

Analytic Subalgebras Associated with Integrable Flows
on von Neumann Algebras

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1. Introduction

Let M be a von Neumann algebra and let $\{\alpha_t\}_{t \in \mathbb{R}}$ be a σ -weakly continuous flow on M ; i.e. suppose that $\{\alpha_t\}_{t \in \mathbb{R}}$ is a one-parameter group of $*$ -automorphisms of M and that for each ρ in the predual, M_* , of M and for each $x \in M$, the function of t , $\rho(\alpha_t(x))$, is continuous on $\mathbb{R}$. In recent years, we have investigated the structure of the subspace of M , $H^\infty(M, \alpha)$, which is defined to be

$$\{x \in M: \rho(\alpha_t(x)) \in H^\infty(\mathbb{R}), \text{ for all } \rho \in M_*\},$$

where $H^\infty(\mathbb{R})$ is the classical Hardy space consisting of the boundary values of functions bounded analytic in the upper half-plane. As in [4, 8, etc.], the elements of $H^\infty(M, \alpha)$ are called analytic with respect to $\{\alpha_t\}_{t \in \mathbb{R}}$ and $H^\infty(M, \alpha)$ itself, is called the analytic subalgebra of M determined by $\{\alpha_t\}_{t \in \mathbb{R}}$. Further, as in [4], $H^\infty(M, \alpha)$ is equal to the set of elements of M such that $\text{Sp}_\alpha(x) \subset [0, \infty)$ where $\text{Sp}_\alpha(x)$ is the Arveson spectrum of x with respect to $\{\alpha_t\}_{t \in \mathbb{R}}$ (cf. [1], [4]).

In this paper, we contribute a partial answer to the following

Question. When is $H^\infty(M, \alpha)$ maximal among the σ -weakly closed subalgebras of M ?

For recent years, we have proved the partial answers of this question (cf. [5, 6, 7, 8, 11, 12, 13, 14, etc.]). In particular, Muhly and the second author in [8] proved that, if M is a crossed product determined by a von Neumann algebra N and a σ -weakly continuous flow $\{\beta_t\}_{t \in \mathbb{R}}$ on N and if $\{\alpha_t\}_{t \in \mathbb{R}}$ is the dual action of $\{\beta_t\}_{t \in \mathbb{R}}$, then $H^\infty(M, \alpha)$ is maximal among the σ -weakly closed subalgebras of M if and only if the fixed point algebra $M^\alpha (= N)$ is a factor. Recall that, if $\{\alpha_t\}_{t \in \mathbb{R}}$ is a dual action, then $\{\alpha_t\}_{t \in \mathbb{R}}$ is integrable in the sense of Connes-Takesaki [2]. Therefore, our aim in this note is to prove the following

Theorem. If $\{\alpha_t\}_{t \in \mathbb{R}}$ is integrable on M , then the fixed point algebra M^α is a factor if and only if $H^\infty(M, \alpha)$ is maximal among the σ -weakly closed subalgebras of M .

After finishing this note, we found the paper by Solel in [15] to study the maximality of $H^\infty(M, \alpha)$ in the general setting. However, we believe that our theory is interesting from a point of view of studying the structure of integrable actions in von Neumann algebras.

2. Preliminaries.

Let M be a von Neumann algebra on a Hilbert space H and let $\{\alpha_t\}_{t \in \mathbb{R}}$ be a σ -weakly continuous flow on M . First, we define the notion of spectral subspaces defined by [1]. We consider

$$\alpha(f)x = \int_{\mathbb{R}} f(t)\alpha_t(x)dt; \quad x \in M, \quad f \in L^1(\mathbb{R}).$$

For $L^1(\mathbb{R})$, we denote by $Z(f)$ the set $\{t \in \mathbb{R}: \hat{f}(t) = 0\}$, where $\hat{f}(t) = \int_{\mathbb{R}} e^{-ist} f(s)ds$. For $x \in M$, we define $Sp_{\alpha}(x)$ to be the set

$$\cap \{Z(f): f \in L^1(\mathbb{R}), \alpha(f)x = 0\}$$

and, for any closed subset S of $\mathbb{R}$, we define the spectral subspace $M^{\alpha}(S)$ to be $\{x \in M: Sp_{\alpha}(x) \subset S\}$. If S is not closed, then $M^{\alpha}(S)$ is defined to be the σ -weak closure of the set $\{x \in M: Sp_{\alpha}(x) \subset S\}$. We refer the reader to [1], [4] and [16] for the basic facts about spectra.

