

A Proof of the M-Convex Intersection Theorem

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Abstract

This short note gives an alternative proof of the M-convex intersection theorem, which is one of the central results in discrete convex analysis. This note is intended to provide a direct simpler proof accessible to nonexperts.

1 M-Convex Intersection Theorem

The M-convex intersection theorem [3, Theorem 8.17] reads as follows, where V is a nonempty finite set, and $\mathbf{Z}$ and $\mathbf{R}$ are the sets of integers and reals, respectively; see §3 for the definitions of M^h -convex functions and notation $\arg \min$. This theorem is equivalent to the M-separation theorem, to the Fenchel-type min-max duality theorem, and to an optimality criterion of the M-convex submodular flow problem.

Theorem 1 (M-convex intersection theorem). *For M^h -convex functions f_1, f_2 and a point $x^* \in \text{dom} f_1 \cap \text{dom} f_2$ we have*

$$f_1(x^*) + f_2(x^*) \leq f_1(x) + f_2(x) \quad (\forall x \in \mathbf{Z}^V) \quad (1)$$

if and only if there exists $p^ \in \mathbf{R}^V$ such that¹*

$$f_1[-p^*](x^*) \leq f_1[-p^*](x) \quad (\forall x \in \mathbf{Z}^V), \quad (2)$$

$$f_2[+p^*](x^*) \leq f_2[+p^*](x) \quad (\forall x \in \mathbf{Z}^V). \quad (3)$$

For such p^ we have*

$$\arg \min(f_1 + f_2) = \arg \min f_1[-p^*] \cap \arg \min f_2[+p^*]. \quad (4)$$

Moreover, if f_1 and f_2 are integer-valued, we can choose integer-valued $p^ \in \mathbf{Z}^V$.*

We shall give a constructive proof of Theorem 1 based on the successive shortest path algorithm. Different proofs available in [3] are:

1. original proof based on negative-cycle cancelling for the M-convex submodular flow problem (§9.5 and Note 9.21 of [3]), and
2. polyhedral proof for the discrete separation theorem based on the separation in convex analysis (Proof of Theorem 8.15 of [3]).

¹Notation: $f_1[-p^*](x) = f_1(x) - \sum_{v \in V} p^*(v)x(v)$, $f_2[+p^*](x) = f_2(x) + \sum_{v \in V} p^*(v)x(v)$.

2 Essence of Theorem 1

The essence of Theorem 1 consists of two assertions:

1. optimality of $x^* \Rightarrow$ existence of p^* ,
2. integrality of $f_1, f_2 \Rightarrow$ integrality of p^* .

To see this we make easier observations in this section.

Observation 1: Existence of p^* with (2) and (3) $\Rightarrow$ optimality (1) of x^* .

(Proof)

$$\begin{aligned} f_1(x^*) + f_2(x^*) &= f_1[-p^*](x^*) + f_2[+p^*](x^*) \\ &\leq f_1[-p^*](x) + f_2[+p^*](x) = f_1(x) + f_2(x). \end{aligned}$$

Observation 2: For any $p^* \in \mathbf{R}^V$ we have

$$\arg \min(f_1 + f_2) \supseteq \arg \min f_1[-p^*] \cap \arg \min f_2[+p^*]. \quad (5)$$

(Proof) This follows from the inequality shown in the proof of Observation 1.

Observation 3: If

$$f_1[-p^*](x^\circ) \leq f_1[-p^*](x) \quad (\forall x \in \mathbf{Z}^V), \quad (6)$$

$$f_2[+p^*](x^\circ) \leq f_2[+p^*](x) \quad (\forall x \in \mathbf{Z}^V) \quad (7)$$

for some x° and p^* , then

$$f_1[-p^*](x^*) \leq f_1[-p^*](x) \quad (\forall x \in \mathbf{Z}^V), \quad (8)$$

$$f_2[+p^*](x^*) \leq f_2[+p^*](x) \quad (\forall x \in \mathbf{Z}^V) \quad (9)$$

for every $x^* \in \arg \min(f_1 + f_2)$. Hence,

$$\arg \min(f_1 + f_2) \subseteq \arg \min f_1[-p^*] \cap \arg \min f_2[+p^*]. \quad (10)$$

