

Perturbation of the Navier-Stokes flow in an annular domain with the non-vanishing outflow condition

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Abstract. The boundary value problem of the Navier-Stokes equations has been studied so far only under the vanishing outflow condition due to Leray. We consider this problem in an annular domain $D = \{\mathbf{x} \in \mathbf{R}^2; R_1 < |\mathbf{x}| < R_2\}$, under the boundary condition with non-vanishing outflow. In a previous paper of the first author, an exact solution is obtained for a simple boundary condition of non-vanishing outflow type: $\mathbf{u} = \frac{\mu}{R_i} \mathbf{e}_r + b_i \mathbf{e}_\theta$ on Γ_i , $i = 1, 2$, where μ, b_1, b_2 are arbitrary constants. In this paper, we show the existence of solutions satisfying the boundary condition: $\mathbf{u} = \{\frac{\mu}{R_i} + \varphi_i(\theta)\} \mathbf{e}_r + \{b_i + \psi_i(\theta)\} \mathbf{e}_\theta$ on Γ_i , $i = 1, 2$, where $\varphi_i(\theta), \psi_i(\theta)$ are 2π -periodic smooth function of θ , under some additional condition.

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Let D be an annular domain in $\mathbf{R}^2$:

$$D = \{\mathbf{x} \in \mathbf{R}^2; R_1 < |\mathbf{x}| < R_2\},$$

where $0 < R_1 < R_2$, and Γ_i its boundary:

$$\Gamma_i = \{\mathbf{x} \in \mathbf{R}^2; |\mathbf{x}| = R_i\}, \quad i = 1, 2.$$

We consider the boundary value problem of the Navier-Stokes equations:

$$(1) \quad \begin{cases} -\nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \frac{1}{\rho} \nabla p = \mathbf{0} & \text{in } D, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } D, \\ \mathbf{u} = \mathbf{b} & \text{on } \partial D, \end{cases}$$

where $\mathbf{u}$ is the fluid velocity, p is the pressure, ν (the kinematic viscosity) and ρ (the density) are given positive constants, and the vector $\mathbf{b}$ is a given boundary value of the velocity.

For the general bounded domain $D \subset \mathbf{R}^n, n \geq 2$, J. Leray [6] showed the existence of the solution to this problem under the following condition:

$$(H) \quad \int_{\Gamma_i} \mathbf{b} \cdot \mathbf{n} \, ds = 0, \quad 1 \leq i \leq k,$$

where $\partial D = \bigcup_{i=1}^k \Gamma_i$, Γ_i is the connected component of ∂D and $\mathbf{n}$ is the unit outward normal to the boundary ∂D . The condition (H) is stronger than the condition

$$(H)_0 \quad \int_{\partial \Omega} \mathbf{b} \cdot \mathbf{n} \, ds = \sum_{i=1}^k \int_{\Gamma_i} \mathbf{b} \cdot \mathbf{n} \, ds = 0,$$

which is to be satisfied by the boundary value $\mathbf{b}$ of a solenoidal vector $\mathbf{u}$.

We are concerned with the problem whether does exist a solution to (1) under the non-vanishing outflow condition $(H)_0$, even if the boundary value does not satisfy the vanishing outflow condition (H) ([2], [6]). In the previous paper [7], the first author showed an exact solution to this equation in an annular domain under the boundary condition with non-vanishing outflow given by,

$$\mathbf{b} = \frac{\mu}{R_i} \mathbf{e}_r + b_i \mathbf{e}_\theta \quad \text{on } \Gamma_i, \quad i = 1, 2,$$

where μ, b_1, b_2 are given constants and $\mathbf{e}_r, \mathbf{e}_\theta$ are the unit vectors in the polar coordinates representation $\{r, \theta\}$.

