh]$. Then, the kernel is formed as

$$\mathcal{G}_1 := \{ (g, h, \sigma) \in (G \times G) \rtimes \mathbb{Z}/2 \mid gh \in [G, G] \}.$$

Further, with respect to $x \in X$ and $(g, h, \sigma) \in \mathcal{G}_1$, we define $x \cdot (g, h, \sigma) := h^{-1}x^\sigma g$, which ensures an action of $\mathcal{G}_1$ on X . Further, consider a subgroup of the form

$$\mathcal{G}_2 := \{ (z, z, \sigma) \in (G \times G) \rtimes \mathbb{Z}/2 \mid z^2 \in [G, G], \quad k^{-1}zk = z^\sigma \text{ for any } k \in G \},$$

which is contained in the center of $\mathcal{G}_1$. Then, the quotient action subject to $\mathcal{G}_2$ is effective.

Proposition 4.13. *There is a group isomorphism $\text{Inn}(X) \cong \mathcal{G}_1/\mathcal{G}_2$.*

Proof. Consider the map $\kappa : X \rightarrow \mathcal{G}_1/\mathcal{G}_2$ which sends g to $[(g, g^{-1}, -1)]$. We claim that this $\mathcal{G}_1/\mathcal{G}_2$ is generated by the image $\mathfrak{S}(\kappa)$. Actually, we can easily verify that any element (g, h, σ) in $\mathcal{G}_1$ with $g_i, h_i \in G$ and $gh = g_1 h_1 g_1^{-1} h_1^{-1} \cdots g_m h_m g_m^{-1} h_m^{-1}$ is decomposed as

$$\kappa(1_G)^{\frac{\sigma+1}{2}} \cdot \kappa(gh^{-1}) \cdot \left((\kappa(g_1 h_1) \cdot \kappa(1_G) \cdot \kappa(g_1^{-1}) \cdot \kappa(h_1)) \cdots (\kappa(g_m h_m) \cdot \kappa(1_G) \cdot \kappa(g_m^{-1}) \cdot \kappa(h_m)) \right).$$

Then, the routine discussion from Lemma 3.1 completes the proof. $\square$

This proposition implies the difficulty to determine $\text{Inn}(X)$, in general. Thus, it also seems hard to determine $\text{As}(X)$. Actually, even if X is a connected core quandle, Proposition 4.13 implies that the kernel $\text{Ker}(\psi)$ is complicated by the reason of the second homology $H_2^{\text{gr}}(G)$ and $H_2^{\text{gr}}(\text{Inn}(X))$. For example, if X is the product of h -copies of the cyclic group $\mathbb{Z}/m$, i.e., X is the Alexander quandle of the form $(\mathbb{Z}/m)^h[T]/(T+1)$, then the kernel $\text{Ker}(\psi)$ stated in Proposition 4.2 is not so simple.

§ 5. On quandle coverings

This section suggests that the results in Section 2 are applicable to quandle coverings.

Let us review coverings in the sense of Eisermann [Eis2, Eis1]. A map $f : Y \rightarrow Z$ between quandles is a *(quandle) homomorphism*, if $f(a \triangleleft b) = f(a) \triangleleft f(b)$ for any $a, b \in Y$. Furthermore, a quandle epimorphism $p : Y \rightarrow Z$ is a *(quandle) covering*, if the equality $p(\tilde{x}) = p(\tilde{y}) \in Z$ implies $\tilde{a} \triangleleft \tilde{x} = \tilde{a} \triangleleft \tilde{y} \in Y$ for any $\tilde{a}, \tilde{x}, \tilde{y} \in Y$.