In this note, we write $H^{\infty}(M, \alpha)$ for $M^{\alpha}(\mathbb{R}_+)$ and $H_0^{\infty}(M, \alpha)$ for $M^{\alpha}(\mathbb{R}_{+0})$, where $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}_{+0} = (0, \infty)$, respectively. Further, we write M_t^{α} for $M^{\alpha}(\{t\})$ and note that

$$M^{\alpha}(\{t\}) = \{x \in M: \alpha_s(x) = e^{its}x, \quad s \in \mathbb{R}\}.$$

In particular, put $M^{\alpha} = M^{\alpha}(\{0\})$.

Let $\mathfrak{N}$ be the set of all $x \in M$ such that there is some $y \in M$ with $y = \int_{\mathbb{R}} \alpha_t(x^*x)dt$. If the linear span of $\mathfrak{N}$ is σ -weakly dense in M , we shall say that $\{\alpha_t\}_{t \in \mathbb{R}}$ is integrable. As in [2, 16], note that $\{\alpha_t\}_{t \in \mathbb{R}}$ is integrable if and only if $\int_{\mathbb{R}} \alpha_t(x)dt, \quad x \in$

M_+ is a faithful normal semifinite operator valued weight on M (cf. [16]). Then we have the following lemma by [16, 21.4 Corollary].

Lemma 1. If $\{\alpha_t\}_{t \in \mathbb{R}}$ is integrable on M , then M is the von Neumann algebra generated by $\{M_t\}_{t \in \mathbb{R}}$ and $H^\infty(M, \alpha)$ is a σ -weakly closed subalgebra of M generated by $\{M_t\}_{t \in \mathbb{R}_+}$.

Let M be a von Neumann algebra on a Hilbert space H and let $\{\alpha_t\}_{t \in \mathbb{R}}$ be a σ -weakly continuous flow on M . Put $\tilde{M} = M \otimes B(L^2(\mathbb{R}))$ and let $\tilde{\alpha}_t = \alpha_t \otimes \text{id}$. Then we easily have the following proposition

Proposition 2. Keep the notations as above. Then,

(i) for every subset S of $\mathbb{R}$, $\tilde{M}^{\tilde{\alpha}}(S) = M^\alpha(S) \otimes B(L^2(\mathbb{R}))$.

(ii) The mapping $A \rightarrow A \otimes B(L^2(\mathbb{R}))$ defines bijective correspondence between the class of σ -weakly closed subspaces of M and the class of σ -weakly closed subspaces of $M \otimes B(L^2(\mathbb{R}))$ with the form $A \otimes B(L^2(\mathbb{R}))$, where A is a σ -weakly closed subspace of M .

(iii) $H^\infty(\tilde{M}, \tilde{\alpha}) = H^\infty(M, \alpha) \otimes B(L^2(\mathbb{R}))$.

(iv) $\tilde{M}^{\tilde{\alpha}} = M^\alpha \otimes B(L^2(\mathbb{R}))$.

Proof. (i). Since $M \otimes B(L^2(\mathbb{R}))$ consists of all operators $x = (x_{ij}) \in B(H \otimes L^2(\mathbb{R}))$ with operators $x_{ij} \in M$, we may consider $\tilde{\alpha}_t(x) = (\alpha_t(x_{ij}))$. Thus, we have $\tilde{\alpha}(f)x = (\alpha(f)x_{ij})$. By the definition of spectra, if $\tilde{\alpha}(f)x = 0$, then $\alpha(f)x_{ij} = 0$ for all i, j . Thus, if $x \in \tilde{M}^{\tilde{\alpha}}(S)$, then $x_{ij} \in M^\alpha(S)$ for all i, j . Hence we have $\tilde{M}^{\tilde{\alpha}}(S) \subset M^\alpha(S) \otimes B(L^2(\mathbb{R}))$. Since the converse inclusion is clear, we have (i).

(ii) is clear and, from (i), we have (iii) and (iv). This .

completes the proof.

By Proposition 2, we have the following corollary.

Corollary 3. Keep the notations as above. Then $H^\infty(M, \alpha)$ is maximal among the σ -weakly closed subalgebras of M if and only if $H^\infty(\tilde{M}, \tilde{\alpha})$ is maximal among the σ -weakly closed subalgebras of $\tilde{M}$.