In this paper, we study the case where the boundary value depends on θ variable, more precisely, the vector $\mathbf{b}$ is given as follows:

$$(2) \quad \mathbf{b} = \{a_i + \varphi_i(\theta)\} \mathbf{e}_r + \{b_i + \psi_i(\theta)\} \mathbf{e}_\theta \quad \text{on } \Gamma_i, \quad i = 1, 2.$$

Remark 1. Since the condition $(H)_0$ has to be satisfied,

$$(A1) \quad a_1 R_1 = a_2 R_2$$

should hold. We denote this common value by μ . If $\mu \neq 0$, the condition (H) does not hold.

On the other hand, without loss of generality, we can suppose the following:

$$(A2) \quad \varphi_i(\theta), \psi_i(\theta) \text{ be } 2\pi\text{-periodic smooth function of } \theta, \text{ satisfying}$$

$$\int_0^{2\pi} \varphi_i(\theta) d\theta = 0, \quad \int_0^{2\pi} \psi_i(\theta) d\theta = 0, \quad i = 1, 2.$$

Finally we put

$$\omega_i = \frac{b_i}{R_i}, \quad i = 1, 2.$$

If the absolute value $|\mu|$ of μ is small, we can show the existence of a solution to (1) (2) by the usual method (c.f. [5], [7]). We show, in the following, the existence of a solution to (1) (2) even for large $|\mu|$.

Theorem 1. Suppose (A1), A(2) hold. If the inequality

$$|\omega_1 - \omega_2| \frac{R_1^2 R_2^2}{R_2^2 - R_1^2} \left(\log \frac{R_2}{R_1} \right)^2 < 2\nu$$

is satisfied and $\varphi_i(\theta), \psi_i(\theta)$ ($i = 1, 2$) are sufficiently small, then there exists at least one strong solution to (1) (2) for any $\mu \in \mathbf{R}$ with at most countable exception.

Remark 2. Let μ, ω_1, ω_2 be constants and $\varphi_i(\theta) = \psi_i(\theta) = 0$ ($i = 1, 2$). Then, we have the following exact solution $\mathbf{u}_0$ to (1) (2) of the form:

$$\mathbf{u}_0 = \frac{\mu}{r} \mathbf{e}_r + b(\mu, r) \mathbf{e}_\theta.$$

(i) For $\mu \neq -2\nu$,

$$b(\mu, r) = \frac{c_1}{r} + c_2 r^{1+\frac{\mu}{\nu}},$$

where

$$c_1 = \frac{\omega_1 R_1^2 R_2^{2+\frac{\mu}{\nu}} - \omega_2 R_2^2 R_1^{2+\frac{\mu}{\nu}}}{R_2^{2+\frac{\mu}{\nu}} - R_1^{2+\frac{\mu}{\nu}}}, \quad c_2 = \frac{\omega_2 R_2^2 - \omega_1 R_1^2}{R_2^{2+\frac{\mu}{\nu}} - R_1^{2+\frac{\mu}{\nu}}}.$$

(ii) For $\mu = -2\nu$,

$$b(\mu, r) = \frac{1}{r} (c_1 + c_2 \log r),$$

where

$$c_1 = \frac{\omega_1 R_1^2 \log R_2 - \omega_2 R_2^2 \log R_1}{\log R_2 - \log R_1}, \quad c_2 = \frac{\omega_2 R_2^2 - \omega_1 R_1^2}{\log R_2 - \log R_1}.$$

The pressure p_0 can be obtained from the equation. This solution is unique if $|\mu|$ and $|\omega_1 - \omega_2|$ (case(i)) ($|\omega_1|, |\omega_2|$ (case(ii))) are sufficiently small (c.f. [7], [8]).

Let us prove Theorem 1 in several steps. Let $C_{0,\sigma}^\infty(D)$ be all smooth solenoidal functions with compact support in the domain D , H_σ the closure of $C_{0,\sigma}^\infty(D)$ in $L^2(D)$, and V the closure of $C_{0,\sigma}^\infty(D)$ in the Sobolev space $H^1(D)$.

Let $\mathbf{u}_0, p_0$ be the solution as above. Let $\mathbf{v}_0$ satisfy the condition

(3) $\operatorname{div} \mathbf{v}_0 = 0$ in D , and $\mathbf{v}_0 = \varphi_i(\theta) \mathbf{e}_r + \psi_i(\theta) \mathbf{e}_\theta$ on Γ_i , $i = 1, 2$.