Let us mention a typical example. Given a connected quandle X with $a \in X$, recall the abelianization $\varepsilon_0 : \text{As}(X) \rightarrow \mathbb{Z}$ in (2.2). Then, the kernel $\text{Ker}(\varepsilon_0)$ has a quandle operation defined by setting

$$g \triangleleft h := e_a^{-1} g h^{-1} e_a h \quad \text{for } g, h \in \text{Ker}(\varepsilon_0).$$

We can easily see the independence of the choice of $a \in X$ up to quandle isomorphisms. One writes $\tilde{X}$ for the quandle $(\text{Ker}(\varepsilon_0), \triangleleft)$, which is considered in [Joy, §7]. When X is of type t_X , so is the extended one $\tilde{X}$ by Lemma 3.4. Furthermore, using the restricted action $X \curvearrowright \text{Ker}(\varepsilon_0) \subset \text{As}(X)$, we see that the map $p : \tilde{X} \rightarrow X$ sending g to $a \cdot g$ is a covering. This p is called *the universal (quandle) covering of X* , according to [Eis2, §5].

As a preliminary, we will explore some properties of quandle coverings.

Proposition 5.1. *For any quandle covering $p : Y \rightarrow Z$, the induced group surjection $p_* : \text{As}(Y) \rightarrow \text{As}(Z)$ is a central extension. Furthermore, if Y and Z are connected and Z is of type t_Z , then the abelian kernel $\text{Ker}(p_*)$ is annihilated by t_Z .*

Proof. Fix a section $\mathfrak{s} : Y \rightarrow Z$. For any $y \in Z$, put arbitrary $y_i \in p^{-1}(y)$. Then,

$$e_{\mathfrak{s}(y)}^{-1} e_b e_{\mathfrak{s}(y)} = e_{b \triangleleft \mathfrak{s}(y)} = e_{b \triangleleft y_i} = e_{y_i}^{-1} e_b e_{y_i} \in \text{As}(Y)$$

for any $b \in Y$. Here the second equality is due to the covering p . Denoting $e_{\mathfrak{s}(y)} e_{y_i}^{-1}$ by z_i , the equalities imply that z_i is central in $\text{As}(Y)$. Since $e_{\mathfrak{s}(y)} = z_i e_{y_i}$, $\text{As}(Y)$ is generated by $e_{\mathfrak{s}(y)}$ with $y \in Y$ and the central elements z_i associated with $y_i \in p^{-1}(y)$; consequently, the surjection p_* is a central extension.

We will show the latter part. Take the inflation-restriction exact sequence, i.e.,

$$H_2^{\text{gr}}(\text{As}(Z)) \longrightarrow \text{Ker}(p_*) \longrightarrow H_1^{\text{gr}}(\text{As}(Y)) \longrightarrow H_1^{\text{gr}}(\text{As}(Z)) \longrightarrow 0 \quad (\text{exact}).$$

By connectivities the third map from $H_1^{\text{gr}}(\text{As}(Y)) = \mathbb{Z}$ is an isomorphism. Since Theorem 2.1 says that $H_2^{\text{gr}}(\text{As}(Z))$ is annihilated by t_Z , so is the kernel $\text{Ker}(p_*)$ as desired. $\square$

Next, we will compute the second homology of $\tilde{X}$ (Theorem 5.4) by showing propositions:

Proposition 5.2. *For any connected quandle X , the extended one $\tilde{X}$ above is also connected.*

Proof. It is enough to show that the identity $1_{\tilde{X}} \in \tilde{X} = \text{Ker}(\varepsilon_0)$ is transitive to any element h in $\tilde{X}$. Expand $h \in \tilde{X} \subset \text{As}(X)$ as $h = e_{x_1}^{\epsilon_1} \cdots e_{x_n}^{\epsilon_n}$ for some $x_i \in X$ and $\epsilon_i \in \mathbb{Z}$. Since $h \in \text{Ker}(\varepsilon_0)$, note $\sum \epsilon_i = 0$. The connectivity of X ensures some $g_i \in \text{As}(X)$ so that $a \cdot g_i^{\epsilon_i} = x_i$. Therefore $g_i^{-\epsilon_i} e_a g_i^{\epsilon_i} = e_{a \cdot g_i^{\epsilon_i}} = e_{x_i}^{\epsilon_i}$ by (2.1). In the sequel, we have

