

Strong solutions of Tsirelson's equation in discrete time taking values in compact spaces with semigroup action

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Abstract

Under the assumption that the infinite product of the evolution process converges almost surely, the set of strong solutions is characterized by a compact space T , which may be regarded as the set of possible initial states. More precisely, any strong solution may be represented as the result of a uniquely specified element of T acted by the infinite product of the evolution process. *Keywords:* Stochastic equation, strong solution, infinite product of random variables

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1. Introduction

Let S and Σ be compact metric spaces with countable bases and suppose that Σ is a topological semigroup acting continuously on S . Denote $\mathbb{N} = \{0, 1, 2, \dots\}$ and consider the following stochastic equation (which we call

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Tsirelson's equation in discrete time):

$$X_k = N_k X_{k-1} \quad \text{for } k \in -\mathbb{N}, \quad (1.1)$$

where $X = (X_k)_{k \in -\mathbb{N}}$ is the unknown process taking values in S and $N = (N_k)_{k \in -\mathbb{N}}$ is the driving noise process taking values in Σ . More precisely, for a given sequence $(\mu_k)_{k \in -\mathbb{N}}$ of laws on Σ , the process

$$\{(X_k)_{k \in -\mathbb{N}}, (N_k)_{k \in -\mathbb{N}}\} \quad (1.2)$$

is called a *solution* of equation (1.1) if it satisfies (1.1) and for each $k \in -\mathbb{N}$ the random variable N_k has law μ_k and is independent of $\sigma(X_j : j \leq k-1)$. We adopt the convention that two solutions are identified if their joint laws on $S^{-\mathbb{N}} \times \Sigma^{-\mathbb{N}}$ are equal in law. For instance, uniqueness always means the uniqueness in law. The process $(X_k)_{k \in -\mathbb{N}}$ evolves forward in time k so that the present state X_k is obtained from X_{k-1} , the state one step before, by being acted by N_k . Here we must keep in mind that the index k varies in $-\mathbb{N}$, the set of non-positive integers, so that there are a priori no initial time nor initial state in this evolution.

Equation (1.1) on the one-dimensional torus $\{z \in \mathbb{C} : |z| = 1\}$ as a multiplicative group has been originally studied by Tsirelson (1975) and further studied by Yor (1992). It was generalized to compact groups by Akahori et al. (2008) and by Hirayama and Yano (2010) (see Yano and Yor (2010) for the related survey). It was generalized to compact spaces with semigroup action by Yano (2010). For other related works, see Takahashi (2009), Raja (2012), Evans and Gordeeva (2011) and Delattre and Rosenbaum (2012). Note also that equation (1.1) can also be found in Furstenberg (1973).

A solution (1.2) of equation (1.1) is called *strong* if, for each $k \in -\mathbb{N}$, the present state X_k is measurable with respect to the past noise $N_k, N_{k-1}, \dots$ up to null sets; or in other words, there exists a Borel function $f_k : \Sigma^{-\mathbb{N}} \rightarrow S$ such that

$$X_k = f_k(N_k, N_{k-1}, \dots) \quad \text{almost surely.} \quad (1.3)$$

Let $(\mu_k)_{k \in -\mathbb{N}}$ be a sequence of laws on Σ and let $(N_k)_{k \in -\mathbb{N}}$ be a sequence of independent random variables such that N_k has law μ_k for each $k \in -\mathbb{N}$. If the infinite product

$$\Phi_k = \lim_{l \rightarrow -\infty} N_k N_{k-1} \cdots N_{l+1} \quad (1.4)$$

converges almost surely for each $k \in -\mathbb{N}$, then, for each $x \in S$, we see that

$$\{(\Phi_k x)_{k \in -\mathbb{N}}, (N_k)_{k \in -\mathbb{N}}\} \quad (1.5)$$

is a strong solution. For this solution, one might think of the various choices of x as corresponding to distinguishable initial states in the sense that different choices of x lead to different distributions, but that this is not always the case, as Theorem 1.3 given below shows. The following is the main result of this paper, which suggests an alternative to the initial states.

