

Stability analysis of numerical methods for delay integro-differential equations

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Abstract

Stability of θ -methods for delay integro-differential equations (DIDEs) is studied on the basis of the linear equation

$$\frac{du}{dt} = \lambda u(t) + \mu u(t - \tau) + \kappa \int_{t-\tau}^t u(\sigma) d\sigma,$$

where λ, μ, κ are complex numbers and τ is a constant delay. It is shown that every A -stable θ -method possesses a similar stability property to P -stability, i.e., the method preserves the delay-independent stability of the exact solution under the condition that τ/h is an integer, where h is a step-size. It is also shown that the method does not possess the same property if τ/h is not an integer. As a result, any θ -method cannot possess a similar stability property to GP -stability with respect to DIDEs.

1. Introduction

We study stability of (2-stage) θ -methods for delay integro-differential equations (DIDEs) on the basis of the linear equation

$$\frac{du}{dt} = \lambda u(t) + \mu u(t - \tau) + \kappa \int_{t-\tau}^t u(\sigma) d\sigma, \quad (1.1)$$

where λ, μ, κ are complex numbers and τ is a constant delay. When $\kappa = 0$, the equation (1.1) coincides with the test equation

$$\frac{du}{dt} = \lambda u(t) + \mu u(t - \tau), \quad (1.2)$$

which was proposed by Barwell [1] to examine stability of numerical methods for delay differential equations (DDEs). As described in [1], if λ, μ satisfy

$$|\mu| < -\operatorname{Re} \lambda, \quad (1.3)$$

the zero solution of (1.2) is asymptotically stable for any $\tau \geq 0$. This asymptotic property is called delay-independent stability, and analogous stability properties of numerical methods are considered on the basis of the condition (1.3). For example,

a numerical method for DDEs is said to be *P*-stable if every numerical solution to (1.2) tends to zero whenever λ, μ satisfy (1.3) and τ/h is an integer, where h is the step-size. A numerical method is said to be *GP*-stable if the same holds for any constant step-size.

In the last two decades, various studies were carried out concerning stability properties of numerical methods for DDEs (see, e.g., [12]). In particular, an earliest study by Watanabe and Roth [10] has revealed that every *A*-stable θ -method is *GP*-stable. To the contrary, little is known about stability properties of numerical methods for DIDEs. It is quite recent that we studied delay-independent stability of linear DIDEs [7], and even stability of θ -methods for (1.1) remains to be investigated.

By Theorem 2 of [7], the zero solution of (1.1) is asymptotically stable for any $\tau \geq 0$ if and only if λ, μ, κ satisfy

$$\lambda + \mu + \kappa\tau \neq 0 \quad \text{for any } \tau \geq 0, \quad (1.4)$$

$$z^2 - z\lambda - \kappa = 0, \quad z \in \mathcal{C}, \quad z \neq 0 \implies \operatorname{Re} z < 0, \quad (1.5)$$

$$\left| \frac{\mu z - \kappa}{z^2 - z\lambda - \kappa} \right| < 1 \quad \text{for any } \operatorname{Re} z = 0 \text{ with } z \neq 0. \quad (1.6)$$

Moreover, the conditions (1.5), (1.6) are rewritten as

$$\operatorname{Re} \lambda < 0 \quad \text{and} \quad \left(\operatorname{Re} \lambda \operatorname{Re}(\lambda \bar{\kappa}) + (\operatorname{Im} \kappa)^2 < 0 \text{ or } \kappa = 0 \right), \quad (1.7)$$

$$\operatorname{Im}[(\lambda + \mu)\bar{\kappa}] = 0 \quad \text{and} \quad \left[|\mu|^2 < (\operatorname{Re} \lambda)^2 + 2 \operatorname{Re} \kappa \right. \\ \left. \text{or } \left(\operatorname{Im} \lambda = 0, |\mu|^2 = (\operatorname{Re} \lambda)^2 + 2 \operatorname{Re} \kappa \right) \right], \quad (1.8)$$

respectively (Sect. 3 in [7]). When λ, μ, κ are all real and $\kappa \neq 0$, these conditions are reduced to the simple condition

$$\lambda < 0, \quad \kappa < 0, \quad \mu^2 \leq \lambda^2 + 2\kappa. \quad (1.9)$$

We study stability properties of θ -methods by comparing the region determined by these conditions with a kind of stability regions of the methods.

