

Theta functions and the tropical Riemann-Roch inequality on tropical abelian surfaces

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1. Abstract

We show that the space of theta functions on tropical tori is identified with a convex polyhedron. We also show a Riemann-Roch inequality for tropical abelian surfaces by calculating the self-intersection numbers of divisors. This poster is due to [arXiv:1809.10987](https://arxiv.org/abs/1809.10987).

2. Preliminaries

Tropical geometry: algebraic geometry over the tropical semifield $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ equipped with the tropical sum " $x + y = \max\{x, y\}$ " and the tropical product " $xy = x + y$ ".

- Tropical polynomial : " $\sum_n a_n x^n = \max\{a_n + n \cdot x\}$ ".
- Regular function : a convex piecewise linear function with integral slopes.
- Tropical line bundle : a $\mathbb{T}$ -bundle with $\mathbb{Z}$ -affine linear transition functions.

Fix the standard lattice $\mathbb{Z}^n$.

Tropical torus : $\mathbb{R}^n / \Lambda$, where Λ is a lattice of rank n .

Facts:

- For a tropical line bundle L on a tropical torus X , we can take a symmetric form Q on $\mathbb{R}^n$ which satisfies $Q(\lambda, \cdot) \in (\mathbb{Z}^n)^*$ and a linear function α on $\mathbb{R}^n$ such that L is isomorphic to the line bundle $\mathbb{R}^n \times \mathbb{R} / (x, r) \sim (x + \lambda, r + Q(\lambda, x) + \frac{1}{2}Q(\lambda, \lambda) + \alpha(\lambda))$. Then we denote $L = L(Q, \alpha)$. Q is called the first Chern class of L and denoted by $c_1(L)$.
- For $\gamma \in (\mathbb{Z}^n)^*$, $L(Q, \alpha + \gamma) \cong L(Q, \alpha)$.
- Let $q: \mathbb{R}^n \rightarrow (\mathbb{R}^n)^*$ be a linear function given by $q(x) = Q(x, \cdot)$ and let $\iota_c: \mathbb{R}^n / \Lambda \rightarrow \mathbb{R}^n / \Lambda$ be a translation given by $x \mapsto x + c$ for $c \in \mathbb{R}^n$. Then $\iota_c^* L(Q, \alpha) \cong L(Q, \alpha + q(c))$.

3. Tropical line bundles on tropical tori

$X = \mathbb{R}^n / \Lambda$: a tropical tori.

$L = L(Q, \alpha)$: a tropical line bundle on X .

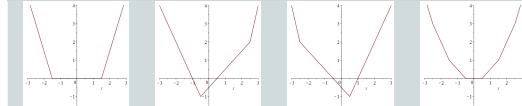
Definition

A theta function for $L = L(Q, \alpha)$ is a regular function on $\mathbb{R}^n$ which satisfies the following quasi-periodicity:

$$\Theta(x + \lambda) = \Theta(x) + Q(\lambda, x) + \frac{1}{2}Q(\lambda, \lambda) + \alpha(\lambda), \quad \lambda \in \Lambda, \quad x \in \mathbb{R}^n$$

Theta functions are identified with global sections of L which are not $-\infty$. $\{\text{Theta functions for } L\} \cup \{-\infty\}$ is identified with $H^0(X, L)$.

Example



Graphs of some theta functions for a line bundle $L(1, 0)$ on $\mathbb{R}/3\mathbb{Z}$.

Remark

Theta functions are regular, and thus convex. If Q has a negative eigenvalue, the convexity is broken. Thus then $H^0(X, L) = \{-\infty\}$.

Mikhalkin-Zharkov(2008) states the following proposition without proof:

Proposition

Suppose that L is ample, that is, Q is positive definite. Then the $\mathbb{T}$ -module $H^0(X, L)$ is generated by just $|(\mathbb{Z}^n)^* / q(\Lambda)|$ elements.

By analyzing theta functions, we get more detailed statement. We set $l = |\text{torsion part of } (\mathbb{Z}^n)^* / q(\Lambda)|$.

Theorem (Main result 1)

Suppose Q : positive semidefinite.

- If α does not lie in $(\mathbb{Z}^n)^* + \text{Im}(q)$, then $H^0(X, L) = \{-\infty\}$.
- If α lies in $(\mathbb{Z}^n)^* + \text{Im}(q)$, then $H^0(X, L)$ is generated by just l elements as a $\mathbb{T}$ -module. Moreover, we can define a "natural" embedding $\varphi: H^0(X, L) \rightarrow \mathbb{T}^l$. Then $H^0(X, L)$ is identified with an l -dimensional convex polyhedron in $\mathbb{T}^l$ via φ and its projectivization is identified with a compact convex polyhedron in $\mathbb{R}^{l-1}$.