$$(\cdots (1_{\tilde{X}} \triangleleft^{\epsilon_1} g_1) \cdots \triangleleft^{\epsilon_n} g_n) = e_a^{\sum \epsilon_i} 1_{\tilde{X}} (g_1^{-\epsilon_1} e_a g_1^{\epsilon_1}) \cdots (g_n^{-\epsilon_n} e_a g_n^{\epsilon_n}) = e_{x_1}^{\epsilon_1} \cdots e_{x_n}^{\epsilon_n} = h.$$

These equalities in $\tilde{X}$ imply the transitivity of $\tilde{X}$ $\square$

Proposition 5.3. *Let X be a connected quandle. Let $p_* : \text{As}(\tilde{X}) \rightarrow \text{As}(X)$ be the epimorphism induced from the covering $p : \tilde{X} \rightarrow X$. Then, under the canonical action of $\text{As}(\tilde{X})$ on $\tilde{X}$, the stabilizer $\text{Stab}(1_{\tilde{X}})$ of $1_{\tilde{X}}$ is equal to $\mathbb{Z} \times \text{Ker}(p_*)$ in $\text{As}(\tilde{X})$. Furthermore, the summand $\mathbb{Z}$ is generated by $1_{\tilde{X}}$.*

Proof. We can easily see that the stabilizer of $1_{\tilde{X}}$ via the action $\text{Ker}(\varepsilon_0) = \tilde{X} \curvearrowright \text{As}(X)$ is $\underline{\text{Stab}}(1_{\tilde{X}}) = \{e_a^n\}_{n \in \mathbb{Z}} \subset \text{As}(X)$ exactly. Notice that any central extension of $\mathbb{Z}$ is trivial; therefore, since p_* is a central extension (Proposition 5.1), the restriction $p_* : \text{Stab}(1_{\tilde{X}}) \rightarrow \underline{\text{Stab}}(1_{\tilde{X}}) = \mathbb{Z}$ implies the required identity $\text{Stab}(1_{\tilde{X}}) = \mathbb{Z} \times \text{Ker}(p_*)$. $\square$

Theorem 5.4. *The second quandle homology of the extended quandle $\tilde{X}$ is isomorphic to the kernel of $p_* : \text{As}(\tilde{X}) \rightarrow \text{As}(X)$. Namely, $H_2^Q(\tilde{X}) \cong \text{Ker}(p_*)$. In particular, it follows from Proposition 5.1 that, if $t_X < \infty$, then $H_2^Q(\tilde{X})$ is annihilated by the type t_X .*

Proof. Note that $\tilde{X}$ is connected (Proposition 5.2) and the kernel $\text{Ker}(p_*)$ is abelian (Proposition 5.1). Accordingly, the desired isomorphism $H_2^Q(\tilde{X}) \cong (\text{Ker}(\varepsilon_{\tilde{X}}) \cap \text{Stab}(1_{\tilde{X}}))_{\text{ab}} = \text{Ker}(p_*)$ follows immediately from Proposition 5.3 and Theorem 4.1. $\square$

Finally, we now discuss the third group homology.

Proposition 5.5. *The universal covering $p : \tilde{X} \rightarrow X$ induces a $[1/t_X]$ -isomorphism $p_* : H_3^{\text{gr}}(\text{As}(\tilde{X})) \cong H_3^{\text{gr}}(\text{As}(X))$.*

Proof. By connectivity of $\tilde{X}$ and Theorem 2.1, $H_2^{\text{gr}}(\text{As}(\tilde{X}))$ and $H_2^{\text{gr}}(\text{As}(X))$ are annihilated by t_X . Furthermore, since the epimorphism $p_* : \text{As}(\tilde{X}) \rightarrow \text{As}(X)$ is a central extension whose kernel is annihilated by t_X (Proposition 5.1), we readily obtain the $[1/t_X]$ -isomorphism $p_* : H_3^{\text{gr}}(\text{As}(\tilde{X})) \cong H_3^{\text{gr}}(\text{As}(X))$ from the Lyndon-Hochschild sequence of p_* . $\square$

These properties played a key role to prove the main theorem in [N1].