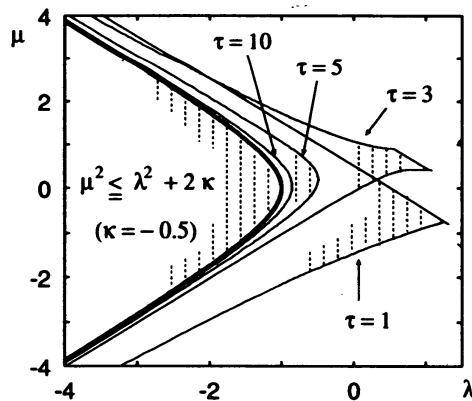


Fig. 1 Delay-independent v.s. delay-dependent stability regions

It should be noted that a considerable number of papers [2, 3, 4, 6, 9] are devoted to stability analysis of θ -methods for DDEs, which does not seem strange from a practical viewpoint. Some important instances of stiff DDEs are obtained from the space-discretization of partial functional differential equations (see, e.g., [13]). The θ -methods have practicality in such a situation.

2. Stability regions of θ -methods

Consider delay integro-differential equations (DIDEs) with a constant delay,

$$\frac{du}{dt} = f\left(t, u(t), u(t - \tau), \int_{t-\tau}^t g(t, \sigma, u(\sigma)) d\sigma\right). \quad (2.1)$$

For a given step-size $h > 0$, let m be the smallest integer greater than or equal to τ/h . Then, the delay τ is represented in the form

$$\tau = (m - \delta)h, \quad 0 \leq \delta < 1, \quad (2.2)$$

and the relation

$$t_n - \tau = t_{n-m} + \delta h \quad (2.3)$$

holds for the step points $t_n = t_0 + nh$, $n \in \mathbb{Z}$.

By approximating the delayed argument and the integrand in (2.1) with linear interpolation, we can adapt a θ -method to (2.1) as follows:

$$u_{n+1} = u_n + h(1 - \theta)f(t_n, u_n, v_n, G_n) + h\theta f(t_{n+1}, u_{n+1}, v_{n+1}, G_{n+1}), \quad (2.4)$$

where, $0 \leq \theta \leq 1$, u_n is an approximate value of $u(t_n)$, and

$$v_n = (1 - \delta)u_{n-m} + \delta u_{n-m+1}, \quad (2.5)$$

$$G_n = \frac{h(1 - \delta)^2}{2} g(t_n, t_{n-m}, u_{n-m}) + \frac{h(2 - \delta^2)}{2} g(t_n, t_{n-m+1}, u_{n-m+1}) \\ + h \sum_{k=2}^{m-1} g(t_n, t_{n-m+k}, u_{n-m+k}) + \frac{h}{2} g(t_n, t_n, u_n). \quad (2.6)$$

As a result, the integral term of (2.1) is approximated with the trapezoidal rule. When $\theta = 1/2$ and $\delta = 0$, the formula (2.4)–(2.6) determines a method that belongs to a class of Runge-Kutta methods discussed in [7]. But, when $\theta \neq 1/2$, it gives another type of numerical method.

In the case of the test equation (1.1), the formula (2.4)–(2.6) is reduced to

$$u_{n+1} = u_n + (1 - \theta)\alpha u_n + \theta\alpha u_{n+1} \\ + \beta \left[(1 - \delta)(1 - \theta)u_{n-m} + (\delta + \theta - 2\delta\theta)u_{n-m+1} + \delta\theta u_{n-m+2} \right] \\ + \gamma \left[\frac{(1 - \delta)^2(1 - \theta)}{2} u_{n-m} + \frac{(2 - \delta^2)(1 - \theta) + (1 - \delta)^2\theta}{2} u_{n-m+1} \right. \\ \left. + \frac{2 - \delta^2\theta}{2} u_{n-m+2} + \sum_{k=3}^{m-1} u_{n-m+k} + \frac{1 + \theta}{2} u_n + \frac{\theta}{2} u_{n+1} \right], \quad (2.7)$$

