

A CLASS OF SECOND ORDER QUASILINEAR EVOLUTION EQUATIONS

Ryo IKEHATA
(池 畠 良)

Department of Mathematics
Science University of Tokyo

1. INTRODUCTION

In this paper, we consider second order quasilinear evolution equations of the form

$$u''(t) + M(|A^{1/2}u(t)|^2)Au(t) = f(u(t)), \quad t > 0, \quad (1.1)$$

$$u(0) = u_0, \quad u'(0) = u_1 \quad (1.2)$$

in a real Hilbert space H with norm $|\cdot|$. Here A is a nonnegative selfadjoint operator in H , f is a nonlinear operator from $D(A^{1/2})$ to H , $M(s)$ is a C^1 -class function satisfying

$$M(s) \geq m_0 > 0, \text{ with } m_0 \text{ constant.}$$

When $f(u(t)) \equiv 0$, the equation (1.1) has its origin in the mathematical description of small amplitude vibrations of an elastic string (see Ames [1]).

In case of $M(r) := 1$ (semi-linear type), there is a lot of literatures (see e.g., Browder [2], Ebihara, Nakao and Nanbu [3], Ishii [8], Ôtani [13], Payne and Sattinger [14], Reed [16] and Tsutsumi [17]).

For general $M(r) \geq m_0 > 0$, when $f(u)(x) := -|u(x)|^\alpha u(x)$

($\alpha \geq 0$), Hosoya and Yamada [4] obtained a local solution by a Galerkin method. Furthermore, Ikehata [6] has got a unique local strong solution to (1.1)-(1.2) by applying the theory of evolution equations and also discussed the blowing-up property of local solutions whose results contain that of Levine [12]. However, in [6], the relations between Ôtani [13] and Ikehata [6] have not been shown clearly.

The first purpose of the present paper is to obtain a local strong solution to (1.1)-(1.2) by applying the theory of quasilinear hyperbolic systems which are given by Kato [11]. This will be an improvement of the result of Ikehata [6].

The second purpose of the present paper is to discuss the blowing-up property of local solutions to the equations:

$$u_{tt}(t,x) - (\alpha + 2\beta \int_{\Omega} |\nabla u(t,y)|^2 dy) \Delta u(t,x) = \mu(u(t,x))^3, \quad (*)$$

where $\alpha > 0$, $\beta \geq 0$, $\mu > 0$ and $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary. The essence of our argument is in taking the coefficient $\mu > 0$ 'sufficiently large'. Therefore, at least in case of (*), we have to take care of how to choose the coefficient μ with delicacy. If in particular $\alpha = 1$, $\beta = 0$ and $\mu = 1$, then the result will become the same as that of Ôtani [13].

2. LOCAL EXISTENCE AND UNIQUENESS

Let H be a real Hilbert space with norm $|\cdot|$ and inner product $(\cdot, \cdot)$. Let us consider the second order quasilinear evolution equation

$$u''(t) + M(|A^{1/2}u(t)|^2)Au(t) = f(u(t)), \quad t > 0, \text{ in } H, \quad (2.1)$$

$$u(0) = u_0, \quad u'(0) = u_1. \quad (2.2)$$

We assume that

(I) A is a nonnegative selfadjoint operator with domain $D(A)$.

It follows from (I) that the square root $A^{1/2}$ of A is well defined and also a nonnegative selfadjoint operator. Note that $V := D(A)$ is a real Hilbert space with the graph norm $\|v\|_V^2 := |v|^2 + |Av|^2$ and $W := D(A^{1/2})$ is a real Hilbert space with the graph norm $\|w\|_W^2 := |w|^2 + |A^{1/2}w|^2$.

(II) $M \in C^1[0, \infty)$ and $M(s) \geq m_0 > 0$ with m_0 constant.