(Proof) Put $x = x^*$ in (6) and (7) to obtain

$$f_1[-p^*](x^\circ) \leq f_1[-p^*](x^*), \quad (11)$$

$$f_2[+p^*](x^\circ) \leq f_2[+p^*](x^*). \quad (12)$$

Adding these yields

$$\begin{aligned} f_1(x^\circ) + f_2(x^\circ) &= f_1[-p^*](x^\circ) + f_2[+p^*](x^\circ) \\ &\leq f_1[-p^*](x^*) + f_2[+p^*](x^*) = f_1(x^*) + f_2(x^*), \end{aligned}$$

whereas $x^* \in \arg \min(f_1 + f_2)$. Hence we have equalities in (11) and (12).

Observation 4: It suffices to consider M-convex functions rather than $M^{\natural}$ -convex functions.

(Proof) This follows from the equivalence between $M^{\natural}$ -convexity and M-convexity; see [3, §6.1].

Thus the proof of Theorem 1 is reduced to showing the following.

Proposition 2. For M -convex functions f_1, f_2 with $\arg \min(f_1 + f_2) \neq \emptyset$, there exist $x^\circ \in \arg \min(f_1 + f_2)$ and $p^* \in \mathbf{R}^V$ such that

$$f_1[-p^*](x^\circ) \leq f_1[-p^*](x) \quad (\forall x \in \mathbf{Z}^V), \quad (13)$$

$$f_2[+p^*](x^\circ) \leq f_2[+p^*](x) \quad (\forall x \in \mathbf{Z}^V). \quad (14)$$

If f_1 and f_2 are integer-valued, we can choose integer-valued $p^* \in \mathbf{Z}^V$.

3 Notation and Basic Facts

We denote by $\mathbf{Z}^V$ the set of integral vectors indexed by V , and by $\mathbf{R}^V$ the set of real vectors indexed by V . For a vector $x = (x(v) : v \in V) \in \mathbf{Z}^V$, where $x(v)$ is the v th component of x , we define the positive support $\text{supp}^+(x)$ and the negative support $\text{supp}^-(x)$ by

$$\text{supp}^+(x) = \{v \in V \mid x(v) > 0\}, \quad \text{supp}^-(x) = \{v \in V \mid x(v) < 0\}.$$

We use notation $x(S) = \sum_{v \in S} x(v)$ for a subset S of V . For each $S \subseteq V$, we denote by χ_S the characteristic vector of S defined by: $\chi_S(v) = 1$ if $v \in S$ and $\chi_S(v) = 0$ otherwise, and write χ_v for $\chi_{\{v\}}$ for all $v \in V$. For a vector $p = (p(v) : v \in V) \in \mathbf{R}^V$ and a function $f : \mathbf{Z}^V \rightarrow \mathbf{R} \cup \{+\infty\}$, we define functions $\langle p, x \rangle$ and $f[p](x)$ in $x \in \mathbf{Z}^V$ by

$$\langle p, x \rangle = \sum_{v \in V} p(v)x(v), \quad f[p](x) = f(x) + \langle p, x \rangle.$$

We also denote the set of minimizers of f and the effective domain of f by

$$\arg \min f = \{x \in \mathbf{Z}^V \mid f(x) \leq f(y) \ (\forall y \in \mathbf{Z}^V)\},$$

$$\text{dom } f = \{x \in \mathbf{Z}^V \mid f(x) < +\infty\}.$$

A function $f : \mathbf{Z}^V \rightarrow \mathbf{R} \cup \{+\infty\}$ with $\text{dom } f \neq \emptyset$ is called M^h -convex if it satisfies

(M^h -EXC) for all $x, y \in \text{dom } f$ and all $u \in \text{supp}^+(x - y)$, there exists $v \in \text{supp}^-(x - y) \cup \{0\}$ such that

$$f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v),$$

where χ_0 is defined to be the zero vector in $\mathbf{Z}^V$.