$$\alpha = h\lambda, \quad \beta = h\mu, \quad \gamma = h^2\kappa. \quad (2.8)$$

The characteristic equation of (2.7) is written as

$$\begin{aligned} z^{m+1} - z^m - (1-\theta)\alpha z^m - \theta\alpha z^{m+1} \\ - \beta \left[(1-\delta)(1-\theta) + (\delta+\theta-2\delta\theta)z + \delta\theta z^2 \right] \\ - \gamma \left[\frac{(1-\delta)^2(1-\theta)}{2} + \frac{(2-\delta^2)(1-\theta) + (1-\delta)^2\theta}{2} z \right. \\ \left. + \frac{2-\delta^2\theta}{2} z^2 + \sum_{k=3}^{m-1} z^k + \frac{1+\theta}{2} z^m + \frac{\theta}{2} z^{m+1} \right] = 0. \end{aligned} \quad (2.9)$$

Using (2.9) we define the sets $S_{\theta,m}^{(\delta)}$ and $S_{\theta}^{(\delta)}$ for $0 \leq \delta < 1$ by

$$S_{\theta,m}^{(\delta)} = \{(\alpha, \beta, \gamma) \in \mathcal{C}^3 : \text{all the roots of (2.9) satisfy } |z| < 1\}, \quad (2.10)$$

$$S_{\theta}^{(\delta)} = \bigcap_{m \geq 1} S_{\theta,m}^{(\delta)}. \quad (2.11)$$

The set $S_{\theta}^{(\delta)}$ is an analogue of the δ -stability region of the θ -method [4].

When $z = 1$, the left hand side of (2.9) is equal to $-(\alpha + \beta + (m - \delta)\gamma)$. Hence, for any $m \geq 1$, $z = 1$ is not a root of (2.9) if and only if

$$(C_0) \quad \alpha + \beta + (m - \delta)\gamma \neq 0 \quad \text{for any } m \geq 1.$$

Substituting $\sum_{k=3}^{m-1} z^k = (z^3 - z^m)/(1 - z)$ into (2.9) and multiplying $(1 - z)$, we get

$$z^m q(z) - p(z) = 0, \quad (2.12)$$

$$q(z) = q_0 z^2 + q_1 z + q_2, \quad (2.13)$$

$$p(z) = p_0 z^3 + p_1 z^2 + p_2 z + p_3, \quad (2.14)$$

where

$$q_0 = \theta\alpha + \frac{\theta}{2}\gamma - 1, \quad q_1 = (1 - 2\theta)\alpha + \frac{\gamma}{2} + 2,$$

$$q_2 = -(1 - \theta)\alpha + \frac{1 - \theta}{2}\gamma - 1,$$

$$p_0 = -\delta\theta\beta + \frac{\delta^2\theta}{2}\gamma, \quad p_1 = (3\delta\theta - \delta - \theta)\beta + \frac{-3\delta^2\theta + \delta^2 + 2\delta\theta + \theta}{2}\gamma,$$

$$p_2 = (-3\delta\theta + 2\delta + 2\theta - 1)\beta + \frac{3\delta^2\theta - 2\delta^2 - 4\delta\theta - 2\delta^2 + 2\delta + 1}{2}\gamma,$$

$$p_3 = (\delta\theta - \delta - \theta + 1)\beta + \frac{-\delta^2\theta + \delta^2 + 2\delta\theta - 2\delta - \theta + 1}{2}\gamma.$$

Moreover, we set

$$r(z) = p(z)/q(z), \quad (2.15)$$

and consider the following conditions.

- (a) $q(z) \neq 0$ for any $|z| \geq 1$.
 ($\hat{a}$) $q(z) \neq 0$ for any $|z| > 1$.
 (b) $|r(z)| < 1$ for any $|z| = 1$ with $z \neq 1$.
 ($\hat{b}$) $|r(z)| \leq 1$ for any $|z| = 1$.