(III) f is a (possibly) nonlinear operator with domain W and there is a nonnegative and nondecreasing function $L \in C[0, \infty)$ such that

$$|f(u) - f(v)| \leq L(\|u\|_W + \|v\|_W)\|u - v\|_W \quad \text{for } u, v \in W.$$

For a real Hilbert space X let us denote by $C^m([0, T]; X)$ the space of all X -valued C^m -functions on $[0, T)$. Then we can introduce

Definition 2.1. A function $u: [0, T) \rightarrow H$ is called a solution to (2.1)-(2.2) on $[0, T)$ if

$$(1) \quad u \in C([0, T); V) \cap C^1([0, T); W) \cap C^2([0, T); H),$$

$$(2) \quad u \text{ satisfies (2.1) on } [0, T) \text{ in } H,$$

$$(3) \quad u(0) = u_0 \text{ and } u'(0) = u_1.$$

Then we can state the following

Theorem 2.2. (Local Existence) Suppose that three conditions (I)-(III) are satisfied. Then for any $u_0 \in V$ and $u_1 \in W$, there exists a number $T_m > 0$ such that the problem (2.1)-(2.2) has a unique solution $u(t)$ on $[0, T_m)$ satisfying either

$$(i) \quad T_m = +\infty \text{ or}$$

(ii) $T_m < +\infty$ and $\lim_{t \uparrow T_m} \{\|u(t)\|_V + \|u'(t)\|_W\} = +\infty$.

Remark 2.3. Our theorem 2.2 refines the result of Browder [2] in case of $M(\cdot) \equiv 1$ and improves the result of Ikehata [6].

3. PROOF OF THEOREM 2.2

The proof will be done by refining results given by Ikehata [6]. We shall give an outline of its proof.

Let $k > 0$ be an arbitrary constant satisfying

$$k \geq \left[\frac{2}{\text{Min}\{1, m_0/2\}} \{ \|u_1\|_W^2 + M(|A^{1/2}u_0|^2) \|A^{1/2}u_0\|_W^2 + M_{00} + 1 \} \right]^{1/2}, \quad (3.1)$$

$$\text{with } M_{00} := 2|f(u_0)| |Au_0| + 4m_0^{-1} |f(u_0)|^2. \quad (3.2)$$

Set

$$C_1 := |f(0)| + (\|u_0\|_W + kT_0)L(\|u_0\|_W + kT_0) + 2m_0^{-1}kL(2\|u_0\|_W + 2kT_0), \quad (3.3)$$

$$C_2 := 4m_0^{-1} [kL(2\|u_0\|_W + 2kT_0)]^2, \quad (3.4)$$

$$C_3 := \text{Max}\{(\|u_0\|_W + kT_0)L(\|u_0\|_W + kT_0), kL(2\|u_0\|_W + 2kT_0)\}, \quad (3.5)$$

$$C_4 := |f(0)| + C_3 + 4m_0^{-1}k^2M_1, \quad (3.6)$$

$$M_0 := \text{Max}\{M(r) : 0 \leq r \leq k^2\} \quad (3.7)$$

and

$$M_1 := \text{Max}\{|M'(r)| : 0 \leq r \leq k^2\}. \quad (3.8)$$

Moreover, let $T_0 > 0$ be a constant satisfying

$$\exp(C_4 T_0) \leq 2, \quad (3.9)$$

$$C_1 T_0 + C_2 T_0^2 \leq 1. \quad (3.10)$$

We consider the initial value problem (2.1)-(2.2) in H on $[0, T_0]$. Setting $u'(t) = v(t)$, the problem can be written in the system in $X := W \times H$:

$$(P.1) \quad \frac{d}{dt} \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 & -I \\ M(|A|^{1/2} u(t)|^2)A & 0 \end{bmatrix} \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} 0 \\ f(u(t)) \end{bmatrix},$$

$$(P.2) \quad \begin{bmatrix} u(0) \\ v(0) \end{bmatrix} = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}.$$

Let $\mathcal{A}(U(t)) := \begin{bmatrix} 0 & -I \\ M(|A|^{1/2} u(t)|^2)A & 0 \end{bmatrix}$ and $F(U(t)) := \begin{bmatrix} 0 \\ f(u(t)) \end{bmatrix}$

with $U(t) := \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}$ and let $U_0 := \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}$. Then the problem (P.1)-(P.2) in X can be written in the 'quasilinear' evolution equation:

$$(P.3) \quad \begin{cases} \frac{d}{dt} U(t) + \mathcal{A}(U(t))U(t) = F(U(t)) \text{ in } X \text{ on } [0, T_0], \\ U(0) = U_0 \in Y, \end{cases}$$

where $Y := V \times W$. Since the problem (2.1)-(2.2) is equivalent to (P.3), we shall simply consider the solvability of (P.3).