Next, we recall that the crossed product $M \rtimes_\alpha \mathbb{R}$ determined by M and $\{\alpha_t\}_{t \in \mathbb{R}}$ is the von Neumann algebra on the Hilbert space $L^2(\mathbb{R}, H)$ generated by the operators $\pi^\alpha(x)$, $x \in M$, and $\lambda(s)$, $s \in \mathbb{R}$, defined by the equations

$$(\pi^\alpha(x)f)(t) = \alpha_{-t}(x)f(t), \quad f \in L^2(\mathbb{R}, H), \quad t \in \mathbb{R},$$

and

$$(\lambda(s)f)(t) = f(t-s), \quad f \in L^2(\mathbb{R}, H), \quad t \in \mathbb{R}.$$

The automorphism group $\{\hat{\alpha}_t\}_{t \in \mathbb{R}}$ of $M \rtimes_\alpha \mathbb{R}$ which is dual to $\{\alpha_t\}_{t \in \mathbb{R}}$ is implemented by the unitary representation of $\mathbb{R}$, $\{S_t\}_{t \in \mathbb{R}}$, defined by the formula

$$(S_t f)(s) = e^{ist} f(s), \quad f \in L^2(\mathbb{R}, H).$$

Further, we recall that the double crossed product $(M \rtimes_\alpha \mathbb{R}) \rtimes_{\hat{\alpha}} \mathbb{R}$ is the von Neumann algebra on $L^2(\mathbb{R}, L^2(\mathbb{R}, H))$ generated by the

operators $\pi^{\hat{\alpha}}(y)$, $y \in M \rtimes_{\alpha} \mathbb{R}$, and $\mu(s)$, $s \in \mathbb{R}$, defined by the equations

$$(\pi^{\hat{\alpha}}(y)g)(t) = \hat{\alpha}_{-t}(y)g(t), \quad g \in L^2(\mathbb{R}, L^2(\mathbb{R}, H)), \quad t \in \mathbb{R},$$

and

$$(\mu(s)g)(t) = g(t-s), \quad g \in L^2(\mathbb{R}, L^2(\mathbb{R}, H)), \quad t \in \mathbb{R}.$$

The automorphism group $\{\hat{\alpha}_t\}_{t \in \mathbb{R}}$ of $(M \rtimes_{\alpha} \mathbb{R}) \rtimes_{\hat{\alpha}} \mathbb{R}$ which is dual to $\{\hat{\alpha}_t\}_{t \in \mathbb{R}}$ is implemented by the unitary representation of $\mathbb{R}$, $\{S_t\}_{t \in \mathbb{R}}$, defined by the formula

$$(\tilde{S}_t g)(s) = e^{ist} g(s), \quad g \in L^2(\mathbb{R}, L^2(\mathbb{R}, H)).$$

For simplicity, we put $N = (M \rtimes_{\alpha} \mathbb{R}) \rtimes_{\hat{\alpha}} \mathbb{R}$. From the definition of spectra, we have easily

Lemma 4. Let p be a projection of $M \rtimes_{\alpha} \mathbb{R}$. Put $\pi^{\hat{\alpha}}(p) = P$ and $\beta^p = \hat{\alpha}|_{N_P}$, where N_P is the reduced von Neumann algebra of N by P . Then, for every subset S of $\mathbb{R}$, $(N_P)^{\beta^p}(S) = P N^{\hat{\alpha}}(S) P$.

3. Proof of Theorem.

Keep the notations and the assumptions as in §2. Suppose that

$\{\alpha_t\}_{t \in \mathbb{R}}$ is integrable on M . Considering $(M \otimes B(L^2(\mathbb{R})), \alpha \otimes \text{id})$, by Proposition 2 and Corollary 3, we may suppose that M^α is properly infinite to prove this theorem. By [10, Theorem 4.1], there exists a projection p in $(M \otimes B(L^2(\mathbb{R})))^{\alpha \otimes \text{Ad}(\rho)}$ ($= M \rtimes_\alpha \mathbb{R}$) such that

$$(M \otimes B(L^2(\mathbb{R})), \alpha \otimes \text{id}) \cong (M \otimes B(L^2(\mathbb{R})), \alpha \otimes \text{Ad}(\rho))_p$$