Theorem 1.1. *Suppose that the infinite product (1.4) converges almost surely for each $k \in -\mathbb{N}$. Then there exist a compact Hausdorff space T with a countable base, a continuous onto mapping $\pi : S \rightarrow T$ and a measurable section $\psi : T \rightarrow S$ of π (i.e., $\pi \circ \psi$ is identity) which satisfy the following conditions:*

- (i) *for any two distinct elements $y_1, y_2 \in T$, the solutions (1.5) for $x = \psi(y_1)$ and $x = \psi(y_2)$ are distinct;*

- (ii) *the solutions (1.5) for $x = \psi(y)$ with y running over T exhaust all strong solutions;*
- (iii) *any solution (1.2) is equal to*

$$\{(\Phi_k \psi(\Xi))_{k \in -\mathbb{N}}, (N_k)_{k \in -\mathbb{N}}\} \quad (1.6)$$

for some T -valued random variable Ξ which is independent of $(N_k)_{k \in -\mathbb{N}}$.

Theorem 1.1 will be proved in Section 3.

Theorem 1.1 provides us with a general framework which unifies the following two earlier studies, which seem in completely different situations.

1°). Suppose that $S = \Sigma = G$ for a compact metric group G with a countable base. (We note that the Ellis theorem (Ellis (1957)) asserts that a topological semigroup which is algebraically a group is necessarily a topological group; in particular, the inversion operation is continuous as well.) We study equation (1.1) where $N_k X_{k-1}$ in the right hand side of (1.1) is considered to be the usual product in G . In Hirayama and Yano (2010), the authors utilized the results of Csiszár (1966) concerning convergence of infinite product of G -valued random variables, and obtained the following result:

Theorem 1.2 (Hirayama and Yano (2010)). *Suppose that there exists a solution (1.2) which is strong. Then there exists a sequence $(\alpha_k)_{k \in -\mathbb{N}}$ of deterministic elements of G such that the “centered processes” defined as*

$$N_k^{(\alpha)} := \alpha_{k+1}^{-1} N_k \alpha_k, \quad X_k^{(\alpha)} := \alpha_{k+1}^{-1} X_k \quad \text{for } k \in -\mathbb{N} \quad (1.7)$$

satisfy the following:

- (i) for each $k \in -\mathbb{N}$, the infinite product $N_k^{(\alpha)} N_{k-1}^{(\alpha)} \cdots N_{l+1}^{(\alpha)}$ converges almost surely as $l \rightarrow -\infty$ to the limit random variable $\Phi_k^{(\alpha)}$;
- (ii) any strong solution is of the form (1.5) for some $x \in G$, where $\Phi_k = \alpha_{k+1} \Phi_k^{(\alpha)}$.

We note that, under the assumptions of Theorem 1.2, Theorem 1.1 holds with $T = G$ and $\pi : G \rightarrow G$ being identity.

2°). Suppose that S has finitely many elements and Σ is the composition semigroup of all mappings from S to itself. We equip S and Σ with discrete topologies. We study equation (1.1) where $N_k X_{k-1}$ in the right hand side of (1.1) is considered to be $N_k(X_{k-1})$, the value of the mapping N_k evaluated at X_{k-1} . Yano (2010) obtained the following result.

Theorem 1.3 (Yano (2010)). *Let μ be a law on Σ . Suppose that the Markov chain whose transition probability is*

$$p(x, y) = \mu(\sigma : \sigma x = y) \quad (1.8)$$

is ergodic. Set $\mu_k = \mu$ for all $k \in -\mathbb{N}$. (In this case, there exists a unique solution.) Then the unique solution is strong if and only if there exists a finite sequence $\{\sigma_0, \sigma_1, \dots, \sigma_r\}$ of the support of μ such that the composition product $\sigma_r \sigma_{r-1} \cdots \sigma_0$ maps S into a singleton. In this case, for each $k \in -\mathbb{N}$, the infinite product (1.4) converges almost surely and is given as

$$\Phi_k = N_k N_{k-1} \cdots N_{k-T_k} \quad (1.9)$$

where

$$T_k = \inf \{n \geq r : N_{k-n+r} = \sigma_r, N_{k-n+r-1} = \sigma_{r-1}, \dots, N_{k-n+0} = \sigma_0\}. \quad (1.10)$$

Consequently, the unique strong solution is given as (1.5) for any choice of $x \in S$.