The norms of $X = W \times H$ and $Y = V \times W$ are respectively defined as follows:

$$\|U\|_X := \{\|u\|_W^2 + |v|^2\}^{1/2},$$

$$\|U\|_Y := \{\|u\|_V^2 + \|v\|_W^2\}^{1/2} \text{ for } U := \begin{bmatrix} u \\ v \end{bmatrix} \in X \text{ or } Y.$$

Let k be a constant satisfying (3.1). Set

$$K := \{V(\cdot) := \begin{bmatrix} \xi(\cdot) \\ \eta(\cdot) \end{bmatrix} : [0, T_0] \rightarrow Y \mid V(0) = U_0, \|A^{1/2} \xi(t)\|_W \leq k, \\ \|\xi'(t)\|_W \leq k \text{ (a.e.)}, \|V(t) - V(s)\|_X \leq \varepsilon |t - s|\}, \quad (3.11)$$

where $\varepsilon := [k^2 + \{|f(u_0)| + kT_0 L(2\|u_0\|_W + 2kT_0) + kM_0\}^2]^{1/2}$.

For each $V(\cdot) := \begin{bmatrix} \xi(\cdot) \\ \eta(\cdot) \end{bmatrix} \in K$, we will consider the linearized problems:

$$(P.4) \quad \begin{cases} \frac{d}{dt}U(t) + \cancel{A}(V(t))U(t) = F(V(t)) \text{ in } X \text{ on } [0, T_0], \\ U(0) = U_0 \in Y. \end{cases}$$

Namely, (P.4) is nothing but (P.3) with $\cancel{A}(U(t))$ and $F(U(t))$ replaced by $\cancel{A}(V(t))$ and $F(V(t))$, respectively.

By the same argument as the proof of Ikehata [6], for each $V(\cdot) := \begin{bmatrix} \xi(\cdot) \\ \eta(\cdot) \end{bmatrix} \in K$, we can obtain a unique solution $U(t)$ on $[0, T_0]$ to (P.4) satisfying

$$U(\cdot) \in C([0, T_0]; Y) \cap C^1([0, T_0]; X). \quad (3.12)$$

We define a mapping $\Phi: K \rightarrow X$ by

$$U = \Phi V \quad (V \in K). \quad (3.13)$$

In order to show that Φ maps K into itself, we need the following lemma 3.1 without proof.

Lemma 3.1. Let $U(\cdot) = \begin{bmatrix} u(\cdot) \\ u'(\cdot) \end{bmatrix}$ be a solution to the problem (P.4) for a given $V(\cdot) = \begin{bmatrix} \xi(\cdot) \\ \eta(\cdot) \end{bmatrix} \in K$. Then the following estimate holds:

$$\|u'(t)\|_W^2 + 2^{-1}m_0 \|A^{1/2}u(t)\|_W^2 \quad (3.14)$$

$$\leq [\|u_1\|_W^2 + M(|A^{1/2}u_0|^2) \|A^{1/2}u_0\|_W^2 + M_{00} + C_1t + C_2t^2] \cdot \exp(C_4t)$$

on $[0, T_0]$, where M_{00} and C_j ($j = 1, 2, 4$) are the constants given by (3.2), (3.3), (3.4) and (3.6), respectively.

Then we get the following

Lemma 3.2. Let Φ be the mapping defined by (3.13). Then Φ maps K into itself.

