

# Structure of the solutions to Fuchsian systems

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**Abstract:** To a certain Volevič system of homogeneous singular partial differential equations in a complex domain, called a Fuchsian system, holomorphic solutions which have singularities only on the initial surface are considered. All the solutions are constructed and parametrized in a good way, without any assumptions on the characteristic exponents.

## 1 Introduction

We consider a system of linear partial differential operators

$$P = tD_t I_m - A(t, x; D_x), \quad (t, x) \in \mathcal{C} \times \mathcal{C}^m, \quad (1.1)$$

where  $I_m$  is the  $m \times m$  unit matrix, and

$$A = (A_{i,j}(t, x; D_x))_{1 \leq i,j \leq m}, \quad A_{i,j} = \sum_{\alpha: \text{finite}} a_{i,j;\alpha}(t, x) D_x^\alpha,$$

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The research was partially supported by the Ministry of Education, Science, Sports and Culture (Japan), Grant-in-Aid for Scientific Research 12640194(2000,2001).

**Key Words:** Fuchsian system, Volevič system, regular singularity, characteristic exponent, characteristic index.

$a_{i,j;\alpha}$  are holomorphic in a neighborhood of the origin  $(0, 0) \in \mathbf{C}^{n+1}$ . We use  $D_t = \frac{\partial}{\partial t}$ ,  $D_x = (D_1, \dots, D_n)$ ,  $D_j = \frac{\partial}{\partial x_j}$ , without dividing by  $\sqrt{-1}$ .

$P$  is called a *Fuchsian system* if  $P$  satisfies the following two conditions.

(A-1) There exists  $n_j \in \mathbf{N} := \{0, 1, 2, \dots\}$  such that

$$\text{ord}_{D_x} A_{i,j}(t, x; D_x) \leq n_i - n_j + 1.$$

(A-2)  $A(0, x; D_x) =: A_0(x)$  is independent of  $D_x$ .

The condition (A-1) is equivalent to each of the following condition ([4], [5]).

$$(A-1)' \quad \max_{\substack{1 \leq p \leq m \\ 1 \leq i_1 < \dots < i_p \leq m}} \left( \frac{1}{p} \max_{\pi \in \mathcal{S}_p} \sum_{k=1}^p \text{ord}_{D_x} A_{i_k, i_{\pi(k)}} \right) =: \rho(A) \leq 1$$

( $\rho(A)$  is called the *matrix order* of  $A$ .)

When this condition is satisfied, the system  $D_t I_m - A(t, x; D_x)$  is called a *kowalevskian system in Volevič's sense* ([4]).

The polynomial

$$\mathcal{C}(x; \lambda) := \det(\lambda I_m - A_0(x))$$

of  $\lambda$  is called the *indicial polynomial* of  $P$ , and a root  $\lambda$  of  $\mathcal{C}(x; \lambda) = 0$  is called a *characteristic exponent* or a *characteristic index* of  $P$  at  $x$ .

The second author ([6], [7]) has shown the following fundamental theorems corresponding to the Cauchy-Kowalevsky theorem and the Holmgren theorem. Let  $\mathcal{O}_{(0,0)}$  denote the germ space of holomorphic functions at  $(0, 0) \in \mathbf{C} \times \mathbf{C}^n$ .

**Theorem 1.1** ([6, Theorem 1.2.10]). *If  $\mathcal{C}(0; j) \neq 0$  ( $j \in \mathbf{N}$ ), then for every  $\vec{f} \in (\mathcal{O}_{(0,0)})^m$ , there exists a unique  $\vec{u} \in (\mathcal{O}_{(0,0)})^m$  such that  $P\vec{u} = \vec{f}(t, x)$ .*

**Theorem 1.2** ([7, Theorem 2]). *Let  $\Omega$  be an open neighborhood of  $0 \in \mathbf{R}^n$  and  $T > 0$ . Let  $L \in \mathbf{R}$  satisfy that if  $\mathcal{C}(x; \lambda) = 0$  ( $x \in \Omega$ ), then  $\text{Re } \lambda < L$ . If  $\vec{u}(t) = \vec{u}(t, x) \in C^1((0, T], \mathcal{D}'(\Omega))^m$  satisfies  $P\vec{u} = \vec{0}$  in  $(0, T) \times \Omega$ , and if  $t^{-L}\vec{u} \in C^0([0, T], \mathcal{D}'(\Omega))^m$ , then  $\vec{u} = \vec{0}$  near  $(0, 0)$  in  $(0, T) \times \Omega$ . Here,  $\mathcal{D}'(\Omega)$  denotes the space of Schwartz distributions on  $\Omega$ .*