where $\{\rho_t\}_{t \in \mathbb{R}}$ is the left regular representation of $\mathbb{R}$ on $L^2(\mathbb{R})$ and $\text{Ad}(\rho)$ is implemented by $\{\rho_t\}_{t \in \mathbb{R}}$. Put $P = \pi^{\hat{\alpha}}(p)$. From the duality theorem of crossed product, we have

$$(M \otimes B(L^2(\mathbb{R})), \alpha \otimes \text{id}) \cong (N_P, \hat{\alpha}|_{N_P}),$$

where N_P is the reduced von Neumann algebra of N by P . Put $\beta^P = \hat{\alpha}|_{N_P}$. That is, there exists an isomorphism Φ of $M \otimes B(L^2(\mathbb{R}))$ onto N_P such that

$$\Phi \circ \tilde{\alpha}_t = \beta_t^P \circ \Phi, \quad t \in \mathbb{R}.$$

Then, for any $X \in N_P$ and $f \in L^1(\mathbb{R})$, we have

$$\begin{aligned} \Phi(\tilde{\alpha}(f)X) &= \Phi\left(\int_{\mathbb{R}} f(t) \tilde{\alpha}_t(X) dt\right) = \int_{\mathbb{R}} f(t) \Phi(\tilde{\alpha}_t(X)) dt \\ &= \int_{\mathbb{R}} f(t) \beta_t^P(\Phi(X)) dt = \beta^P(f)(\Phi(X)). \end{aligned}$$

Thus, we have the following

Proposition 5. For every subset S of $\mathbb{R}$, $\Phi(\tilde{M}^{\tilde{\alpha}}(S)) = (N_P)^{\beta^P}(S)$.

Let $(M \rtimes_{\alpha} R) \rtimes_{\hat{\alpha}} R_+$ be the σ -weakly closed subalgebra generated by $\pi^{\hat{\alpha}}(M \rtimes_{\alpha} R)$ and $\{\mu(t)\}_{t \in \mathbb{R}_+}$. As in [8], we call it the analytic crossed products determined by $M \rtimes_{\alpha} R$ and $\{\hat{\alpha}_t\}_{t \in \mathbb{R}}$. By [8, Proposition 5.11], three spaces $H^{\infty}(N, \hat{\alpha})$, $H_0^{\infty}(N, \hat{\alpha})$ and $(M \rtimes_{\alpha} R) \rtimes_{\hat{\alpha}} R_+$ coincide. Then, by Lemma 4 and Propositions 2 and 5, we have

Proposition 6. (i) $\Phi(H^{\infty}(\tilde{M}, \tilde{\alpha})) = H^{\infty}(N_P, \beta^P) = P H^{\infty}(N, \hat{\alpha}) P$.

(ii) $\Phi(\tilde{M}^{\tilde{\alpha}}) = (N_P)^{\beta^P} = P \pi^{\hat{\alpha}}(M \rtimes_{\alpha} R) P = \pi^{\hat{\alpha}}((M \rtimes_{\alpha} R)_p)$, where $(M \rtimes_{\alpha} R)_p$ is the reduced von Neumann algebra of $M \rtimes_{\alpha} R$.

To prove Theorem, by Proposition 6, it is sufficient to prove that $(M \rtimes_{\alpha} R)_p$ is a factor if and only if $H^{\infty}(N_P, \beta^P) (= P H^{\infty}(N, \hat{\alpha}) P)$ is maximal among the σ -weakly closed subalgebras of N_P . Let $c(p)$ be the central projection of p in $M \rtimes_{\alpha} R$. Then we have $(M \rtimes_{\alpha} R)_p' = ((M \rtimes_{\alpha} R)')_p$ and $(M \rtimes_{\alpha} R)_{c(p)}' = ((M \rtimes_{\alpha} R)')_{c(p)}$. Since $((M \rtimes_{\alpha} R)')_p$ is isomorphic to $((M \rtimes_{\alpha} R)')_{c(p)}$, $(M \rtimes_{\alpha} R)_p$ is a factor if and only if $(M \rtimes_{\alpha} R)_{c(p)}$ is a factor.

Suppose that M^{α} is a factor, that is, $(M \rtimes_{\alpha} R)_{c(p)}$ is a factor. This implies that $c(p)$ is a minimal projection in the center $\mathcal{Z}(M \rtimes_{\alpha} R)$ of $M \rtimes_{\alpha} R$. Since $\hat{\alpha}_t(c(p))$ is a minimal projection in $\mathcal{Z}(M \rtimes_{\alpha} R)$ for all $t \in \mathbb{R}$, $\hat{\alpha}_t(c(p))c(p) = 0$ or $\hat{\alpha}_t(c(p))$.