Proof. Let $U(\cdot) := \begin{bmatrix} u(\cdot) \\ u'(\cdot) \end{bmatrix}$ be a solution to the problem (P.4) for each $V(\cdot) := \begin{bmatrix} \xi(\cdot) \\ \eta(\cdot) \end{bmatrix} \in K$. We have to show that $U(\cdot) \in K$. We see from lemma 3.1 and (3.9)-(3.10) that

$$\begin{aligned} & \text{Min}\{2^{-1}m_0, 1\} [\|u'(t)\|_W^2 + \|A^{1/2}u(t)\|_W^2] \\ & \leq 2[\|u_1\|_W^2 + M(|A^{1/2}u_0|^2) \|A^{1/2}u_0\|_W^2 + M_{00} + 1] \text{ on } [0, T_0]. \end{aligned}$$

Therefore, it follows from (3.1) that

$$\|u'(t)\|_W^2 + \|A^{1/2}u(t)\|_W^2 \leq k^2$$

and hence

$$\|u'(t)\|_W \leq k \text{ and } \|A^{1/2}u(t)\|_W \leq k. \quad (3.15)$$

On the other hand, since $U(\cdot) := \begin{bmatrix} u(\cdot) \\ u'(\cdot) \end{bmatrix}$ is a solution to the problem (P.4) for $V(\cdot) := \begin{bmatrix} \xi(\cdot) \\ \eta(\cdot) \end{bmatrix} \in K$, $u(t)$ satisfies

$$u''(t) + M(|A^{1/2}\xi(t)|^2)Au(t) = f(\xi(t)), \quad (3.16)$$

$$u(0) = u_0, \quad u'(0) = u_1.$$

Therefore, by (3.7), (3.11), (3.15) and (3.16) we have

$$\begin{aligned} \|U'(t)\|_X^2 &= \|u'(t)\|_W^2 + |u''(t)|^2 \\ &\leq k^2 + |-M(|A|^{1/2}\xi(t)|^2)Au(t) + f(\xi(t))|^2 \\ &\leq k^2 + \{M_0|Au(t)| + |f(\xi(t))|\}^2 \\ &\leq k^2 + \{M_0k + |f(u_0)| + kT_0L(2\|u_0\|_W + 2kT_0)\}^2 = \varepsilon^2. \end{aligned}$$

Here we have used the fact that

$$|f(\xi(t))| \leq |f(u_0)| + kT_0L(2\|u_0\|_W + 2kT_0).$$

Therefore, it holds that

$$\begin{aligned} \|U(t) - U(s)\|_X &= \left\| \int_s^t U'(r) dr \right\|_X \\ &\leq \left| \int_s^t \|U'(r)\|_X dr \right| \leq \varepsilon |t - s|. \end{aligned} \quad (3.17)$$

(3.15), (3.17) and the fact that $U(0) = U_0$ and $U(\cdot) \in C([0, T_0]; Y)$ imply $U(\cdot) \in K$, i.e., Φ maps K into itself. Q.E.D.

Furthermore, we get the following

Lemma 3.3. Let $U_1(\cdot)$ and $U_2(\cdot)$ be solutions to the problem

(P.4) for given $V_1(\cdot) := {}^t[\xi(\cdot), \zeta_1(\cdot)] \in K$ and

$V_2(\cdot) := {}^t[\eta(\cdot), \zeta_2(\cdot)] \in K$, respectively. Then

$$d(U_1, U_2) \leq C_6(T_0 + T_0^3)^{1/2} \exp(C_5 T_0) d(V_1, V_2), \quad (3.18)$$

where $d(V, W) := \sup\{\|V(t) - W(t)\|_X : 0 \leq t \leq T_0\}$,

$$C_5 := \frac{1}{2} L(2\|u_0\|_W + 2kT_0) + k^2 M_1 \max\{m_0^{-1}, 1\}$$

and

$$C_6^2 := [2k^2 M_1 + L(2\|u_0\|_W + 2kT_0)][\min\{1, m_0\}]^{-1}.$$

Finally we assume that $T_0 > 0$ satisfies

$$C_6(T_0 + T_0^3)^{1/2} \exp(C_5 T_0) < 1. \quad (3.19)$$

Then lemma 3.3 with (3.19) implies that $\Phi: K \rightarrow K$ defined in (3.13) becomes a strict contraction. Though it is not expected that K is complete with respect to the metric $d(U, V)$, we can show by iteration that there is a function $U \in K$ such that U is a unique fixed point of $\Phi: K \rightarrow K$, i.e., $\Phi U = U$ and is a unique strong solution to (P.3) on $[0, T_0]$, or equivalently to (2.1)-(2.2) on $[0, T_0]$ (for details, see Ikehata [6]). This completes the proof of theorem 2.2.