The existence of such function is known (c.f. [1]) but for our convenience, we choose:

$$\begin{aligned} \mathbf{v}_0 = & \left[R_1 r^{-1} \int_r^{R_2} \alpha(t) dt \varphi_1(\theta) + R_2 r^{-1} \int_{R_1}^r \alpha(t) dt \varphi_2(\theta) \right. \\ & \left. - r^{-1} \int_{R_1}^r \beta_1(t) dt \psi_1'(\theta) - r^{-1} \int_{R_1}^r \beta_2(t) dt \psi_2'(\theta) \right] \mathbf{e}_r \\ & + \left[\int_0^\theta \{R_1 \varphi_1(t) - R_2 \varphi_2(t)\} dt \alpha(r) + \beta_1(r) \psi_1(\theta) + \beta_2(r) \psi_2(\theta) \right] \mathbf{e}_\theta, \end{aligned} \quad (4)$$

where, $\alpha(r)$, $\beta_i(\theta)$ ($i = 1, 2$) are smooth functions such that

$$\alpha(R_1) = \alpha(R_2) = 0, \quad \int_{R_1}^{R_2} \alpha(t) dt = 1,$$

$$\beta_i(R_j) = \delta_{ij} \quad (i, j = 1, 2), \quad \int_{R_1}^{R_2} \beta_i(t) dt = 0 \quad (i = 1, 2).$$

Then, we have the following estimate.

Lemma 1. *There exists a positive constant c_0 such that*

$$\|\mathbf{v}_0\|_{C^2(D)} \leq c_0 \sum_{i=1}^2 (\|\varphi_i\|_{C^2(I)} + \|\psi_i\|_{C^3(I)})$$

holds, where I is the closed interval $[0, 2\pi]$.

Suppose $\mathbf{u} = \mathbf{w} + \mathbf{u}_0 + \mathbf{v}_0$ satisfy (1) with $\mathbf{b}$ given in (2). Then, the equation for $\mathbf{w}$ is as follows:

$$(5) \quad \begin{cases} -\nu \Delta \mathbf{w} + (\mathbf{w} \cdot \nabla) \mathbf{u}_0 + (\mathbf{u}_0 \cdot \nabla) \mathbf{w} + (\mathbf{w} \cdot \nabla) \mathbf{w} + \frac{1}{\rho} \nabla q \\ \quad + (\mathbf{w} \cdot \nabla) \mathbf{v}_0 + (\mathbf{v}_0 \cdot \nabla) \mathbf{w} + \mathbf{f}_0 = \mathbf{0} \quad \text{in } D, \\ \operatorname{div} \mathbf{w} = 0 \quad \text{in } D, \\ \mathbf{w} = \mathbf{0} \quad \text{on } \partial D, \end{cases}$$

where $\mathbf{f}_0 = -\nu \Delta \mathbf{v}_0 + (\mathbf{v}_0 \cdot \nabla) \mathbf{v}_0 + (\mathbf{v}_0 \cdot \nabla) \mathbf{u}_0 + (\mathbf{u}_0 \cdot \nabla) \mathbf{v}_0$.

Let P be the orthogonal projection from $L^2(D)$ onto H_σ and $A = -P\Delta$ be the Stokes operator. Applying the orthogonal projection P to the first equation in (5), we get:

$$(6) \quad \begin{cases} A\mathbf{w} + \frac{1}{\nu} P\{(\mathbf{w} \cdot \nabla) \mathbf{u}_0 + (\mathbf{u}_0 \cdot \nabla) \mathbf{w}\} \\ \quad + \frac{1}{\nu} P\{(\mathbf{w} \cdot \nabla) \mathbf{w} + (\mathbf{w} \cdot \nabla) \mathbf{v}_0 + (\mathbf{v}_0 \cdot \nabla) \mathbf{w} + \mathbf{f}_0\} = \mathbf{0}. \end{cases}$$