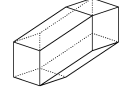
4. Example

$$Q := \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad \Lambda := q^{-1}(2(\mathbb{Z}^2)^*), \quad X := \mathbb{R}^2 / \Lambda.$$

The tropical projectivization of $\varphi(H^0(X, L(Q, 0)))$ is identified with

$$\bigcup_{-\frac{1}{6} \leq a \leq \frac{1}{6}} \left\{ (r, s, t) + (a, a, a) \mid |r| \leq \frac{1}{6}, |s| \leq \frac{1}{6}, |t| \leq \frac{1}{6} \right\}.$$

This is the convex polyhedron that has 14 vertices.



5. Divisors and intersection numbers

$X = \mathbb{R}^2 / \Lambda$: a 2-dim tropical torus.

Definition

A divisor on X is a tropical curve on X , that is, a graph Γ on X with weight function w : (edges of Γ) $\rightarrow \mathbb{Z}$ satisfying "the balancing condition".

There exists a 1 to 1 correspondence between equivalent classes of divisors and equivalent classes of line bundles as in the classical case.

$\mathcal{O}(D)$: line bundle given by D .

Definition

D_1, D_2 : divisors on a 2-dimensional tropical tori X ,

p : an intersection point of D_1, D_2 ,

E_i : an edge of D_i containing p .

The intersection number of D_1 and D_2 at p is

$$(D_1, D_2)_p = w(E_1)w(E_2) | \text{cross product of slopes of } E_1 \text{ and } E_2 |.$$

This is independent on the choice of E_i .

$$\text{The intersection number of } D_1 \text{ and } D_2 \text{ is } (D_1, D_2) = \sum_{p \in D_1 \cap D_2} (D_1, D_2)_p$$

Tropical abelian surface : a 2-dim tropical torus that has an ample line bundle.

Proposition

$X = \mathbb{R}^2 / \Lambda$: a tropical abelian surface, D : a divisor on X .

$$\Rightarrow D^2 = 2 \det Q \cdot \text{vol}(\mathbb{R}^2 / \Lambda).$$

Here $Q = c_1(\mathcal{O}(D))$ and $\det Q$ is the determinant associated to the standard basis of $\mathbb{R}^2$

Remark

$$\text{rk}(Q) = 2 \Rightarrow \det Q \cdot \text{vol}(\mathbb{R}^2 / \Lambda) = \pm |(\mathbb{Z}^n)^* / q(\Lambda)|.$$

6. Main result 2

Combining Main result 1 and the above proposition, we have

Theorem

$X = \mathbb{R}^2 / \Lambda$: a tropical abelian surface, D : a divisor on X .

If we define $h^0(X, \mathcal{O}(D)) := \dim H^0(X, \mathcal{O}(D))$, we have

$$h^0(X, \mathcal{O}(D)) + h^0(X, \mathcal{O}(-D)) \geq \frac{1}{2} D^2.$$

The classical Riemann-Roch formula for abelian surfaces is

$$h^0(X, L) - h^1(X, L) + h^2(X, L) = \frac{1}{2} L^2.$$

Thus we can call the above inequality Riemann-Roch inequality.

7. Validity

The Riemann-Roch theorem on tropical curves is the following.

Theorem (Gathmann-Kerber(2008) and Mikhalkin-Zharkov, 2008)

C : compact tropical curve with genus g , D : divisor on C ,

K : canonical divisor of C . Then

$$r(D) - r(K - D) = \deg D - g + 1.$$

Here

$$r(D) = \max \{ k \in \mathbb{Z} \mid \forall E \in \text{Div}_k^+(C), |D - E| \neq \emptyset \} \quad \text{if } |D| \neq \emptyset \\ = -1 \quad \text{if } |D| = \emptyset$$

In general, the number $r(D)$ is different from $\dim |D|$ as a polyhedral complex. Gathmann-Kerber(2008) gave an example where

$r(D) + 1 \neq \dim H^0(C, \mathcal{O}(D))$. In their example, $H^0(C, \mathcal{O}(D))$ was not pure-dimensional; it seems that $r(D) + 1 = \dim H^0(C, \mathcal{O}(D))$ if $H^0(C, \mathcal{O}(D))$ is a pure-dimensional polyhedral complex. Thus we study the space of global sections of line bundles over tropical tori and show that it is always pure-dimensional.