Now, we introduce the following notation.

$$\begin{aligned}
\mathcal{O}(\Omega) &:= \{ \text{holomorphic functions on } \Omega \} , \\
B_R &:= \{ x \in \mathbb{C}^m : |x| < R \} , \quad \Delta_T := \{ t \in \mathbb{C} : |t| < T \} \quad (T > 0) , \\
\mathcal{O}_0 &:= \bigcup_{R>0} \mathcal{O}(B_R) , \quad \mathcal{O}_{(0,0)} := \bigcup_{R>0, T>0} \mathcal{O}(\Delta_T \times B_R) , \\
S_{\infty, T} &:= \mathcal{R}(\Delta_T \setminus \{0\}) \quad (\text{the universal covering of } \Delta_T \setminus \{0\}) , \\
S_{\theta, T} &:= \{ t \in S_{\infty, T} : |\arg t| \leq \theta \} , \quad \tilde{\mathcal{O}} := \bigcup_{T>0, R>0} \mathcal{O}(S_{\infty, T} \times B_R) .
\end{aligned}$$

Now, we consider solutions of  $P\vec{u} = \vec{0}$  which are singular only at  $t = 0$ , that is,  $\vec{u} \in (\tilde{\mathcal{O}})^m$ . Under the assumption that the characteristic exponents  $\lambda_j(x)$  ( $j = 1, 2, \dots, m$ ) of  $P$  do *not* differ by integers, that is,  $\lambda_i(0) - \lambda_j(0) \notin \mathbb{Z}$  ( $i \neq j$ ), the structure of the kernel  $\text{Ker}_{(\tilde{\mathcal{O}})^m} P$  of the map  $P : (\tilde{\mathcal{O}})^m \rightarrow (\tilde{\mathcal{O}})^m$  has been studied by the second author([6]).

Our purpose of this talk is to construct a solution map, that is, a linear isomorphism

$$(\mathcal{O}_0)^m \xrightarrow{\sim} \text{Ker}_{(\tilde{\mathcal{O}})^m} P := \{ \vec{u} \in (\tilde{\mathcal{O}})^m : P\vec{u} = \vec{0} \} , \quad (1.2)$$

rather explicitly, with *no* assumptions on the characteristic exponents (Theorem 2.2).

In the case of *single* Fuchsian partial differential equations, the first author([2]) have constructed a good solution map. These single equations can be reduced to our Fuchsian systems as follows.

*Remark 1.3.* Let  $P'$  be a single Fuchsian partial differential operator with weight 0 ([1], [6], [2], etc.); that is,  $P' = (tD_t)^m + \sum_{j=1}^m P'_j(t, x; D_x)(tD_t)^{m-j}$ ,  $\text{ord}_{D_x} P'_j \leq j$ , and  $P'_j(0, x; D_x) =: a_j(x)$  is a function of  $x$ . Then, by  $u_j = (tD_t)^{j-1}u$  ( $1 \leq j \leq m$ ),

the equation  $Pu = f$  is reduced to

$$\left( tD_t I_m - \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -P'_m & -P'_{m-1} & -P'_{m-2} & \dots & -P'_1 \end{pmatrix} \right) \vec{u} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f \end{pmatrix}.$$

Since this system satisfies (A-1) with  $n_j = j$  and (A-2), it is a Fuchsian system. Further, this system has the same indicial polynomial  $\mathcal{C}(x; \lambda)$  as  $P'$ , where the indicial polynomial of  $P'$  is defined by

$$\mathcal{C}[P'](x; \lambda) := \lambda^m + \sum_{j=1}^m a_j(x) \lambda^{m-j} = [t^{-\lambda} P'(t^\lambda)]|_{t=0}.$$

## 2 Construction of the solution map

Let  $\mu_l$  ( $l = 1, \dots, d$ ) be all the distinct roots of  $\mathcal{C}(0; \lambda) = 0$ , and let  $r_l$  be the multiplicity of  $\mu_l$ . There exists  $Q(x) \in GL_m(\mathcal{O}_0)$  such that