Since $\{\hat{\alpha}_t\}_{t \in \mathbb{R}}$ is σ -weakly continuous, $\hat{\alpha}_t(c(p))$ converges to p σ -weakly as $t \rightarrow 0$. It follows that $\hat{\alpha}_t(c(p)) = c(p)$ for all t in a neighborhood of 0 and, therefore, for all $t \in \mathbb{R}$. Put $Q = \pi^{\hat{\alpha}}(c(p))$. Then we have

$$\mu(t)Q\mu(t)^* = \mu(t)\pi^{\hat{\alpha}}(c(p))\mu(t)^* = \pi^{\hat{\alpha}}(\hat{\alpha}_t(c(p))) = \pi^{\hat{\alpha}}(c(p)) = Q.$$

This implies that Q is in the center $\mathcal{Z}(N)$ of N . Since the reduced von Neumann algebra N_Q is generated by $\pi^{\hat{\alpha}}((M \rtimes_{\alpha} R)_{c(p)})$ and $\mu(t)Q$, we have $N_Q \cong (M \rtimes_{\alpha} R)_{c(p)} \rtimes_{\gamma} R$, where $\gamma = \hat{\alpha}|_{(M \rtimes_{\alpha} R)_{c(p)}}$ and the crossed product $(M \rtimes_{\alpha} R)_{c(p)} \rtimes_{\gamma} R$ is considered on the Hilbert space $L^2(R, c(p)L^2(R, H))$. Since $(M \rtimes_{\alpha} R)_{c(p)}$ is a factor, by [8, Theorem 5.2], $H^{\infty}(N, \hat{\alpha})Q (= H^{\infty}(N_Q, \beta^{c(p)}))$ is maximal among the σ -weakly closed subalgebras of N_Q .

We now prove that $H^{\infty}(N_P, \beta^p)$ is maximal among the σ -weakly closed subalgebras of N_P . Let B be a σ -weakly closed subalgebra of N_P containing $H^{\infty}(N_P, \beta^p)$ properly. We construct the σ -weakly closed subalgebra $\tilde{B}$ of N_Q generated by $H^{\infty}(N, \hat{\alpha})Q$ and B . Since $\tilde{B} \supsetneq H^{\infty}(N, \hat{\alpha})Q$ clearly, we have $\tilde{B} = N_Q$. It is clear that $P\tilde{B}P = B$ and $(N_Q)_P = N_P$. Thus, $B = N_P$. Therefore, $H^{\infty}(N_P, \beta^p)$ is maximal among the σ -weakly closed subalgebras of N_P and so $H^{\infty}(M, \alpha)$ is maximal among the σ -weakly closed subalgebras of M .

Conversely, we suppose that $H^{\infty}(M, \alpha)$ is maximal among the σ -weakly closed subalgebras of M , that is, we suppose that $H^{\infty}(N_P, \beta^p)$ is maximal among the σ -weakly closed subalgebras of N_P .

Further, suppose that $(M \rtimes_{\alpha} \mathbb{R})_p$ is not a factor. Let $c(p)$ be the central projection of p in $3(M \rtimes_{\alpha} \mathbb{R})$. Put $q = \bigvee_{t \in \mathbb{R}} \hat{\alpha}_t(c(p))$. Then $\hat{\alpha}_t(q) = q$ and so $\pi^{\hat{\alpha}}(q) \in 3(N)$. Putting $Q = \pi^{\hat{\alpha}}(q)$, then N_Q is isomorphic to the crossed product $(M \rtimes_{\alpha} \mathbb{R})_q \rtimes_{\gamma} \mathbb{R}$ defined by $(M \rtimes_{\alpha} \mathbb{R})_q$ and $\gamma_t (= \hat{\alpha}|_{(M \rtimes_{\alpha} \mathbb{R})_q})$ in such a way that $H^{\infty}(N_Q, \beta^q)$ is carried onto the analytic crossed product $(M \rtimes_{\alpha} \mathbb{R})_q \rtimes_{\gamma} \mathbb{R}_+$.