4. BLOWING-UP OF SOLUTIONS

In this section we consider the blowing-up property to the Problem 4.1. Consider the mixed problems:

$$u_{tt}(t, x) - (\alpha + 2\beta \int_{\Omega} |\nabla u(t, y)|^2 dy) \Delta u(t, x) = \mu(u(t, x))^3 \quad (4.1)$$

$$\text{for } x \in \Omega, \quad t > 0,$$

$$u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in \Omega, \quad (4.2)$$

$$u(t, x)|_{\partial\Omega} = 0, \quad t > 0. \quad (4.3)$$

Here $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary, $\alpha > 0$, $\beta \geq 0$ and $\mu \in \mathbb{R}$. On the 'local' solvability to the problems (4.1)-(4.3), we can apply our theorem 2.2.

Let H be the real $L^2(\Omega)$, and let $\|\cdot\|_p$ be the usual real $L^p(\Omega)$ -norms ($1 \leq p \leq \infty$). We define a positive definite selfadjoint operator A in H as follows:

$$Au := -\Delta u \text{ for } u \in D(A) = H^2(\Omega) \cap H_0^1(\Omega).$$

Then $\|A^{1/2}u\|_2 = \|\nabla u\|_2$ for $u \in D(A^{1/2}) = H_0^1(\Omega)$. Moreover, a nonlinear operator f in H can be defined as follows ($\mu \in \mathbb{R}$):

$$f(u)(x) := \mu(u(x))^3 \text{ for } u \in H_0^1(\Omega). \quad (4.4)$$

Note that (4.4) is well defined by means of the well known Sobolev inequality (note $N = 3$):

Lemma 4.2.(Sobolev) If $1 \leq r \leq 6$, then

$$\|u\|_r \leq C \|\nabla u\|_2 \text{ for } u \in H_0^1(\Omega)$$

with some constant $C > 0$.

Furthermore, if we set

$$C(r, \Omega) := \sup\{\|u\|_r / \|\nabla u\|_2 : u \in H_0^1(\Omega), u \neq 0\}, \quad (SC)$$

then the best constant $C(r, \Omega) > 0$ is finite by means of lemma 4.2.

Next we can easily make sure that a nonlinear operator f defined by (4.4) satisfies the condition (III) in Section 2 and also, in problem 4.1, we have only to consider the case of

$$M(r) := \alpha + 2\beta r.$$

So the problem 4.1 with $\alpha > 0$, $\beta \geq 0$ and $\mu \in \mathbb{R}$ has a unique local strong solution $u(t, \cdot)$ belonging to the class

$$C([0, T_m]; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0, T_m]; H_0^1(\Omega)) \cap C^2([0, T_m]; L^2(\Omega))$$

for some $T_m > 0$ by applying our theorem 2.2.

The purpose in this Section 4 is to discuss the "blowing-up" property of a local solution $u(t, x)$ to the problem (4.1)-(4.3).

In the following paragraph, we further assume that $\mu \in \mathbb{R}$ in (4.1) satisfies

$$\mu > 2\beta C(4, \Omega)^{-4}, \quad (4.5)$$

where $C(4, \Omega)$ is a constant defined in (SC).

Let

$$J(u) := \frac{\alpha}{2} \|\nabla u\|_2^2 + \frac{\beta}{2} \|\nabla u\|_2^4 - \frac{1}{4} \mu \|u\|_4^4 \quad \text{for } u \in H_0^1(\Omega) \quad (4.6)$$

and let

$$d := \inf_{\lambda \in \mathbb{R}} \{ \sup_{u \in H_0^1(\Omega), u \neq 0} J(\lambda u) \}. \quad (4.7)$$