A function $f : \mathbf{Z}^V \rightarrow \mathbf{R} \cup \{+\infty\}$ with $\text{dom } f \neq \emptyset$ is called M -convex if it satisfies

(M -EXC) for all $x, y \in \text{dom } f$ and all $u \in \text{supp}^+(x - y)$, there exists $v \in \text{supp}^-(x - y)$ such that

$$f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v).$$

A nonempty set $B \subseteq \mathbf{Z}^V$ is called M -convex if it satisfies

(B-EXC) for all $x, y \in B$ and all $u \in \text{supp}^+(x - y)$, there exists $v \in \text{supp}^-(x - y)$ such that $x - \chi_u + \chi_v, y + \chi_u - \chi_v \in B$.

The minimizers of an M-convex function have a good characterization.

Lemma 3 ([3, Theorem 6.26]). *For an M-convex function f and $x \in \text{dom } f$, $x \in \arg \min f$ if and only if $f(x) \leq f(x - \chi_u + \chi_v)$ for all $u, v \in V$.*

Lemma 4 ([3, Proposition 6.29]). *For an M-convex function f , $\arg \min f$ is an M-convex set if not empty.*

An M-convex set has the following property. (See [1, Lemma 4.5] and [2, Lemma 2.3.22, Remark 3.3.24]. This is a special case of [3, Proposition 9.23].)

Lemma 5 (“no-short cut lemma”). *Let B be an M-convex set. For any $x \in B$ and any distinct $u_1, v_1, u_2, v_2, \dots, u_r, v_r \in V$, if $x - \chi_{u_i} + \chi_{v_i} \in B$ for all $i = 1, \dots, r$ and $x - \chi_{u_i} + \chi_{v_j} \notin B$ for all i, j with $i < j$, then $y = x - \sum_{i=1}^r (\chi_{u_i} - \chi_{v_i}) \in B$.*

4 Proof of Proposition 2 by SSP

We give a proof of Proposition 2 on the basis of the successive shortest path algorithm (SSP) [3, §10.3.4] as adapted to finding a minimizer of $f_1 + f_2$. We may assume that the effective domains of f_1 and f_2 are bounded.

Let x_1 and x_2 be arbitrary minimizers of f_1 and f_2 , respectively. We construct a directed graph $G(f_1, f_2, x_1, x_2) = (V, A)$ and an arc length $\ell \in \mathbf{R}^A$ as follows. Arc set A is the union of two disjoint parts:

$$\begin{aligned} A_1 &= \{(u, v) \mid u, v \in V, u \neq v, x_1 - \chi_u + \chi_v \in \text{dom } f_1\}, \\ A_2 &= \{(v, u) \mid u, v \in V, u \neq v, x_2 - \chi_u + \chi_v \in \text{dom } f_2\}, \end{aligned} \quad (15)$$

and $\ell \in \mathbf{R}^A$ is defined by

$$\ell(a) = \begin{cases} f_1(x_1 - \chi_u + \chi_v) - f_1(x_1) & \text{if } a = (u, v) \in A_1, \\ f_2(x_2 - \chi_u + \chi_v) - f_2(x_2) & \text{if } a = (v, u) \in A_2. \end{cases} \quad (16)$$

The length function ℓ is nonnegative due to Lemma 3.

Put $S = \text{supp}^+(x_1 - x_2)$ and $T = \text{supp}^-(x_1 - x_2)$. A path exists from S to T by Lemma 6 below. Let P be a shortest path from S to T in G with a minimum number of arcs, and let $t \in T$ be the terminal vertex of P .