We note that, under the assumptions of Theorem 1.3, Theorem 1.1 holds with T being a singleton.

Note that Theorem 1.3 is related to the *coupling from the past*; see, e.g., Häggström (2002) and also Diaconis and Freedman (1999). For other related works, see Yano and Yasutomi (2011, 2012).

This paper is organized as follows. In Section 2, we show that equation (1.1) can be reduced to convolution equation. Section 3 is devoted to the proof of Theorem 1.1.

2. Convolution equation

For general theory of topological semigroups, see, e.g., Berglund and Hofmann (1967), Mukherjea and Tserpes (1976) and Högnäs and Mukherjea (1995).

Let $\mathcal{B}(S)$ denote the set of all Borel sets of S , and let $\mathcal{P}(S)$ denote the set of all probability laws on S . We introduce $\mathcal{B}(\Sigma)$ and $\mathcal{P}(\Sigma)$ similarly. We equip $\mathcal{P}(S)$ and $\mathcal{P}(\Sigma)$ with the topology of weak convergence. Since S and Σ are compact, they are compactly metrizable. For $\mu_1, \mu_2, \mu \in \mathcal{P}(\Sigma)$ and $\lambda \in \mathcal{P}(S)$, we define the *convolutions* $\mu_1 * \mu_2 \in \mathcal{P}(\Sigma)$ and $\mu * \lambda \in \mathcal{P}(S)$ by

$$(\mu_1 * \mu_2)(A) = \int_{\Sigma} \mu_1(d\sigma_1) \int_{\Sigma} \mu_2(d\sigma_2) 1_{\{\sigma_1 \sigma_2 \in A\}}, \quad A \in \mathcal{B}(\Sigma), \quad (2.1)$$

$$(\mu * \lambda)(A) = \int_{\Sigma} \mu(d\sigma) \int_S \lambda(dx) 1_{\{\sigma x \in A\}}, \quad A \in \mathcal{B}(S). \quad (2.2)$$

By the semigroup structure of Σ , we see that

$$\mu_1 * (\mu_2 * \mu_3) = (\mu_1 * \mu_2) * \mu_3, \quad \mu_1, \mu_2, \mu_3 \in \mathcal{P}(\Sigma). \quad (2.3)$$

By the associativity of the Σ -action on S , we see that

$$\mu_1 * (\mu_2 * \lambda) = (\mu_1 * \mu_2) * \lambda. \quad (2.4)$$

Since S and Σ are compact, the convolutions $(\mu_1, \mu_2) \mapsto \mu_1 * \mu_2$ and $(\mu, \lambda) \mapsto \mu * \lambda$ are jointly continuous.

Let $(\mu_k)_{k \in -\mathbb{N}}$ be a sequence in $\mathcal{P}(\Sigma)$. Let (1.2) be a solution of equation (1.1) and let $(\lambda_k)_{k \in -\mathbb{N}}$ denote its marginal law system, i.e., λ_k is the law of X_k for each $k \in -\mathbb{N}$. Then it follows by the definition of a solution that the convolution equation

$$\lambda_k = \mu_k * \lambda_{k-1} \quad \text{for } k \in -\mathbb{N} \quad (2.5)$$

holds. The following proposition, which generalizes Lemma 4.3 of Akahori et al. (2008), asserts that equation (1.1) can be reduced to the convolution equation (2.5).