Lemma 4.3. Let μ and β in (4.1) satisfy (4.5). Then

$$d \geq 4^{-1} \alpha^2 (C(4, \Omega)^4 \mu - 2\beta)^{-1} > 0.$$

$$\text{Let } W^* := \{u \in H_0^1(\Omega) : J(u) < d, \alpha \|\nabla u\|_2^2 + 2\beta \|\nabla u\|_2^4 < \mu \|u\|_4^4\},$$

$$E(0) := \frac{1}{2} [\|u_1\|_2^2 + \alpha \|\nabla u_0\|_2^2 + \beta \|\nabla u_0\|_2^4] \text{ and } F(u_0) := \frac{\mu}{4} \|u_0\|_4^4.$$

By using lemma 4.3, we get the main theorem of this section.

Theorem 4.4. Let μ and β in (4.1) satisfy (4.5). Let $u(t, x)$ be a local solution to (4.1)-(4.3) on $[0, T_m)$ with initial data $u_0 \in W^* \cap H^2(\Omega)$ and $u_1 \in H_0^1(\Omega)$ satisfying $E(0) - F(u_0) < d$. Then $T_m < +\infty$ (i.e., $u(t, x)$ can not be continued to $[0, +\infty)$ as a solution to (4.1)-(4.3)).

Remark 4.5. When $\alpha = 1$, $\beta = 0$ and $\mu = 1$, our result coincides with that of Ôtani [13].

In order to prove Theorem 4.4, we prepare some lemmas.

Lemma 4.6. Let μ and β in (4.1) satisfy (4.5). Let $u(t, x)$ be a local solution on $[0, T_m)$ to (4.1)-(4.3) with initial data $u_0 \in W^* \cap H^2(\Omega)$ and $u_1 \in H_0^1(\Omega)$ satisfying $E(0) - F(u_0) < d$. Then $u(t, \cdot) \in W^*$ on $[0, T_m)$.

Lemma 4.7. Let $u(t, x)$ be a local solution to (4.1)-(4.3) satisfying (4.5). Then $\alpha \|\nabla u(t, \cdot)\|_2^2 > 4d$ whenever $u(t, \cdot) \in W^*$.

Remark 4.8. In lemma 4.7, we can take $\beta = 0$ in (4.1) which result coincides with that of Ôtani [13] (i.e., the case of semi-linear wave equations). However, we cannot take $\alpha = 0$ which are essential in this paper.

Proof of Theorem 4.4. Suppose $T_m = +\infty$ and let $u(t) := u(t, \cdot)$ be a 'global' solution to (4.1)-(4.3) with (4.5).

First note that the identity:

$$\frac{1}{2} \frac{d^2}{dt^2} \|u(t)\|_2^2 = \|u'(t)\|_2^2 + (u''(t), u(t)). \quad (4.8)$$

Here (f, g) means usual $L^2(\Omega)$ -inner products. Multiplying (4.1) by $u(t) = u(t, \cdot)$ and integrating it over Ω , we have

$$(u''(t), u(t)) = \mu \|u(t)\|_2^4 - M(\|\nabla u(t)\|_2^2) \|\nabla u(t)\|_2^2, \quad (4.9)$$

where $M(r) = \alpha + 2\beta r$. (4.8) and (4.9) give

$$\frac{1}{2} \frac{d^2}{dt^2} \|u(t)\|_2^2 = \|u'(t)\|_2^2 + \mu \|u(t)\|_2^4 - M(\|\nabla u(t)\|_2^2) \|\nabla u(t)\|_2^2. \quad (4.10)$$

It follows from the definition of W^* , lemma 4.6 and (4.10) that

$$\frac{1}{2} \frac{d^2}{dt^2} \|u(t)\|_2^2 \geq \|u'(t)\|_2^2 \geq 0$$

which means the convexity of a function $t \rightarrow \|u(t)\|_2^2$.