If $\{\gamma_t\}_{t \in \mathbb{R}}$ is not ergodic on $3(M \rtimes_{\alpha} \mathbb{R})_q$, then there exists a $\{\gamma_t\}_{t \in \mathbb{R}}$ -invariant projection p_1 in $(M \rtimes_{\alpha} \mathbb{R})_q$ such that $0 \leq p_1 \leq q$. Since q is the least, $\{\gamma_t\}$ -invariant central projection in $(M \rtimes_{\alpha} \mathbb{R})_q$ containing p , it is clear that $0 \leq p_1 p \leq p$. Put

$$\tilde{B} = \pi^{\hat{\alpha}}(p_1) H^{\infty}(N_Q, \beta^q) \oplus \pi^{\hat{\alpha}}(q - p_1) N_Q.$$

Then $\tilde{B}$ is a proper σ -weakly closed subalgebra of N_Q containing $H^{\infty}(N_Q, \beta^q)$ properly. Put $B = \pi^{\hat{\alpha}}(p) \tilde{B} \pi^{\hat{\alpha}}(p)$. If $B = H^{\infty}(N_P, \beta^p)$, then we have

$$\pi^{\hat{\alpha}}(p) \pi^{\hat{\alpha}}(q - p_1) \pi^{\hat{\alpha}}(M \rtimes_{\alpha} \mathbb{R}) \mu(t) \pi^{\hat{\alpha}}(p) = 0, \text{ for all } t < 0.$$

Thus, $\pi^{\hat{\alpha}}(p) \pi^{\hat{\alpha}}(q - p_1) \mu(t) \pi^{\hat{\alpha}}(p) = 0$ for all $t < 0$ and so $(p - p p_1) \hat{\alpha}_t(p) = 0$ for all $t < 0$. Since $\{\hat{\alpha}_t\}_{t \in \mathbb{R}}$ is σ -weakly continuous, we have $(p - p p_1) p = 0$ and so $p = p p_1$. This is a contradiction. Then $B \neq H^{\infty}(N_P, \beta^p)$. Similarly, we have $B \neq N_P$. Therefore B is a properly σ -weakly closed subalgebra of N_P containing $H^{\infty}(N_P, \beta^p)$ properly. This is a contradiction. Consequently, without loss of generality, we may suppose that

$\{\gamma_t\}_{t \in \mathbb{R}}$ is ergodic on $3(M \rtimes_{\alpha} \mathbb{R})_q$. Then we need the following lemma as in [8].

Lemma 7. If $(M \rtimes_{\alpha} \mathbb{R})_p$ is not a factor and if $\{\gamma_t\}_{t \in \mathbb{R}}$ acts ergodically on the center $3(M \rtimes_{\alpha} \mathbb{R})_q$ of $(M \rtimes_{\alpha} \mathbb{R})_q$, then there is a strongly continuous family $\{e_t\}_{t < 0}$ of projections in $3(M \rtimes_{\alpha} \mathbb{R})_q$ such that

$$e_{t+s} = e_t \gamma_t(e_s), \quad s, t < 0,$$

and $0 \leq e_t p \leq e_0 p \leq p$ for some $t < 0$, where $e_0 = \lim_{t \uparrow 0} e_t$.

Proof. As in [8, Lemma 5.6], we note that $3(M \rtimes_{\alpha} \mathbb{R})_q$ is nonatomic and that there exists a faithful normal state on $3(M \rtimes_{\alpha} \mathbb{R})$. By Cohen's factorization theorem,

$$\{\gamma(f)x : f \in L^1(\mathbb{R}), \quad x \in 3(M \rtimes_{\alpha} \mathbb{R})_q\}$$

is a $\{\gamma_t\}_{t \in \mathbb{R}}$ -invariant, σ -weakly dense, C^* -subalgebra of $3(M \rtimes_{\alpha} \mathbb{R})_q$ on which $\{\gamma_t\}_{t \in \mathbb{R}}$ is strongly continuous. If Ω is the maximal ideal space of this subalgebra, then there is a continuous one-parameter group of homeomorphisms, $\{T_t\}_{t \in \mathbb{R}}$, of Ω , and, there is a nonatomic, quasi-invariant, ergodic, probability measure μ on Ω , with $\text{supp}(\mu) = \Omega$, such that