Proposition 2.1. *The following statements hold:*

- (i) *Two solutions whose marginal law systems coincide are equal.*
- (ii) *Let $(\lambda_k)_{k \in -\mathbb{N}}$ be a sequence in $\mathcal{P}(S)$ such that (2.5) holds. Then there uniquely exists a solution (1.2) whose marginal law system is $(\lambda_k)_{k \in -\mathbb{N}}$.*

Proof. (i) Let two solutions be given such that their marginal law systems coincide. Then, by equation (2.5), their finite dimensional distributions coincide, which implies that they are equal.

(ii) Let $(\lambda_k)_{k \in -\mathbb{N}}$ be a sequence in $\mathcal{P}(S)$ such that (2.5) holds. For each $k \in -\mathbb{N}$, we define a law Λ_{k+1} on $(\Sigma \times S)^{-k}$ as follows. Let $N_0, N_{-1}, \dots, N_{k+1}, X_k$ be independent random variables such that N_j has law μ_j for $j = 0, -1, \dots, k+1$ and that X_k has law λ_k . For $j = 0, -1, \dots, k+1$, we set $X_j = N_{j,k} X_k$, where

$$N_{j,k} = N_j N_{j-1} \cdots N_{k+1} \quad \text{for } j > k. \quad (2.6)$$

Define Λ_{k+1} be the law of $(X_j, N_j : j = 0, -1, \dots, k+1)$ on $(\Sigma \times S)^{-k}$. Then, by (2.5), it is obvious that the family $\{\Lambda_{k+1} : k \in -\mathbb{N}\}$ is consistent. Thus, by Kolmogorov's extension theorem, we obtain existence of a solution. $\square$

3. Proof of main theorem

Since $\mathcal{P}(S)$ is a compact Hausdorff space with a countable base, so is the countable direct product space $\mathcal{P}(S)^{-\mathbb{N}}$. Since $\mathcal{P}(S)$ is equipped with a convex structure in the usual way, so is $\mathcal{P}(S)^{-\mathbb{N}}$. Let us write $\mathcal{P}^{\text{cvl}}$ for the set of all solutions $(\lambda_k)_{k \in -\mathbb{N}}$ of equation (2.5).

Lemma 3.1. *The set $\mathcal{P}^{\text{cvl}}$ is a compact convex subset of $\mathcal{P}(S)^{-\mathbb{N}}$.*

Proof. It is obvious by equation (2.5) that $\mathcal{P}^{\text{cvl}}$ is a convex subset of $\mathcal{P}(S)^{-\mathbb{N}}$. Let us prove that $\mathcal{P}^{\text{cvl}}$ is closed in $\mathcal{P}(S)^{-\mathbb{N}}$. Let $(\lambda_k^{(n)})_{k \in -\mathbb{N}} \subset \mathcal{P}^{\text{cvl}}$ be such that $(\lambda_k^{(n)})_{k \in -\mathbb{N}} \rightarrow (\lambda_k)_{k \in -\mathbb{N}}$ as $n \rightarrow \infty$ for some $(\lambda_k)_{k \in -\mathbb{N}} \in \mathcal{P}(S)^{-\mathbb{N}}$. Then we have

$$\lambda_k^{(n)} = \mu_k * \lambda_{k-1}^{(n)} \quad \text{for all } k \in -\mathbb{N} \text{ and } n \in \mathbb{N}. \quad (3.1)$$

Since $\lambda_k^{(n)} \rightarrow \lambda_k$ as $n \rightarrow \infty$ for all $k \in -\mathbb{N}$, we see, by the continuity of convolutions, that equation (2.5) holds, which implies that $(\lambda_k)_{k \in -\mathbb{N}} \in \mathcal{P}^{\text{cvl}}$. Since $\mathcal{P}(S)^{-\mathbb{N}}$ has a countable base, we conclude that $\mathcal{P}^{\text{cvl}}$ is closed. $\square$

Now we proceed to prove Theorem 1.1.