On the other hand, multiplying (4.1) by $u'(t) := u_t(t, \cdot)$ and integrating it over Ω , we have

$$\frac{1}{2} \frac{d}{dt} [\|u'(t)\|_2^2 + \bar{M}(\|\nabla u(t)\|_2^2)] = \frac{d}{dt} F(u(t)), \quad (4.11)$$

where $F(u(t)) := 4^{-1} \mu \|u(t)\|_4^4$ and $\bar{M}(r) := \alpha r + \beta r^2$. Integrating the both sides of (4.11) on $[0, t]$, we have

$$\frac{1}{2} \|u'(t)\|_2^2 + \frac{1}{2} \bar{M}(\|\nabla u(t)\|_2^2) - F(u(t)) = E(0) - F(u_0).$$

Thus we get

$$2\|u'(t)\|_2^2 + 2\bar{M}(\|\nabla u(t)\|_2^2) - 4F(u(t)) = 4(E(0) - F(u_0)). \quad (4.12)$$

(4.10) and (4.12) give

$$\begin{aligned} \frac{1}{2} \frac{d^2}{dt^2} \|u(t)\|_2^2 &= 3\|u'(t)\|_2^2 + 2\bar{M}(\|\nabla u(t)\|_2^2) - M(\|\nabla u(t)\|_2^2) \|\nabla u(t)\|_2^2 \\ &\quad - 4(E(0) - F(u_0)). \end{aligned}$$

Noting that

$$2\bar{M}(r) - M(r)r = 2\alpha r + 2\beta r^2 - (\alpha + 2\beta r)r = \alpha r,$$

we have

$$\frac{1}{2} \frac{d^2}{dt^2} \|u(t)\|_2^2 = 3\|u'(t)\|_2^2 + \alpha \|\nabla u(t)\|_2^2 - 4(E(0) - F(u_0)). \quad (4.13)$$

It follows from lemma 4.7 and (4.13) that

$$\frac{1}{2} \frac{d^2}{dt^2} \|u(t)\|_2^2 \geq 4d - 4(E(0) - F(u_0)) = 4[d - (E(0) - F(u_0))].$$

Integrating this inequality on $[0, t]$, we get

$$\frac{d}{dt} \|u(t)\|_2^2 \geq 2(u_0, u_1) + 8[d - (E(0) - F(u_0))]t.$$

Since $d - (E(0) - F(u_0)) > 0$ by assumption, there exists a constant $t_1 > 0$ such that $\frac{d}{dt} \|u(t_1)\|_2^2 > 0$. With the aid of the convexity of $t \rightarrow \|u(t)\|_2^2$, we find that the function $t \rightarrow \|u(t)\|_2^2$ is monotone increasing on $[t_1, \infty)$.

Furthermore, it follows from (4.13) and the Poincaré inequality that

$$\frac{1}{2} \frac{d^2}{dt^2} \|u(t)\|_2^2 \geq 6\|u'(t)\|_2^2 + 2\alpha\lambda_1 \|u(t)\|_2^2 - 8(E(0) - F(u_0)), \quad (4.14)$$

where λ_1 is the first eigen value of $-\Delta$ (with Dirichlet null conditions). Since the function $t \rightarrow 2\alpha\lambda_1 \|u(t)\|_2^2 - 8(E(0) - F(u_0))$ is monotone increasing on $[t_1, \infty)$, there is a constant $t_2 > t_1$ such that

$$2\alpha\lambda_1 \|u(t)\|_2^2 - 8(E(0) - F(u_0)) > 0 \text{ on } [t_2, \infty). \quad (4.15)$$

By (4.14) and (4.15), we have

$$\frac{1}{2} \frac{d^2}{dt^2} \|u(t)\|_2^2 \geq 6\|u'(t)\|_2^2 \text{ on } [t_2, \infty). \quad (4.16)$$

Set $P(t) := \|u(t)\|_2^2$. Then by (4.16), we obtain

$$\begin{aligned} P(t)P''(t) - \frac{3}{2} \cdot (P'(t))^2 &\geq 6\|u(t)\|_2^2 \cdot \|u'(t)\|_2^2 - \frac{3}{2}[2(u(t), u'(t))]^2 \\ &= 6\{\|u(t)\|_2^2 \cdot \|u'(t)\|_2^2 - [(u(t), u'(t))]^2\} \text{ on } [t_2, \infty). \end{aligned}$$

So the Schwarz inequality gives

$$P(t)P''(t) - \frac{3}{2} \cdot (P'(t))^2 \geq 0 \text{ on } [t_2, \infty).$$

Therefore, it follows from the standard 'concavity argument' (see Levine [12]) that there is a constant $T_0 > t_2$ such that

$$\lim_{t \uparrow T_0} \|u(t)\|_2 = +\infty$$

which contradicts to $T_m = +\infty$.