These are regarded as conditions for α, β, γ . We also write

- (c) $(\alpha, \beta, \gamma) \in S_{\theta}^{(\delta)}$.

Under this notation, we can characterize $S_{\theta}^{(\delta)}$ as follows.

Theorem 2.1 *The following implications hold:*

$$(C_0) \text{ and (a) and (b)} \implies (c) \implies (\hat{a}) \text{ and } (\hat{b}).$$

If, in addition,

- (C₁) $p(z), q(z)$ have no common zero on $|z| = 1$,

then (c) implies (a).

Proof. Assume (C₀), (a) and (b). We first show that $\hat{r}(z) = r(z)/z$ satisfies $|\hat{r}(z)| < 1$ for any $|z| \geq 1$ with $z \neq 1$.

The linear fractional transformation

$$z = \frac{w+1}{w-1} \quad (2.16)$$

maps $\operatorname{Re} w > 0$ conformally onto $|z| > 1$, with $w = \infty$ corresponding to $z = 1$. The function $\hat{R}(w) = \hat{r}[(w+1)/(w-1)]$ is represented in the form

$$\hat{R}(w) = \hat{P}(w)/\hat{Q}(w), \quad (2.17)$$

$$\begin{aligned} \hat{P}(w) &= [\gamma w^2 + (-2\beta + 2\delta\gamma - \gamma)w + 2(1 - 2\delta)\beta - 2\delta(1 - \delta)\gamma] \\ &\quad \times [w - (1 - 2\theta)], \end{aligned} \quad (2.18)$$

$$\hat{Q}(w) = (w+1) \left\{ \gamma w^2 + [2\alpha - (1 - 2\theta)\gamma]w - 2(1 - 2\theta)\alpha - 4 \right\}. \quad (2.19)$$

Then, it follows from (a) that $\hat{R}(w)$ is a bounded, holomorphic function in $\operatorname{Re} w > 0$. Hence, by the Phragmén-Lindelöf theorem (see, e.g., [8], p. 168), it follows from (b) that $|\hat{R}(w)| < 1$ for any $\operatorname{Re} w > 0$, which implies that $|\hat{r}(z)| < 1$ for any $|z| \geq 1$ with $z \neq 1$.

If $|z| \geq 1$ and $z \neq 1$, then

$$z^m q(z) - p(z) = q(z)z [z^{m-1} - \hat{r}(z)] \neq 0,$$

which, together with (C_0) , implies (c) .

Assume (c) . If $q(z_0) = 0$ for some $|z_0| > 1$, then there exists $\varepsilon > 0$ such that $C(z_0, \varepsilon) \subset \{|z| > 1\}$ and $q(z) \neq 0$ on $C(z_0, \varepsilon)$, where

$$C(z_0, \varepsilon) = \{z \in \mathcal{C} : |z - z_0| = \varepsilon\}.$$

By Rouché's theorem, the polynomial $z^m q(z) - p(z)$ has a root in the interior of $C(z_0, \varepsilon)$ for sufficiently large m , which contradicts (c) . Therefore, $(\hat{a})$ holds.

Moreover, if $|r(z_0)| > 1$ for some $|z_0| = 1$, then the equation $z^m = r(z)$ has a solution with $|z| > 1$ for sufficiently large m . This is verified by applying Proposition 7 of [11] to $\psi(z) = 1/r(z)$. In fact, there exists $\varepsilon > 0$ such that $|r(z)| > 1$ for any $z \in \overline{V_\varepsilon}$, where $V_\varepsilon = \{z \in \mathcal{C} : |z - (1 + \varepsilon)z_0| < \varepsilon\}$. Hence,

$$\rho = \max_{z \in \overline{V_\varepsilon}} |\psi(z)| < 1,$$

and $1 \in \mathcal{C} \setminus B(0, \rho)$, where $B(0, \rho) = \{z \in \mathcal{C} : |z| \leq \rho\}$. On the other hand, we have

$$\mathcal{C} \setminus B(0, \rho) \subset \bigcup_{m \geq 1} \{z^m \psi(z) : z \in V_\varepsilon\}, \quad (2.20)$$

by Proposition 7 of [11]. Since $|z| > 1$ for any $z \in V_\varepsilon$, it follows from (2.20) that $z^m = r(z)$ holds for some $m \geq 1$ and $|z| > 1$, which contradicts (c) . Therefore, $(\hat{b})$ holds.