- $Q(x)^{-1} A_0(x) Q(x) = A_1(x) \otimes \dots \otimes A_d(x) := \begin{pmatrix} A_1(x) & O & \dots & O \\ O & A_2(x) & O & \vdots \\ \vdots & O & \ddots & \vdots \\ O & \dots & O & A_d(x) \end{pmatrix},$
- $A_l \in M_{r_l}(\mathcal{O}_0)$  ( $l = 1, \dots, d$ ),
- $\det(\lambda I_{r_l} - A_l(0)) = (\lambda - \mu_l)^{r_l}$  ( $l = 1, \dots, d$ ).

Corresponding to the blocks of  $Q(x)^{-1} A_0(x) Q(x)$ , we denote the  $l$ -th block of  $\vec{u}$  by  $\vec{u}^{(l)} \in \mathbb{C}^{r_l}$ , that is,  $\vec{u} = \begin{pmatrix} \vec{u}^{(1)} \\ \vdots \\ \vec{u}^{(d)} \end{pmatrix}$ . Conversely, for an  $r_l$ -vector  $\vec{v} \in \mathbb{C}^{r_l}$ , we

denote by  $\vec{v}^{*(l)} \in \mathbb{C}^m$  the  $m$ -vector

$$\vec{v}^{*(l)} = \begin{pmatrix} 0 \\ \vdots \\ \vec{v} \\ \vdots \\ 0 \end{pmatrix} \quad (l \text{ th block})$$

with the entries  $\vec{v}$  in the  $l$ -th block and the entries 0 in the other blocks.

Set

$$\Lambda_P := \{ \mu_l - j \in \mathbb{C} : 1 \leq l \leq d, j \in \mathbb{N} \}. \quad (2.1)$$

Take  $\epsilon \geq 0$  as  $\operatorname{Re} \mu_l - \epsilon \notin \mathbb{Z}$  for all  $l$ . For each  $l$ , take  $L_l \in \mathbb{Z}$  as  $L_l + \epsilon < \operatorname{Re} \mu_l < L_l + \epsilon + 1$ .

**Lemma 2.1.** (1) *For each  $l$ , there exists a domain  $D_l$  in  $\mathbb{C}$  enclosed by a simple closed curve  $\Gamma_l$  such that*

- (a)  $\mu_l \in D_l$  ( $1 \leq l \leq d$ ),
- (b)  $\overline{D_l} \cap \overline{D_{l'}} = \emptyset$  ( $l \neq l'$ ), where  $\overline{D}$  denotes the closure of  $D$ .
- (c)  $\overline{D_l} \cap \Lambda_P = \{ \mu_l \}$  for every  $l$ .
- (d)  $\overline{D_l} \subset \{ \lambda \in \mathbb{C} : L_l + \epsilon < \operatorname{Re} \lambda < L_l + \epsilon + 1 \}$  for every  $l$ .

(2) *There exists  $R_0 > 0$  such that*

- (e)  $\mathcal{C}(x; \lambda + j) \neq 0$  for every  $x \in B_{R_0}$ , every  $\lambda \in \bigcup_{l=1}^d \Gamma_l$ , and every  $j \in \mathbb{N}$ .

The main result is

**Theorem 2.2.** *For every  $l$  and every  $\vec{\varphi}_l \in (\mathcal{O}_0)^{r_l}$ , there exists a unique  $\vec{V} = \vec{V}[l, \vec{\varphi}_l](t, x; \lambda) \in \mathcal{O}(\{(0, 0)\} \times (\bigcup_{l=1}^d \Gamma_l))^m$  such that*

$$P(t^\lambda \vec{V}) = t^\lambda Q(x) \vec{\varphi}_l^{*(l)}(x) \quad (2.2)$$

in a neighborhood of  $\{(0, 0)\} \times (\bigcup_{l=1}^d \Gamma_l)$ .

Set  $\vec{u}_l[\vec{\varphi}_l](t, x) := \frac{1}{2\pi\sqrt{-1}} \int_{\Gamma_l} t^\lambda \vec{V}[l, \vec{\varphi}_l](t, x; \lambda) d\lambda$ . Then, the map

$$(\mathcal{O}_0)^m \ni \begin{pmatrix} \vec{\varphi}_1 \\ \vdots \\ \vec{\varphi}_d \end{pmatrix} \mapsto \sum_{l=1}^d \vec{u}_l[\vec{\varphi}_l] \in \text{Ker}_{(\mathcal{O})^m} P \quad (2.3)$$

is a linear isomorphism.