Q.E.D.

In theorem 4.4, the condition (4.5) plays an essential role to get a 'blowing-up' property. Indeed, we get the following

Proposition 4.9. Let $u(t, x)$ be a local solution on $[0, T_m)$ to the problem (4.1)-(4.3) with μ satisfying

$$0 \leq \mu \leq 2\beta C(4, \Omega)^{-4}. \quad (4.17)$$

If the initial data $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $u_1 \in H_0^1(\Omega)$ satisfies $F(u_0) < E(0)$, then there is a constant $C > 0$ such that

$$\|\nabla u(t, \cdot)\|_2 \leq C \text{ and } \|u_t(t, \cdot)\|_2 \leq C \text{ on } [0, T_m).$$

Remark 4.10. Of course, we have to consider the cases of $J_0 = E(0) - F(u_0) \leq 0$. However, when $J_0 < 0$, (4.1)-(4.3) has no solutions and also when $J_0 = 0$, it follows that $u(t, x) \equiv 0$ is a unique solution to (4.1)-(4.3) with (4.17). So we have only to treat the case of $J_0 > 0$ in the argument of proposition 4.9.

REFERENCES

1. W.F. Ames, Nonlinear Partial Differential Equations in Engineering, I. Academic Press (1965).
2. F.E. Browder, On nonlinear wave equations, Math. Z., 80 (1962), 249-264.
3. Y. Ebiara, M. Nakao and T. Nanbu, On the existence of global classical solution of initial-boundary value problem for

- $\square u - u^3 = f$, Pacific J. of Math., 60 (1975), 63-70.
4. M. Hosoya and Y. Yamada, On some nonlinear wave equations: local existence and regularity of solutions, preprint.
 5. T.J.R. Hughes, T. Kato and J.E. Marsden, Well- posed Quasilinear Second-order Hyperbolic Systems with Applications to Nonlinear Elastodynamics and General Relativity, Arch. Rational Mech. Anal., 63 (1977), 273-294.
 6. R. Ikehata, On solutions to some quasilinear hyperbolic equations with nonlinear inhomogeneous terms, Nonlinear Analysis, 17 (1991), 181-203.
 7. R. Ikehata and N. Okazawa, A class of second order quasilinear evolution equations, in preparation.
 8. H. Ishii, Asymptotic Stability and Blowing-Up of Solutions of Some Nonlinear Equations, Journal of Differential Equations, 26 (1977), 291-319.
 9. T. Kato, Linear evolution equations of "hyperbolic" type, J. Fac. Sci. Univ. Tokyo, Sect. I, 17 (1970), 241-258.
 10. T. Kato, Quasi-linear Equations of Evolution, with Applications to Partial differential Equations, Springer Lecture Notes in Mathematics 448, 25-70 (1975).
 11. T. Kato, Abstract Differential Equations and Nonlinear Mixed Problems, Scuola Normale Superiore, Pisa (1985).
 12. H.A. Levine, Instability and nonexistence of global solutions to nonlinear wave equations of the form $Pu_{tt} = -Au + \mathcal{F}(u)$, Trans. Amer. Soc., 192 (1974), 1-21.
 13. M. Ôtani, An Introduction to Nonlinear Evolution Equations, Summer Seminar Note, (1983) (in Japanese).
 14. L.E. Payne and D.H. Sattinger, Saddle points and unstability of nonlinear hyperbolic equations, Israel J. of Math., 22 (1975), 273-303.
 15. A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Applied Mathematical Sciences 44. Springer, Berlin (1983).
 16. M. Reed, "Abstract Non-Linear Wave Equations", Lecture Notes in Math., 507 Springer-Verlag (1976).

17. M. Tsutsumi, On solutions of semilinear differential equations in a Hilbert space, Math. Japon., 17 (1972), 173-193.