It is easy to see that $(\hat{a})$ and $(\hat{b})$ imply (a) under the condition (C_1) . $\square$

3. Stability regions in the case $\delta = 0$

We consider the case $\delta = 0$. Since $q(1) = \gamma$, $z = 1$ satisfies $q(z) = 0$ if and only if $\gamma = 0$. We assume that $\gamma \neq 0$ for a while, and rewrite the conditions (a) , $(\hat{a})$, (b) , $(\hat{b})$ by making use of the linear fractional transformation (2.16).

The function $R(w) = r[(w + 1)/(w - 1)]$ is represented in the form

$$R(w) = P(w)/Q(w), \quad (3.1)$$

$$P(w) = (\gamma w - 2\beta)[w - (1 - 2\theta)], \quad (3.2)$$

$$Q(w) = \gamma w^2 + [2\alpha - (1 - 2\theta)\gamma]w - 2(1 - 2\theta)\alpha - 4. \quad (3.3)$$

Hence, (a) , $(\hat{a})$, (b) , $(\hat{b})$ are equivalent to

$$(A) \quad Q(w) \neq 0 \text{ for any } \operatorname{Re} w \geq 0,$$

$$(\hat{A}) \quad Q(w) \neq 0 \text{ for any } \operatorname{Re} w > 0,$$

$$(B) \quad |R(w)| < 1 \text{ for any } \operatorname{Re} w = 0,$$

$$(\hat{B}) \quad |R(w)| \leq 1 \text{ for any } \operatorname{Re} w = 0,$$

respectively.

When α, γ are real, $(\mathbf{A}), (\widehat{\mathbf{A}})$ are reduced to

$$\gamma[2\alpha - (1 - 2\theta)\gamma] > 0, \quad \gamma[-4 - 2(1 - 2\theta)\alpha] > 0, \quad (3.4)$$

$$\gamma[2\alpha - (1 - 2\theta)\gamma] \geq 0, \quad \gamma[-4 - 2(1 - 2\theta)\alpha] \geq 0, \quad (3.5)$$

respectively. In addition, putting $w = iy, y \in \mathbb{R}$, we have

$$\begin{aligned} |Q(w)|^2 - |P(w)|^2 &= 4 \operatorname{Im}[(\alpha + \beta)\bar{\gamma}] y^3 + 4(|\alpha|^2 - |\beta|^2 + 2 \operatorname{Re} \gamma) y^2 \\ &\quad + \left\{ 16 \operatorname{Im} \alpha + 4(1 - 2\theta)^2 \operatorname{Im}[(\alpha + \beta)\bar{\gamma}] \right\} y \\ &\quad + |4 + 2(1 - 2\theta)\alpha|^2 - |2(1 - 2\theta)\beta|^2. \end{aligned} \quad (3.6)$$

When α, β, γ are real, it is reduced to

$$|Q(w)|^2 - |P(w)|^2 = 4(\alpha^2 - \beta^2 + 2\gamma)y^2 + 4\eta, \quad (3.7)$$

$$\eta = [(1 - 2\theta)(\alpha + \beta) + 2][(1 - 2\theta)(\alpha - \beta) + 2]. \quad (3.8)$$

Hence, in this case, $(\mathbf{B}), (\widehat{\mathbf{B}})$ are equivalent to

$$\beta^2 \leq \alpha^2 + 2\gamma, \quad \eta > 0, \quad (3.9)$$

$$\beta^2 \leq \alpha^2 + 2\gamma, \quad \eta \geq 0, \quad (3.10)$$

respectively.