As is well known, A is a self-adjoint positive operator in H_σ and the inverse A^{-1} is a compact operator on H_σ (e.g., [5], [10]). Applying A^{-1} to the equation (6), we obtain:

$$(7) \quad \mathbf{w} - T(\mu)\mathbf{w} + \frac{1}{\nu} A^{-1} P\{(\mathbf{w} \cdot \nabla) \mathbf{w} + (\mathbf{w} \cdot \nabla) \mathbf{v}_0 + (\mathbf{v}_0 \cdot \nabla) \mathbf{w} + \mathbf{f}_0\} = \mathbf{0},$$

where

$$T(\mu)\mathbf{w} \equiv -\frac{1}{\nu} A^{-1} P\{(\mathbf{w} \cdot \nabla) \mathbf{u}_0 + (\mathbf{u}_0 \cdot \nabla) \mathbf{w}\}.$$

Since A^{-1} is compact operator in H_σ , and its range is the domain of the operator A which is compactly imbedded in V , we obtain the following:

Lemma 2. *The operator $T(\mu)$ is a compact linear operator on V .*

Let

$$\mu_n = -2\nu + \frac{2n\pi\nu}{\log R_2 - \log R_1} i, \quad n \in \mathbf{Z}, \quad i = \sqrt{-1}.$$

We define $b(\mu, r)$ for all $\mu \in \mathbf{C}$ letting $b(\mu, r) = b(-2\nu, r)$ for $\mu = \mu_n$. Then, $b(\mu, r)$ is continuous and holomorphic in $\mu \in \mathbf{C}$ even at the points $\mu = \mu_n$. Therefore, we have:

Lemma 3. (c.f. Kato[4]) *The compact operator $T(\mu)$ is an entire function of μ .*

Lemma 4. *If $|\omega_1 - \omega_2|$ is sufficiently small, then 1 is not the eigenvalue of the operator $T(0)$.*

Proof. Suppose that 1 is an eigenvalue of $T(0)$, i.e., there exists a nonzero $\mathbf{w} \in V$ such that $T(0)\mathbf{w} = \mathbf{w}$. Then,

$$-\frac{1}{\nu}A^{-1}P\{(\mathbf{w} \cdot \nabla)\tilde{\mathbf{u}}_0 + (\tilde{\mathbf{u}}_0 \cdot \nabla)\mathbf{w}\} = \mathbf{w}$$

holds, that is,

$$-\nu A\mathbf{w} = P\{(\mathbf{w} \cdot \nabla)\tilde{\mathbf{u}}_0 + (\tilde{\mathbf{u}}_0 \cdot \nabla)\mathbf{w}\}$$

holds, where $\tilde{\mathbf{u}}_0 = b(0, r)\mathbf{e}_\theta$ (See Remark 2). Taking the inner product with $\mathbf{w}$, we have

$$\nu \|\nabla \mathbf{w}\|^2 = ((\mathbf{w} \cdot \nabla)\tilde{\mathbf{u}}_0, \mathbf{w}).$$

The right hand side is equal to:

$$2c_1(0) \int_D \frac{w_r w_\theta}{r} dr d\theta, \quad \text{where} \quad c_1(0) = \frac{\omega_1 - \omega_2}{R_2^2 - R_1^2} R_1^2 R_2^2.$$

After some calculation, we have :

$$|((\mathbf{w} \cdot \nabla)\tilde{\mathbf{u}}_0, \mathbf{w})| \leq \frac{|\omega_1 - \omega_2|}{2} \frac{R_1^2 R_2^2}{R_2^2 - R_1^2} \left(\log \frac{R_2}{R_1} \right)^2 \|\nabla \mathbf{w}\|^2.$$

If $|\omega_1 - \omega_2|$ is sufficiently small, then the inequality

$$\frac{|\omega_1 - \omega_2|}{2} \frac{R_1^2 R_2^2}{R_2^2 - R_1^2} \left(\log \frac{R_2}{R_1} \right)^2 < \nu$$

holds, and contradiction.

c.q.f.d.