### 3 Expansion of the solutions

Expand the operator  $A$  and the vector  $\vec{V}$  as follows.

$$A(t, x; D_x) = A_0(x) + \sum_{l=1}^{\infty} t^l B_l(x; D_x) ,$$

$$\vec{V}[l, \vec{\varphi}_l](t, x; \lambda) = \sum_{j=0}^{\infty} t^j \vec{V}_j(x; \lambda) .$$

Then, the equation (2.2) for  $\vec{V}$  is equivalent to

$$(\lambda I_m - A_0(x)) \vec{V}_0(x; \lambda) = Q(x) \vec{\varphi}_l(x) , \quad (3.1)$$

$$((\lambda + j)I_m - A_0(x)) \vec{V}_j(x; \lambda) = \sum_{l=1}^j B_l(x; D_x) \vec{V}_{j-l}(x; \lambda) \quad (j \geq 1) . \quad (3.2)$$

From these equations, we can determine  $\vec{V}_j$  by Lemma 2.1 (e), and we get an expansion of  $\vec{u}_l[\vec{\varphi}_l]$  as follows.

$$\vec{u}_l[\vec{\varphi}_l](t, x) = \sum_{j=0}^{\infty} t^j \vec{u}_{l,j}(t, x) ,$$

$$\vec{u}_{l,j}(t, x) := \frac{1}{2\pi\sqrt{-1}} \int_{\Gamma} t^\lambda \vec{V}_j(x; \lambda) d\lambda .$$

Especially, the leading term of  $\vec{u}_l[\vec{\varphi}_l]$  is

$$\vec{u}_{l,0}(t, x) = t^{A_0(x)} Q(x) \vec{\varphi}_l^{\sharp(l)}(x) = Q(x) \{t^{A_l(x)} \vec{\varphi}_l(x)\}^{\sharp(l)} . \quad (3.3)$$

## 4 Sketch of the proof of the existence of $\vec{V}$

We change the letter  $\lambda$  to  $\zeta$ . Then, the system  $P(t^\zeta \vec{V}) = t^\zeta Q(x) \vec{\varphi}_l^{\sharp(l)}(x)$  is equivalent to another system

$$\tilde{P} \vec{V} := ((tD_t + \zeta)I_m - A(t, x; D_x)) \vec{V} = Q(x) \vec{\varphi}_l^{\sharp(l)}(x) .$$

This is also a Fuchsian system in  $(t, x, \zeta)$ . Note that we consider  $(x, \zeta)$  as the space variables. Further, the indicial polynomial of  $\tilde{P}$  is

$$\mathcal{C}[\tilde{P}](x, \zeta; \lambda) = \mathcal{C}[P](x; \lambda + \zeta) .$$

Since  $\mathcal{C}[\tilde{P}](0, \zeta; j) \neq 0$  ( $\zeta \in \Gamma_l$ ,  $j \in \mathbf{N}$ ) by Lemma 2.1 (e), we can use Theorem 1.1 to this new system. Thus, there exists a unique  $\vec{V} = \vec{V}[l, \vec{\varphi}_l](t, x; \zeta) \in \mathcal{O}(\{(0, 0)\} \times \Gamma_l)^m$  such that  $\tilde{P} \vec{V} = Q(x) \vec{\varphi}_l^{\sharp(l)}(x)$ .

## 5 Function spaces to estimate the order

**Definition 5.1.** ([2, Definition 5.1]) For  $a \in \mathbf{R}$ , we set

$$W^{(a)} := \bigcup_{R>0, T>0} \left\{ \phi \in \mathcal{O}(S_{\infty, T} \times B_R) : \sup_{|x|<R} |\phi(t, x)| \rightarrow 0 \text{ (as } t \rightarrow 0 \text{ in } S_{\theta, T} \text{ for every } \theta > 0) \right\}$$

**Lemma 5.2.** ([2, Lemma 5.2]) (1)  $a' < a \implies W^{(a)} \subset W^{(a')}$ .

(2)  $t \times W^{(a)} \subset W^{(a+1)}$ ,  $\partial_t(W^{(a)}) \subset W^{(a-1)}$ .