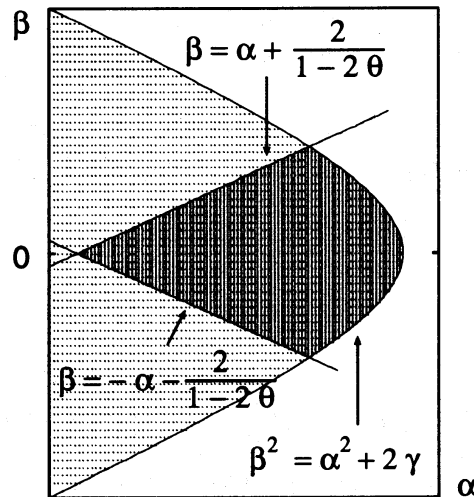


Fig. 2 γ -section of $S_\theta^{(0)} \cap \mathbb{R}^3$ ($0 \leq \theta < 1/2$)

Let $\alpha < 0$ and $\gamma < 0$. The conditions (3.4), (3.9) are reduced to

$$\alpha > -\frac{2}{1-2\theta}, \quad \gamma > \frac{2\alpha}{1-2\theta}, \quad \beta^2 \leq \alpha^2 + 2\gamma, \quad |\beta| < \alpha + \frac{2}{1-2\theta}, \quad (3.11)$$

when $0 \leq \theta < 1/2$ (**Fig. 2**), and

$$\beta^2 \leq \alpha^2 + 2\gamma, \quad (3.12)$$

when $1/2 \leq \theta \leq 1$. If $\alpha(< 0)$, β satisfy $\beta^2 \leq \alpha^2 + 2\gamma$ for $\gamma < 0$, then $\alpha + \beta < 0$, and (C_0) holds. Hence, by Theorem 2.1, these determine the region

$$S_\theta^{(0)} \cap \{(\alpha, \beta, \gamma) \in \mathbb{R}^3 : \alpha < 0, \gamma < 0\}, \quad (3.13)$$

except for ambiguity of the boundary.

We now denote by Ω the set of all the triplicate (λ, μ, κ) for which the zero solution of (1.1) is asymptotically stable for any $\tau \geq 0$, i.e.,

$$\Omega = \{(\lambda, \mu, \kappa) \in \mathcal{C}^3 : (1.4), (1.5), (1.6) \text{ are satisfied}\}. \quad (3.14)$$

It is easy to see that

$$(\lambda, \mu, \kappa) \in \Omega \implies (h\lambda, h\mu, h^2\kappa) \in \Omega \quad \text{for any } h > 0. \quad (3.15)$$

The following theorem shows that A -stable θ -methods possess a similar stability property to P -stability with respect to DIDEs.

Theorem 3.2 *If $1/2 \leq \theta \leq 1$, then $\Omega \subset S_\theta^{(0)}$.*

Proof. The inclusion $\Omega \cap \{\gamma = 0\} \subset S_\theta^{(0)}$ follows from the known result as in the case of DDEs (see, e.g., Theorem 2.6 in [6]). We consider the case $\gamma \neq 0$.

Let $(\alpha, \beta, \gamma) \in \Omega$. The condition (C_0) follows from (1.4). Moreover, it follows from (3.6) and $\text{Im}[(\alpha + \beta)\bar{\gamma}] = 0$ that for $w = iy$, $y \in \mathbb{R}$,

$$\begin{aligned} |Q(w)|^2 - |P(w)|^2 &= \eta_0 y^2 + 2\eta_1 y + \eta_2, \\ \eta_0 &= 4(|\alpha|^2 - |\beta|^2 + 2\text{Re } \gamma), \quad \eta_1 = 8\text{Im } \alpha, \\ \eta_2 &= |2(1-2\theta)\alpha + 4|^2 - |2(1-2\theta)\beta|^2. \end{aligned} \quad (3.16)$$

Since

$$\eta_2 = 16 + 16(1-2\theta)\text{Re } \alpha + 4(1-2\theta)^2(|\alpha|^2 - |\beta|^2) \geq 16, \quad (3.17)$$

$$\begin{aligned} \eta_1^2 - \eta_0 \eta_2 &\leq 64(\text{Im } \alpha)^2 - 64(|\alpha|^2 - |\beta|^2 + 2\text{Re } \gamma) \\ &= -64[(\text{Re } \alpha)^2 + 2\text{Re } \gamma - |\beta|^2], \end{aligned} \quad (3.18)$$

we have

$$|Q(w)| > |P(w)| \quad \text{for any } \text{Re } w = 0, \quad (3.19)$$

which implies (B).