Let $d : V \rightarrow \mathbf{R} \cup \{+\infty\}$ denote the shortest distance from S to all vertices with respect to ℓ . Then we have

$$\ell(a) + d(u) - d(v) \geq 0$$

for all arcs $a = (u, v) \in A$. Define $p \in \mathbf{R}^V$ by $p(v) = \min\{d(v), d(t)\}$ for all $v \in V$. It follows from the nonnegativity of ℓ that

$$\ell(a) + p(u) - p(v) \geq 0$$

for all arcs $a = (u, v) \in A$. The above system of inequalities is equivalent to

$$\begin{aligned} f_1(x_1 - \chi_u + \chi_v) - f_1(x_1) + p(u) - p(v) &\geq 0, \\ f_2(x_2 - \chi_u + \chi_v) - f_2(x_2) - p(u) + p(v) &\geq 0 \end{aligned}$$

for all $u, v \in V$, which is further equivalent to

$$x_1 \in \arg \min f_1[-p], \quad x_2 \in \arg \min f_2[+p],$$

by Lemma 3. Note that for all arcs $a = (u, v) \in A$,

$$\ell_p(a) = \ell(a) + p(u) - p(v)$$

are the lengths of a in the graph $G(f_1[-p], f_2[+p], x_1, x_2)$ associated with $f_1[-p]$, $f_2[+p]$, x_1 , and x_2 .

Since $\ell_p(a) = 0$ for all $a \in P$, we have

$$\begin{aligned} x_1 - \chi_u + \chi_v &\in \arg \min f_1[-p] && \text{for all } (u, v) \in P \cap A_1, \\ x_2 - \chi_u + \chi_v &\in \arg \min f_2[+p] && \text{for all } (v, u) \in P \cap A_2. \end{aligned} \quad (17)$$

Since P has a minimum number of arcs, we also have

$$x_1 - \chi_u + \chi_w \notin \arg \min f_1[-p], \quad x_2 - \chi_w + \chi_u \notin \arg \min f_2[+p] \quad (18)$$

for all vertices u and w of P such that $(u, w) \notin P$ and u appears earlier than w in P .

Furthermore, arcs of A_1 and A_2 appear alternately in P . This can be proved as follows. Suppose that consecutive two arcs $(u, v), (v, w) \in P$ belong to, say, A_1 . Then, by (M-EXC),

$$f_1(x_1 + \chi_u - \chi_v) + f_1(x_1 + \chi_v - \chi_w) \geq f_1(x_1) + f_1(x_1 + \chi_u - \chi_w),$$

which yields

$$\ell(u, v) + \ell(v, w) \geq \ell(u, w),$$

a contradiction to the minimality (with respect to the number of arcs) of P . Consequently, we have

$$\begin{aligned} a_1 = (u_1, v_1), a_2 = (u_2, v_2) \in P \cap A_1, a_1 \neq a_2 &\implies \{u_1, v_1\} \cap \{u_2, v_2\} = \emptyset, \\ a_1 = (u_1, v_1), a_2 = (u_2, v_2) \in P \cap A_2, a_1 \neq a_2 &\implies \{u_1, v_1\} \cap \{u_2, v_2\} = \emptyset. \end{aligned} \quad (19)$$

From Lemmas 4 and 5 together with (17), (18), and (19), we have

$$x'_1 \equiv x_1 - \sum_{(u,v) \in P \cap A_1} (\chi_u - \chi_v) \in \arg \min f_1[-p], \quad (20)$$

$$x'_2 \equiv x_2 - \sum_{(v,u) \in P \cap A_2} (\chi_u - \chi_v) \in \arg \min f_2[+p]. \quad (21)$$

Thus the modification of (f_1, f_2, x_1, x_2) to (f'_1, f'_2, x'_1, x'_2) , where $f'_1 = f_1[-p]$ and $f'_2 = f_2[+p]$, keeps the conditions

$$x'_1 \in \arg \min f'_1, \quad x'_2 \in \arg \min f'_2.$$

We have

$$x'_1 - x'_2 = (x_1 - x_2) - (\chi_s - \chi_t)$$

with $s \in \text{supp}^+(x_1 - x_2)$ and $t \in \text{supp}^-(x_1 - x_2)$, since P is a path from $\text{supp}^+(x_1 - x_2)$ to $\text{supp}^-(x_1 - x_2)$ and arcs of A_1 and A_2 appear alternately in P . This implies that $\sum_{v \in V} |x_1(v) - x_2(v)|$ is decreased by two. Repeating the modification above we eventually arrive at $x_1 = x_2$, when we have

$$x_1 \in \arg \min f_1[-p] \cap \arg \min f_2[+p].$$

Finally note that, if the functions f_1 and f_2 are integer-valued, the length function ℓ is integer-valued, and hence p is also integer-valued.