Let K be any compact subset of $\mathbf{C}$. According to Theorem 1.9 in Chapter VII §1 of [4], there exist a finite set $\{\mu_1^*, \mu_2^*, \dots, \mu_k^*\}$ such that for any μ in $K \setminus \{\mu_1^*, \mu_2^*, \dots, \mu_k^*\}$, $(T(\mu) - 1)^{-1}$ exists and is bounded on V . Let $\mu \in K \setminus \{\mu_1^*, \mu_2^*, \dots, \mu_k^*\}$. From (7), we obtain:

$$(8) \quad \mathbf{w} = \frac{1}{\nu}(T(\mu) - 1)^{-1}A^{-1}P\{(\mathbf{w} \cdot \nabla)\mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{v}_0 + (\mathbf{v}_0 \cdot \nabla)\mathbf{w} + \mathbf{f}_0\}.$$

Let us denote the right hand side of (8) by $N(\mu)\mathbf{w}$:

$$(9) \quad N(\mu)\mathbf{w} = \frac{1}{\nu}(T(\mu) - 1)^{-1}A^{-1}P\{(\mathbf{w} \cdot \nabla)\mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{v}_0 + (\mathbf{v}_0 \cdot \nabla)\mathbf{w} + \mathbf{f}_0\},$$

and

$$\sigma = \sum_{i=1}^2 (\|\varphi_i\|_{C^2(I)} + \|\psi_i\|_{C^3(I)}).$$

According to Lemma 1, we have:

$$(10) \quad \left\| \frac{1}{\nu} (T(\mu) - 1)^{-1} A^{-1} P \{ (\mathbf{w} \cdot \nabla) \mathbf{v}_0 + (\mathbf{v}_0 \cdot \nabla) \mathbf{w} \} \right\|_V \leq \frac{C\sigma}{\nu} \|\mathbf{w}\|_V,$$

$$(11) \quad \left\| \frac{1}{\nu} (T(\mu) - 1)^{-1} A^{-1} P \mathbf{f}_0 \right\|_V \leq \frac{C\sigma}{\nu} \{ (\nu + c_0\sigma) |D|^{1/2} + \|\mathbf{u}_0\|_V \},$$

where c_0 is the constant in Lemma 1, $C = c_0 \|(1 - T(\mu))^{-1} A^{-1/2}\|$ and $|D|$ is the measure of D .

Lemma 5. *There exists a positive constant c_D such that the estimate*

$$\|A^{-1} P(\mathbf{v} \cdot \nabla) \mathbf{w}\|_V \leq c_D \|\mathbf{v}\|_V \|\mathbf{w}\|_V, \quad \forall \mathbf{v}, \mathbf{w} \in V$$

holds.

It is known that there exists an absolute constant c such that

$$\|A^{-1/4} P(\mathbf{v} \cdot \nabla) \mathbf{w}\| \leq c \|A^{1/2} \mathbf{v}\| \|A^{1/2} \mathbf{w}\|, \quad \forall \mathbf{v}, \mathbf{w} \in C_{0,\sigma}^\infty(D)$$

holds. See, e.g., Fujita-Kato [3]. Using this inequality, we obtain Lemma 5 easily.

Put

$$(12) \quad \rho_0 = \max \left[\frac{C}{\nu}, \frac{c_D}{\nu} \|(T(\mu) - 1)^{-1}\|, \frac{C}{\nu} \{ (\nu + c_0\sigma) |D|^{1/2} + \|\mathbf{u}_0\|_V \} \right],$$

where c_0, c_D, C are constants given in Lemma 1, in Lemma 5, and in (10), respectively. Now we have the following estimate for the nonlinear operator $N(\mu)$:

$$(13) \quad \|N(\mu) \mathbf{w}\|_V \leq \rho_0 (\|\mathbf{w}\|_V^2 + \sigma \|\mathbf{w}\|_V + \sigma).$$

$$\text{Let } \sigma_0 = \frac{1}{\rho_0} \{ 1 + 2\rho_0 - \sqrt{(1 + 2\rho_0)^2 - 1} \}.$$