When $\theta = 1/2$, it holds that

$$Q(w) = \gamma w^2 + 2\alpha w - 4 = -w^2 \left[(2/w)^2 - \alpha(2/w) - \gamma \right]. \quad (3.20)$$

Hence, (A) for $\theta = 1/2$ follows from (1.5).

The condition (A) for $\theta = 1/2$, together with (3.19), implies (A) for $1/2 < \theta \leq 1$. In fact, if $Q(w) = 0$ has a solution with $\operatorname{Re} w \geq 0$ for some $1/2 < \theta \leq 1$, then it follows from (A) for $\theta = 1/2$ that there exists $1/2 < \theta_0 \leq \theta$ such that $Q(w) = 0$ for $\theta = \theta_0$ has a solution with $\operatorname{Re} w = 0$. But this is impossible by (3.19). $\square$

4. Stability regions in the case $\delta \neq 0$

The same result as in Theorem 3.2 does not hold in the case $\delta \neq 0$. As a result, any θ -method cannot possess a similar stability property to GP -stability with respect to DIDEs.

Theorem 4.3 *If $0 < \delta < 1$, there exists $(\alpha, \beta, \gamma) \in \Omega$ which does not belong to $S_\theta^{(6)}$.*

Proof. The function $R(w) = r[(w+1)/(w-1)]$ can be written as

$$R(w) = \tilde{P}(w)/\tilde{Q}(w), \quad (4.1)$$

$$\begin{aligned} \tilde{P}(w) = & \left[\gamma w^2 + (-2\beta + 2\delta\gamma - \gamma)w + 2(1 - 2\delta)\beta - 2\delta(1 - \delta)\gamma \right] \\ & \times [w - (1 - 2\theta)], \end{aligned} \quad (4.2)$$

$$\tilde{Q}(w) = (w - 1) \left\{ \gamma w^2 + [2\alpha - (1 - 2\theta)\gamma]w - 2(1 - 2\theta)\alpha - 4 \right\}. \quad (4.3)$$

When α, β, γ are real, we have for $w = iy, y \in \mathbb{R}$,

$$\begin{aligned} |\tilde{Q}(w)|^2 - |\tilde{P}(w)|^2 = & 4(y^2 + 1) \left[(\alpha^2 - \beta^2 + 2\gamma)y^2 + \eta \right] \\ & + 4\delta(1 - \delta)(2\beta - \delta\gamma) \left[2\beta + (1 - \delta)\gamma \right] \left[y^2 + (1 - 2\theta)^2 \right], \end{aligned} \quad (4.4)$$

$$\eta = \left[(1 - 2\theta)(\alpha + \beta) + 2 \right] \left[(1 - 2\theta)(\alpha - \beta) + 2 \right]. \quad (4.5)$$

When $\alpha = -\sqrt{-2\gamma}$ and $\beta = 0$, (4.4) is a quadratic function of y and the coefficient of y^2 is given by

$$4 \left[-(1 - 2\theta)\sqrt{-2\gamma} + 2 \right]^2 - 4\delta^2(1 - \delta)^2\gamma^2. \quad (4.6)$$

If $0 < \delta < 1$ and $-\gamma$ is sufficiently large, the value (4.6) is negative. This implies that (b) does not hold near $(\alpha, \beta) = (-\sqrt{-2\gamma}, 0)$, a point on the hyperbola $\beta^2 = \alpha^2 + 2\gamma$, if $-\gamma$ is sufficiently large. Therefore, by Theorem 2.1, there are points in Ω which do not belong to $S_\theta^{(6)}$. $\square$

In some cases, the region $S_\theta^{(\delta)} \cap \mathbb{R}^3$ is determined on the basis of Theorem 2.1. Let $1/2 \leq \theta \leq 1$, and assume that $\alpha < 0$ and $\gamma < 0$. Then, (a), which does not depend on δ , is satisfied, and (C_0) holds if $\beta^2 \leq \alpha^2 + 2\gamma$. Moreover, (b) is rewritten as

$$(\tilde{\mathbf{B}}) \quad |\tilde{Q}(w)| > |\tilde{P}(w)| \quad \text{for any } \operatorname{Re} w = 0.$$