(3) If  $B(t, x; D_x)$  is a partial differential operator in  $x$  with  $\mathcal{O}_{(0,0)}$  coefficients, then  $B(t, x; D_x)(W^{(a)}) \subset W^{(a)}$ .

## 6 Keys to the proof of the theorem

The first key is the temperedness of the solutions in  $(\tilde{\mathcal{O}})^m$ .

**Proposition 6.1.** *There exists  $a \in \mathbf{R}$  such that if  $\vec{u} \in (\tilde{\mathcal{O}})^m$  and  $P\vec{u} = \vec{0}$ , then  $\vec{u} \in (W^{(a)})^m$ .*

The **second key** is an estimate of the remainder terms of our solutions  $\vec{u}_l[\vec{\varphi}_l](t, x)$ .

**Lemma 6.2.** *For  $\vec{\varphi}_l \in (\mathcal{O}_0)^{r_l}$ , we have*

$$\vec{u}_l[\vec{\varphi}_l](t, x) = Q(x)\{t^{A_l(x)}\vec{\varphi}_l(x)\}^{\sharp(l)} + t \cdot \vec{r}_l[\vec{\varphi}_l](t, x),$$

and  $\vec{r}_l[\vec{\varphi}_l] \in (W^{(L_l+\epsilon)})^m$ . Note that  $t^{A_l(x)}\vec{\varphi}_l(x) \in (W^{(L_l+\epsilon)})^{r_l}$  and  $\vec{u}_l[\vec{\varphi}_l] \in (W^{(L_l+\epsilon)})^m$ .

The **third key** is the two facts on the Euler system  $(tD_t - A_0(x))\vec{u} = \vec{f}(t, x)$  with holomorphic parameters  $x$ .

**Lemma 6.3.** *If  $\vec{u} \in (\tilde{\mathcal{O}})^m$  and  $(tD_t I_m - A_0(x))\vec{u} = \vec{0}$ , then there exists  $\vec{\varphi}_l \in (\mathcal{O}_0)^{r_l}$  ( $1 \leq l \leq d$ ) such that*

$$\vec{u} = \sum_{l=1}^d Q(x)\{t^{A_l(x)}\vec{\varphi}_l(x)\}^{\sharp(l)} = Q(x) \begin{pmatrix} t^{A_1(x)}\vec{\varphi}_1(x) \\ \vdots \\ t^{A_d(x)}\vec{\varphi}_d(x) \end{pmatrix}.$$

Further, if  $L \in \mathbf{Z}$  and  $\vec{u} \in \widetilde{W}^{(L+\epsilon)}(\theta, R)^m$ , then  $\vec{\varphi}_l = 0$  for all  $l$  such that  $L_l < L$ .

**Proposition 6.4.** *For any  $L \in \mathbf{Z}$  and any  $\vec{g} \in W^{(L+\epsilon)}$ , there exists  $\vec{v} \in W^{(L+\epsilon)}$  such that  $(tD_t I_m - A_0(x))\vec{v} = \vec{g}(t, x)$ .*

If a root  $\lambda(x)$  of  $\mathcal{C}(x; \lambda) = 0$  touches the line  $\operatorname{Re} \lambda = L + \epsilon$  in  $\lambda$ -plane, then this proposition does not hold, as the simplest example  $tD_t v = \frac{1}{\log t}$  shows ( $m = 1$ ,  $L + \epsilon = 0$ , no parameter  $x$ ). This proposition is the reason why we took  $\epsilon$ .

## 7 Proof of the injectivity of the solution map

Assume that  $\vec{\varphi}_l \in (\mathcal{O}_0)^{r_l}$  ( $1 \leq l \leq d$ ),  $\sum_{l=1}^d \vec{u}_l[\vec{\varphi}_l] = \vec{0}$ , and that there exists  $l$  such that  $\vec{\varphi}_l \neq \vec{0}$ . Take  $l_0$  as  $L_{l_0} = \min\{L_l : \vec{\varphi}_l \neq \vec{0}\}$ .