$$\Gamma(\gamma_t(x))(\omega) = \Gamma(x)(T_t \omega) \quad \text{a.e.}(\mu),$$

where Γ is the canonical extension of the Gelfand transform to all of $3(M \rtimes_{\alpha} R)_q$, mapping isometrically onto $L^{\infty}(\Omega, \mu)$. Since $3(M \rtimes_{\alpha} R)_p$ is isomorphic to $3(M \rtimes_{\alpha} R)_{c(p)}$, and, since $3(M \rtimes_{\alpha} R)_p$ is not a factor, there exists a projection e in $3(M \rtimes_{\alpha} R)_q$ such that $0 \leq e c(p) \leq c(p)$. Then there is a measurable subset E of Ω such that $\Gamma(e) = 1_E$. Since μ is regular on Ω , we may suppose that E is open in Ω . As in [8, Lemma 5.6], for each $t < 0$, put $E_t = \bigcap_{s \leq 0} T_s E$. If we define $e_t = \Gamma^{-1}(1_{E_t})$, $t < 0$ and $e_0 = s\text{-}\lim_{t \uparrow 0} e_t$, then $e_0 \leq e$. Then we obtain the desired property of $\{e_t\}_{t < 0}$. This completes the proof.

If $\{\gamma_t\}_{t \in \mathbb{R}}$ is ergodic on $3(M \rtimes_{\alpha} R)_q$, then, by Lemma 7, there exists a strongly continuous family, $\{e_t\}_{t < 0}$, of projections in $3(M \rtimes_{\alpha} R)_q$ such that

$$e_{t+s} = e_t \gamma_t(e_s), \text{ for all } s, t < 0,$$

and $0 \leq e_t p \leq e_0 p \leq p$ for some $t < 0$, where $e_0 = s\text{-}\lim_{t \uparrow 0} e_t$. Let $\tilde{B}$ denote the σ -weak closure of the linear span of $H^{\infty}(N_Q, \beta^q)$ and $\{\pi^{\hat{\alpha}}(e_t) \pi^{\hat{\alpha}}(M \rtimes_{\alpha} R) \mu(t)\}_{t < 0}$. Then, as in the proof of [8, Theorem 5.21], $\tilde{B}$ is a properly, σ -weakly closed subalgebra of N_Q containing $H^{\infty}(N_Q, \beta^q)$ properly. Put $B = \pi^{\hat{\alpha}}(p) \tilde{B} \pi^{\hat{\alpha}}(p)$. If $B = H^{\infty}(N_P, \beta^p)$, then

$$\pi^{\hat{\alpha}}(p) \pi^{\hat{\alpha}}(e_t) \pi^{\hat{\alpha}}(M \rtimes_{\alpha} R) \mu(t) \pi^{\hat{\alpha}}(p) = \{0\} \text{ for all } t < 0.$$

and so $\pi^{\hat{\alpha}}(p) \pi^{\hat{\alpha}}(e_t) \mu(t) \pi^{\hat{\alpha}}(p) = 0$ for all $t < 0$. Thus we have

$pe_t\hat{\alpha}_t(p) = 0$ for all $t < 0$. As $t \uparrow 0$, $pe_0p = e_0p = 0$. This is a contradiction. This implies that $B \not\supseteq H^\infty(N_P, \beta^P)$. On the other hand, if $B = N_P$, then we have for all $t < 0$,

$$\pi^{\hat{\alpha}}(p)\pi^{\hat{\alpha}}(e_t)\pi^{\hat{\alpha}}(M \rtimes_{\alpha} \mathbb{R})\mu(t)\pi^{\hat{\alpha}}(p) = \pi^{\hat{\alpha}}(p)\pi^{\hat{\alpha}}(M \rtimes_{\alpha} \mathbb{R})\mu(t)\pi^{\hat{\alpha}}(p),$$

and so, multiplying both left side by $\pi^{\hat{\alpha}}(q-e_t)$, we have

$$\pi^{\hat{\alpha}}(q-e_t)\pi^{\hat{\alpha}}(p)\pi^{\hat{\alpha}}(M \rtimes_{\alpha} \mathbb{R})\mu(t)\pi^{\hat{\alpha}}(p) = 0 \text{ for all } t < 0.$$

Therefore, we have $(q-e_t)p\hat{\alpha}_t(p) = 0$ for all $t < 0$. As $t \uparrow 0$, $p-pe_0 = 0$. This contradiction implies that $B \subsetneq N_P$. This implies that $H^\infty(N_P, \beta^P)$ is not maximal among the σ -weakly closed subalgebras of N_P . This is a contradiction. Therefore, $(M \rtimes_{\alpha} \mathbb{R})_p$ is a factor and so M^{α} is a factor. This completes the proof of Theorem.

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