Remark 3. $\rho_0 \sigma_0$ is the smallest positive root of the equation

$$X^2 - 2(1 + 2\rho_0)X + 1 = 0.$$

The inequality $0 < \sigma_0 < 1$ follows easily. If $0 < \sigma < \sigma_0$, then the equation

$$\rho_0 (X^2 + \sigma X + \sigma) = X$$

has two positive roots. Let r_σ be the smaller one:

$$r_\sigma = \frac{1}{2\rho_0} \left\{ 1 - \rho_0 \sigma - \sqrt{(1 - \rho_0 \sigma)^2 - 4\rho_0^2 \sigma} \right\}.$$

Lemma 6. *If $0 < \sigma < \sigma_0$, then the operator $N(\mu)$ maps the ball*

$$B(r_\sigma) \equiv \{ \mathbf{w} \in V ; \|\mathbf{w}\|_V \leq r_\sigma \}$$

into itself.

Proof. Let $\mathbf{w} \in B(r_\sigma)$. Then,

$$\|N(\mu)\mathbf{w}\|_V \leq \rho_0(\|\mathbf{w}\|_V^2 + \sigma\|\mathbf{w}\|_V + \sigma) \leq \rho_0(r_\sigma^2 + \sigma r_\sigma + \sigma) = r_\sigma.$$

c.q.f.d.

Lemma 7. If $0 < \sigma < \sigma_0$, the operator $N(\mu)$ is a contraction operator on $B(r_\sigma)$.

Proof. Let $\mathbf{w}_1, \mathbf{w}_2$ be arbitrary elements in $B(r_\sigma)$. Then, we have:

$$\begin{aligned} N(\mu)\mathbf{w}_1 - N(\mu)\mathbf{w}_2 &= \frac{1}{\nu}(T(\mu) - 1)^{-1}A^{-1}P\{(\mathbf{w}_1 \cdot \nabla)\mathbf{w}_1 - (\mathbf{w}_2 \cdot \nabla)\mathbf{w}_2 \\ &\quad + ((\mathbf{w}_1 - \mathbf{w}_2) \cdot \nabla)\mathbf{v}_0 + (\mathbf{v}_0 \cdot \nabla)(\mathbf{w}_1 - \mathbf{w}_2)\}. \end{aligned}$$

Since

$$(\mathbf{w}_1 \cdot \nabla)\mathbf{w}_1 - (\mathbf{w}_2 \cdot \nabla)\mathbf{w}_2 = ((\mathbf{w}_1 - \mathbf{w}_2) \cdot \nabla)\mathbf{w}_1 + (\mathbf{w}_2 \cdot \nabla)(\mathbf{w}_1 - \mathbf{w}_2),$$

therefore,

$$\begin{aligned} \|N(\mu)\mathbf{w}_1 - N(\mu)\mathbf{w}_2\|_V &\leq \frac{C_D}{\nu} \|(T(\mu) - 1)^{-1}\| \|\mathbf{w}_1 - \mathbf{w}_2\|_V (\|\mathbf{w}_1\|_V + \|\mathbf{w}_2\|_V) + \frac{C\sigma}{\nu} \|\mathbf{w}_1 - \mathbf{w}_2\|_V \\ &\leq \rho_0(\|\mathbf{w}_1\|_V + \|\mathbf{w}_2\|_V + \sigma) \|\mathbf{w}_1 - \mathbf{w}_2\|_V, \end{aligned}$$

where we used (10) and Lemma 5. Since $\mathbf{w}_1, \mathbf{w}_2 \in B(r_\sigma)$ and $\sigma < \sigma_0$, we have:

$$\rho_0(\|\mathbf{w}_1\|_V + \|\mathbf{w}_2\|_V + \sigma) \leq \rho_0(2r_\sigma + \sigma) = 1 - \sqrt{(1 - \rho_0\sigma)^2 - 4\rho_0^2\sigma} < 1.$$

Therefore the operator $N(\mu)$ is a contraction and has a fixed point in the ball $B(r_\sigma)$. Theorem 1 is thus proved.

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