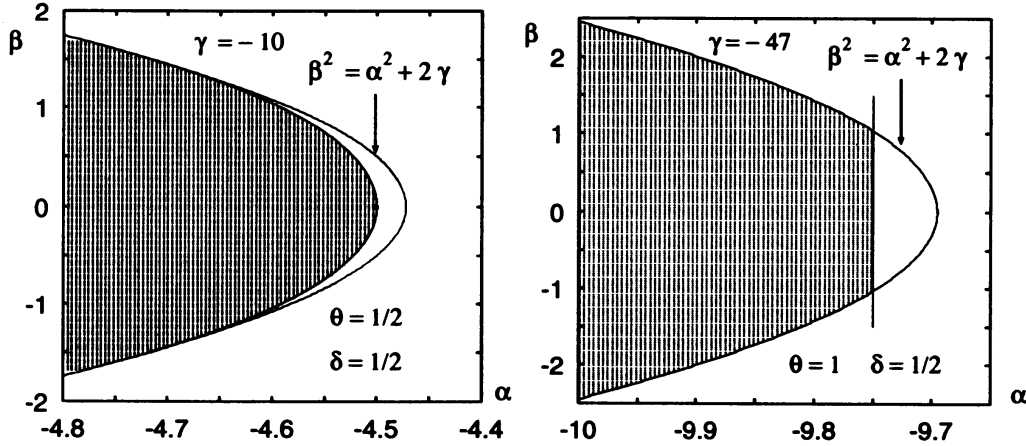


Fig. 3 Examples of γ -sections of $S_\theta^{(\delta)} \cap \mathbb{R}^3$ ($\delta = 1/2$)

In the case $\theta = 1/2$ (the trapezoidal rule), we have for $w = iy$, $y \in \mathbb{R}$,

$$|\tilde{Q}(w)|^2 - |\tilde{P}(w)|^2 = 4[(y^2 + 1)(ay^2 + 4) + by^2], \quad (4.7)$$

$$a = \alpha^2 - \beta^2 + 2\gamma, \quad (4.8)$$

$$b = \delta(1 - \delta)(2\beta - \delta\gamma)[2\beta + (1 - \delta)\gamma]. \quad (4.9)$$

From (4.7) it is easy to verify that $(\tilde{\mathbf{B}})$ holds if and only if $a \geq 0$ and

$$a + b + 4 \geq 0, \quad \text{or} \quad [a + b + 4 < 0 \text{ and } 16a > (a + b + 4)^2]. \quad (4.10)$$

When $\delta = 1/2$, this condition is represented as

$$\begin{cases} \beta^2 \leq \alpha^2 + 2\gamma & (\alpha^2 \geq \frac{\gamma^2}{16} - 2\gamma - 4), \\ \beta^2 < \frac{1}{16} \left[15(\alpha^2 + 2\gamma) - \frac{\gamma^2}{16} - 4 \right] & (\alpha^2 < \frac{\gamma^2}{16} - 2\gamma - 4). \end{cases} \quad (4.11)$$

In the case $\theta = 1$ (the backward Euler method), we have for $w = iy$, $y \in \mathbb{R}$,

$$|\tilde{Q}(w)|^2 - |\tilde{P}(w)|^2 = 4(y^2 + 1)(ay^2 + c), \quad (4.12)$$

$$c = (2 - \alpha)^2 - \beta^2 + \delta(1 - \delta)(2\beta - \delta\gamma)[2\beta + (1 - \delta)\gamma]. \quad (4.13)$$

The condition $(\tilde{\mathbf{B}})$ holds if and only if $a \geq 0$, $c > 0$, which is equivalent to

$$\beta^2 \leq \alpha^2 + 2\gamma, \quad \alpha < \frac{\gamma}{4} + 2, \quad (4.14)$$

when $\delta = 1/2$.

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