The SSP algorithm is summarized below.

Algorithm SSP (f_1, f_2 : M-convex)

Step 0. Find $x_1 \in \arg \min f_1$ and $x_2 \in \arg \min f_2$. Set $p := 0$.

Step 1. If $x_1 = x_2$ then stop.

Step 2. Construct G and compute ℓ for $f_1[-p]$, $f_2[+p]$, x_1 and x_2 by (15) and (16).
Set $S := \text{supp}^+(x_1 - x_2)$, $T := \text{supp}^-(x_1 - x_2)$.

Step 3. Compute the shortest distances $d(v)$ from S to all $v \in V$ in G with respect to ℓ .
Find a shortest path P from S to T with a minimum number of arcs, and let t be the terminal vertex of P .

Step 4. For all $v \in V$, set $p(v) := p(v) + \min\{d(v), d(t)\}$.
Update x_1 and x_2 by (20) and (21).
Go to Step 1.

Lemma 6. If $\text{dom } f_1 \cap \text{dom } f_2 \neq \emptyset$ and $x_1 \neq x_2$, then there exists a path from $S = \text{supp}^+(x_1 - x_2)$ to $T = \text{supp}^-(x_1 - x_2)$.

Proof: To prove by contradiction, suppose that there exists no path from S to T and let W be the set of the vertices reachable from S . Then $W \supseteq S$ and $W \cap T = \emptyset$.

Define set functions $\rho_i : 2^V \rightarrow \mathbf{Z} \cup \{+\infty\}$ as

$$\rho_i(X) = \sup\{z(X) \mid z \in \text{dom } f_i\}$$

for $i = 1, 2$. For $z \in \text{dom } f_i$ we obviously have²

$$z(X) \leq \rho_i(X) \quad (\forall X \subseteq V).$$

²As is well known (see [3, §4.4]), the M-convexity of $\text{dom } f_i$ implies that ρ_i is submodular and

$$\text{dom } f_i = \{z \in \mathbf{Z}^V \mid z(X) \leq \rho_i(X) \ (\forall X \subset V), z(V) = \rho_i(V)\}.$$

However, we do not need this fact for the proof of Lemma 6.

We also have $z(V) = \rho_i(V)$ since $y(V)$ is constant for all $y \in \text{dom } f_i$. Hence, for all $z \in \text{dom } f_1 \cap \text{dom } f_2$ we have

$$\rho_1(V) = z(V) = z(V \setminus X) + z(X) \leq \rho_1(V \setminus X) + \rho_2(X) \quad (\forall X \subseteq V). \quad (22)$$

Since $x_1 \in \text{dom } f_1$ and there exists no arc of A_1 from W to $V \setminus W$, we have

$$x_1(V \setminus W) = \rho_1(V \setminus W)$$

by Lemma 3 applied to an M-convex function

$$f(z) = \begin{cases} -z(V \setminus W) & \text{if } z \in \text{dom } f_1, \\ +\infty & \text{otherwise.} \end{cases}$$

Symmetrically, since $x_2 \in \text{dom } f_2$ and there exists no arc of A_2 from W to $V \setminus W$, we have

$$x_2(W) = \rho_2(W).$$

Adding these yields

$$x_1(V) - [x_1(W) - x_2(W)] = \rho_1(V \setminus W) + \rho_2(W).$$

This contradicts (22), since $x_1(V) = \rho_1(V)$ and $[x_1(W) - x_2(W)] > 0$ by $W \supseteq S$ and $W \cap T = \emptyset$.

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