§ 6. Proof of Theorem 2.1.

The purpose of this section is to prove Theorem 2.1. Let us begin by reviewing the rack space introduced by Fenn-Rourke-Sanderson [FRS]. Let X be a quandle with discrete topology. We set up a disjoint union $\bigcup_{n \geq 0} ([0, 1] \times X)^n$, and consider the relations given by

$$(t_1, x_1, \dots, x_{j-1}, 0, x_j, t_{j+1}, \dots, t_n, x_n) \sim (t_1, x_1, \dots, t_{j-1}, x_{j-1}, t_{j+1}, x_{j+1}, \dots, t_n, x_n).$$

$$(t_1, x_1, \dots, x_{j-1}, 1, x_j, t_{j+1}, \dots, t_n, x_n) \sim$$

$$(t_1, x_1 \triangleleft x_j, \dots, t_{j-1}, x_{j-1} \triangleleft x_j, t_{j+1}, x_{j+1}, \dots, t_n, x_n),$$

Then, the *rack space* BX is defined to be the quotient space. By construction, we have a cell decomposition of BX by regarding the projection $\bigcup_{n \geq 0} ([0, 1] \times X)^n \rightarrow BX$ as characteristic maps. From the 2-skeleton of BX , we have $\pi_1(BX) \cong \text{As}(X)$. Considering the Eilenberg-MacLane space $K(\pi_1(BX), 1)$, we have the classifying map $c : BX \hookrightarrow K(\pi_1(BX), 1)$, i.e., an inclusion obtained by killing the higher homotopy groups of BX .

Theorem 6.1. *Let X be a connected quandle of type t , and let $t < \infty$. For $n = 2$ and 3 , the induced map $c_* : H_n(BX) \rightarrow H_n^{\text{gr}}(\text{As}(X))$ is annihilated by t .*

Remark. This is still more powerful and general than a result of Clauwens [Cla1, Proposition 4.4], which stated that, if a quandle X of finite order satisfies a certain condition, then the composite $(\psi_X)_* \circ c_* : H_n(BX) \rightarrow H_n^{\text{gr}}(\text{As}(X)) \rightarrow H_n^{\text{gr}}(\text{Inn}(X))$ is annihilated by $|\text{Inn}(X)|/|X|$ for any $n \in \mathbb{N}$. Here note from Lemma 3.5 that t is a divisor of the order $|\text{Inn}(X)|/|X|$.

Since the induced map $c_* : H_2(BX) \rightarrow H_2^{\text{gr}}(\text{As}(X))$ with $n = 2$ is known to be surjective (cf. Hopf's theorem [Bro, II.5]), Theorem 2.1 is immediately obtained from Theorem 6.1 and the inflation-restriction exact sequence of (2.3). Hence, we may turn into proving Theorem 6.1.

To this end, we give a brief review of the rack and quandle homology. Let $C_n^R(X)$ be the free right $\mathbb{Z}$ -module generated by X^n . Define a boundary $\partial_n^R : C_n^R(X) \rightarrow C_{n-1}^R(X)$ by

$$\begin{aligned} \partial_n^R(x_1, \dots, x_n) = \sum_{1 \leq i \leq n} (-1)^i & \left((x_1 \triangleleft x_i, \dots, x_{i-1} \triangleleft x_i, x_{i+1}, \dots, x_n) \right. \\ & \left. - (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \right). \end{aligned}$$