For each  $l$  with  $\vec{\varphi}_l \neq \vec{0}$ , consider  $(\vec{v}_l)^{b(l_0)}$  : the  $l_0$ -th block of  $\vec{v}_l := Q^{-1}\vec{u}_l[\vec{\varphi}_l]$ . Then, we have by Lemma 6.2

$$(\vec{v}_{l_0})^{b(l_0)} = t^{A_{l_0}(x)}\vec{\varphi}_{l_0}(x) + (W^{(L_{l_0}+1+\epsilon)})^{r_{l_0}} .$$

On the other hand, if  $l \neq l_0$ , then  $L_l \geq L_{l_0}$  and hence

$$(\vec{v}_l)^{b(l_0)} \in (W^{(L_l+1+\epsilon)})^{r_{l_0}} \subset (W^{(L_{l_0}+1+\epsilon)})^{r_{l_0}} .$$

Thus,

$$\vec{0} = \sum_{l=1}^d (Q^{-1}\vec{u}_l[\vec{\varphi}_l])^{b(l_0)} = t^{A_{l_0}(x)}\vec{\varphi}_{l_0}(x) + (W^{(L_{l_0}+1+\epsilon)})^{r_{l_0}} .$$

Namely,  $t^{A_{l_0}(x)}\vec{\varphi}_{l_0}(x) \in (W^{(L_{l_0}+1+\epsilon)})^{r_{l_0}}$ . It is easy to show that this implies  $\vec{\varphi}_{l_0} = \vec{0}$ , which contradicts the definition of  $l_0$ .

## 8 Proof of the surjectivity of the solution map

Let  $\vec{u} \in (\tilde{\mathcal{O}})^m$  and  $P\vec{u} = \vec{0}$ . Decompose  $A(t, x; D_x) = A_0(x) + tB(t, x; D_x)$ .

(I) By Proposition 6.1, there exists  $L \in \mathbb{Z}$  such that  $\vec{u} \in (W^{(L+\epsilon)})^m$ .

By Lemma 5.2, we have  $tB(\vec{u}) \in (W^{(L+1+\epsilon)})^m$ .

(II) By Proposition 6.4, there exists  $\vec{v} \in (W^{(L+1+\epsilon)})^m$  such that  $(tD_t I_m - A_0(x))\vec{v} = tB(\vec{u}) = (tD_t I_m - A_0(x))\vec{u}$ .

(III) Since  $(tD_t I_m - A_0(x))(\vec{u} - \vec{v}) = \vec{0}$  and  $\vec{u} - \vec{v} \in (W^{(L+\epsilon)})^m$ , there exists  $\vec{\varphi}_l[1] \in (\mathcal{O}_0)^{r_l}$  such that  $\vec{\varphi}_l[1] = \vec{0}$  if  $L_l < L$ , and that

$$\vec{u} - \vec{v} = \sum_{l=1}^d Q(x) \{t^{A_l(x)}\vec{\varphi}_l[1](x)\}^{\sharp(l)} .$$

by Lemma 6.3.

(IV) Set

$$\vec{u}[1] := \vec{u} - \sum_{l=1}^d \vec{u}_l[\vec{\varphi}_l[1]] \in (W^{(L+1+\epsilon)})^m .$$

Then, we have  $P(\vec{u}[1]) = \vec{0}$ ,  $\vec{u}[1] \in (W^{(L+1+\epsilon)})^m$ .

Now, we can return to the step (I) taking  $\vec{u}[1]$  instead of  $\vec{u}$ , with order  $L+1+\epsilon$  instead of  $L+\epsilon$ .

Repeating such arguments, we have  $\vec{\varphi}_i[j] \in (\mathcal{O}_0)^{r_i}$  ( $j = 2, 3, \dots$ ) such that  $\vec{\varphi}_i[j] = \vec{0}$  if  $L_i < L+j-1$ , and that

$$\vec{u}[j] := \vec{u}[j-1] - \sum_{l=1}^d \vec{u}_l[\vec{\varphi}_i[j]] \left( = \vec{u} - \sum_{k=1}^j \sum_{l=1}^d \vec{u}_l[\vec{\varphi}_i[k]] \right) \in (W^{(L+j+\epsilon)})^m,$$

and  $P(\vec{u}[j]) = \vec{0}$ .

By Theorem 1.2, we have  $\vec{u}[M] = \vec{0}$  for sufficiently large  $M$ . Thus, we get  $\vec{\varphi}_i := \sum_{k=1}^M \vec{\varphi}_i[k] \in (\mathcal{O}_0)^{r_i}$ .

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