Proof of Theorem 1.1. Let us assume that the infinite product (1.4) converges almost surely for each $k \in -\mathbb{N}$. We then see that

$$\mu_{k,l} := \mu_k * \mu_{k-1} * \cdots * \mu_{l+1} \xrightarrow{l \rightarrow -\infty} \nu_k \quad \text{for all } k \in -\mathbb{N}, \quad (3.2)$$

where ν_k is the law of Φ_k for all $k \in -\mathbb{N}$.

Let $T = \mathcal{P}^{\text{cvl}}$. By Lemma 3.1, we see that T is a compact Hausdorff space with a countable base. For $x \in S$, we write $\pi(x) = (\nu_k * \delta_x)_{k \in -\mathbb{N}}$. It is now obvious that the mapping $\pi : S \rightarrow T$ is continuous. By Theorem 6.9.7 of Bogachev (2007), there exists a measurable section $\psi : T \rightarrow S$ of π . Claim (i) is obvious by Proposition 2.1.

Let $\{(X_k)_{k \in -\mathbb{N}}, (N_k)_{k \in -\mathbb{N}}\}$ be an arbitrary strong solution. Let λ_k denote the law of X_k for all $k \in -\mathbb{N}$. We then have $(\lambda_k)_{k \in -\mathbb{N}} \in \mathcal{P}^{\text{cvl}}$. Since we have

$$X_k = N_{k,l} X_l \quad (3.3)$$

by iterating equation (1.1), the conditional law of X_k given $\mathcal{F}_{k,l} := \sigma(N_k, N_{k-1}, \dots, N_{l+1})$ is written as

$$P(X_k \in \cdot | \mathcal{F}_{k,l}) = \delta_{N_{k,l}} * \lambda_l. \quad (3.4)$$

Letting $l \rightarrow -\infty$, we see that the left hand side converges to δ_{X_k} since X_k is measurable with respect to $\mathcal{F}_{k,-\infty} := \sigma(N_k, N_{k-1}, \dots)$ up to null sets. Taking

a subsequence if necessary, we may assume that $\lambda_l \rightarrow \lambda$ for some $\lambda \in \mathcal{P}(S)$, we obtain

$$\delta_{X_k} = \delta_{\Phi_k} * \lambda = \int_S \lambda(dx) (\delta_{\Phi_k} * \delta_x). \quad (3.5)$$

This shows that

$$(\lambda_k)_{k \in -\mathbb{N}} = \pi(x) \quad \text{for } \lambda\text{-almost every } x \in S, \quad (3.6)$$

which implies, in particular, that $(\lambda_k)_{k \in -\mathbb{N}} \in T$. This proves Claim (ii).

Let $\{(X_k)_{k \in -\mathbb{N}}, (N_k)_{k \in -\mathbb{N}}\}$ be an arbitrary solution. Let λ_k denote the law of X_k for all $k \in -\mathbb{N}$. We then have $(\lambda_k)_{k \in -\mathbb{N}} \in \mathcal{P}^{\text{cvl}}$. Since we have (3.3) by iterating equation (1.1), we have

$$\lambda_k = \mu_{k,l} * \lambda_l \quad \text{for } k > l. \quad (3.7)$$

Taking a subsequence if necessary, we may assume that $\lambda_l \rightarrow \lambda$ for some $\lambda \in \mathcal{P}(S)$, so that we obtain

$$\lambda_k = \nu_k * \lambda = \int_S \lambda(dx) (\nu_k * \delta_x) \quad \text{for } k \in -\mathbb{N}. \quad (3.8)$$

Now we obtain

$$(\lambda_k)_{k \in -\mathbb{N}} = \int_S \lambda(dx) \pi(x) = \int_T \tilde{\lambda}(dy) y, \quad (3.9)$$

where $\tilde{\lambda} = \lambda \circ \pi^{-1}$. Taking an independent random variable Ξ whose law is $\tilde{\lambda}$, we obtain Claim (iii).

The proof is now complete. $\square$

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