The composite $\partial_{n-1}^R \circ \partial_n^R$ is known to be zero. The homology is denoted by $H_n^R(X)$ and is called *the rack homology*. As is known, the cellular complex of the rack space BX is isomorphic to the complex $(C_*^R(X), \partial_*^R)$. In particular, we have the isomorphism $H_*(BX) \cong H_*^R(X)$. Furthermore, following [CJKLS], let $C_n^D(X)$ be a submodule of $C_n^R(X)$ generated by n -tuples $(x_1, \dots, x_n)$ with $x_i = x_{i+1}$ for some $i \in \{1, \dots, n-1\}$. It can be easily seen that the submodule $C_n^D(X)$ is a subcomplex of $C_n^R(X)$. Then the *quandle homology*, $H_n^Q(X)$, is defined to be the homology of the quotient complex $C_n^R(X)/C_n^D(X)$.

Furthermore, we now observe concretely the map $c_* : H_n(BX) \rightarrow H_n^{\text{gr}}(\text{As}(X))$ for $n \leq 3$. Let us recall the (non-homogenous) standard complex $C_n^{\text{gr}}(\text{As}(X))$ of $\text{As}(X)$; see e.g. [Bro, §I.5]. The map c_* can be described in terms of their complexes. In fact, Kabaya [Kab, §8.4] considered homomorphisms $c_n : C_n^R(X) \rightarrow C_n^{\text{gr}}(\text{As}(X))$, where the map c_n for $n \leq 3$ are defined by setting

$$\begin{aligned} c_1(x) &= e_x, \\ c_2(x, y) &= (e_x, e_y) - (e_y, e_{x \triangleleft y}), \\ c_3(x, y, z) &= (e_x, e_y, e_z) - (e_x, e_z, e_{y \triangleleft z}) + (e_y, e_z, e_A) - (e_y, e_{x \triangleleft y}, e_z) + (e_z, e_{x \triangleleft z}, e_{y \triangleleft z}) \\ &\quad - (e_z, e_{y \triangleleft z}, e_A), \end{aligned}$$

where we denote $(x \triangleleft y) \triangleleft z \in X$ by A for short. As is shown (see [Kab, §8.4]), the induced map on homology coincides with the map above c_* up to homotopy.

We will construct a chain homotopy between $t \cdot c_n$ and zero, when X is connected and of type t . Define a homomorphism $h_i : C_i^R(X) \rightarrow C_{i+1}^{\text{gr}}(\text{As}(X))$ by setting

$$\begin{aligned} h_1(x) &= \sum_{1 \leq j \leq t-1} (e_x, e_x^j), \\ h_2(x, y) &= \sum_{1 \leq j \leq t-1} (e_x, e_y, e_{x \triangleleft y}^j) - (e_x, e_x^j, e_y) - (e_y, e_{x \triangleleft y}, e_{x \triangleleft y}^j) + (e_y, e_y^j, e_y), \\ h_3(x, y, z) &= \sum_{1 \leq j \leq t-1} ((e_x, e_y, e_z, e_A^j) - (e_x, e_z, e_{y \triangleleft z}, e_A^j) - (e_x, e_y, e_{x \triangleleft y}, e_z) - (e_y, e_{x \triangleleft y}, e_z, e_A^j) \\ &\quad + (e_x, e_z, e_{x \triangleleft z}, e_{y \triangleleft z}^j) + (e_z, e_{x \triangleleft z}, e_{y \triangleleft z}, e_A^j) + (e_x, e_x^j, e_y, e_z) - (e_x, e_x^j, e_z, e_{y \triangleleft z}) \\ &\quad + (e_y, e_z, e_A, e_A^j) - (e_z, e_{y \triangleleft z}, e_A, e_A^j) - (e_z, e_{x \triangleleft z}, e_{x \triangleleft z}^j, e_{y \triangleleft z}) + (e_y, e_{x \triangleleft y}, e_{x \triangleleft y}^j, e_z)). \end{aligned}$$

Lemma 6.2. *Let X be as above. Then we have the equality $h_1 \circ \partial_2^R - \partial_3^{\text{gr}} \circ h_2 = t \cdot c_2$.*

Proof. Compute the both terms $h_1 \circ \partial_2^R$ and $\partial_3^{\text{gr}} \circ h_2$ in the left hand side as

$$\begin{aligned} h_1 \circ \partial_2^R(x, y) &= \sum (e_x, e_x^j) - (e_{x \triangleleft y}, e_{x \triangleleft y}^j). \\ \partial_3^{\text{gr}} \circ h_2(x, y) &= \partial_3^{\text{gr}} \sum ((e_x, e_y, e_{x \triangleleft y}^j) - (e_x, e_x^j, e_y) - (e_y, e_{x \triangleleft y}, e_{x \triangleleft y}^j) + (e_y, e_y^j, e_y)) \\ &= \left(\sum (e_y, e_{x \triangleleft y}^j) - (e_x e_y, e_{x \triangleleft y}^j) + (e_x, e_x^j e_y) - (e_x, e_y) - (e_x^j, e_y) + (e_x^{j+1}, e_y) - (e_x, e_x^j e_y) \right. \\ &\quad \left. + (e_x, e_x^j) - (e_{x \triangleleft y}, e_{x \triangleleft y}^j) + (e_x e_y, e_{x \triangleleft y}^j) - (e_y, e_{x \triangleleft y}^{j+1}) + (e_y, e_{x \triangleleft y}) \right) + (e_y, e_y^t) - (e_y^t, e_y) \\ &= t((e_y, e_{x \triangleleft y}) - (e_x, e_y)) + (e_y^t, e_y) - (e_y, e_{x \triangleleft y}^t) - (e_y^t, e_y) + (e_y, e_y^t) + h_1 \circ \partial_2^R(x, y) \\ &= -t \cdot c_2(x, y) + h_1 \circ \partial_2^R(x, y). \end{aligned}$$

Here we use Lemma 3.4 for the last equality. Hence, the equalities complete the proof. $\square$

Lemma 6.3. *Let X be as above. The difference $h_2 \circ \partial_3^R - \partial_4^{\text{gr}} \circ h_3$ is chain homotopic to $t \cdot c_3$.*

Proof. This is similarly proved by a direct calculation. To this end, recalling the notation $A = (x \triangleleft y) \triangleleft z$, we remark two identities

$$e_z e_A = e_{x \triangleleft y} e_z, \quad e_{y \triangleleft z} e_A = e_{x \triangleleft z} e_{y \triangleleft z} \in \text{As}(X).$$

Using them, a tedious calculation can show that the difference $(t \cdot c_3 - h_2 \circ \partial_3^R - \partial_4^{\text{gr}} \circ h_3)(x, y, z)$ is equal to

$$(e_y, e_z, e_A^t) - (e_x^t, e_y, e_z) + (e_x^t, e_z, e_{e \triangleleft z}) - (e_y, e_{x \triangleleft y}^t)$$

$$+(e_z, e_{x \triangleleft z}^t, e_{y \triangleleft z}) - (e_z, e_{y \triangleleft z}, e_A^t) + \sum_{1 \leq j \leq t-1} (e_y, e_y^j, e_y) - (e_{y \triangleleft z}, e_{y \triangleleft z}^j, e_{y \triangleleft z}).$$

Note that this formula is independent of any $x \in X$ since the identity $(e_a)^t = (e_b)^t$ holds for any $a, b \in X$ by Lemma 3.4. However, the map $c_3(x, y, z)$ with $x = y$ is zero by definition. Hence, the map $t \cdot c_3$ is null-homotopic as desired. $\square$

Proof of Theorem 6.1. The map $t \cdot c_*$ are obviously null-homotopic by Lemmas 6.2 and 6.3. $\square$

The proof was an ad hoc computation in an algebraic way; however the theorem should be easily shown by a topological method:

Problem Does the t -vanishing of the map $c_* : H_n(BX) \rightarrow H_n^{\text{gr}}(\text{As}(X))$ hold for any $n \in \mathbb{N}$? Provide its topological proof. Further, how about non-connected quandles?

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