

SPHERICAL CONVEX BODY OF CONSTANT WIDTH AND ITS APPROXIMATION

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1. BASIC DEFINITIONS

Throughout this note, let $\mathbb{S}^n$ denote the unit sphere of the $(n + 1)$ -dimensional Euclidean space $\mathbb{R}^{n+1}$. For any given point P of $\mathbb{S}^n$, we denote by $H(P)$ the *hemisphere* whose center is P , namely,

$$H(P) = \{Q \in \mathbb{S}^n \mid P \cdot Q \geq 0\}.$$

Here the dot in the center stands for the scalar product of P, Q in $\mathbb{R}^{n+1}$. A non-empty subset W of $\mathbb{S}^n$ is *hemispherical* if there exists a point P of $\mathbb{S}^n$ such that the intersection set $W \cap H(P)$ is the empty set. A hemispherical set W of $\mathbb{S}^n$ is said to be *spherical convex* if the arc PQ between any two points $P, Q \in W$ lies in the W . Here

$$PQ = \left\{ \frac{tP + (1-t)Q}{\|tP + (1-t)Q\|} \in \mathbb{S}^n \mid 0 \leq t \leq 1 \right\}.$$

Denote the boundary of W is denoted by ∂W . A spherical convex set W of $\mathbb{S}^n$ is said to be *spherical convex body* if W has an interior point and closed. For any subset W of $\mathbb{S}^n$, the *spherical polar set* of W , denoted by W° , is the following set,

$$\bigcap_{P \in W} H(P).$$

For any non-empty closed hemispherical subset $W \subset \mathbb{S}^n$, the equality $\text{s-conv}(W) = (\text{s-conv}(W))^{\circ\circ}$ holds ([15]), where $\text{s-conv}(W)$ is the *spherical convex hull* of W , that is, the smallest spherical convex body contains W . The *diameter* of a spherical convex body W is defined by

$$\max\{|PQ| \mid P, Q \in W\}.$$

A spherical convex body W is said to be *constant diameter* τ , if the diameter of W is τ , and for every point $P \in \partial W$ there exists a point Q of ∂W such that $|PQ| = \tau$ ([12]). We say a hemisphere $H(Q)$ *supports* W at P if W is a subset of $H(Q)$ and P is a point of $\partial W \cap \partial H(Q)$. The hemisphere $H(Q)$ as defined above is called a *supporting hemisphere* of W at P . For any two points P, Q of $\mathbb{S}^n$, the intersection

$$H(P) \cap H(Q)$$

is called a *lune*, where $P \neq -Q$. The *thickness* of lune $H(P) \cap H(Q)$ is the real number $\pi - |PQ|$, denoted by $\Delta(H(P) \cap H(Q))$. Namely,

$$\Delta(H(P) \cap H(Q)) = \pi - |PQ|.$$

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It is clear that thickness of any lune is greater than 0 and less than π . Let $H(P)$ be a supporting hemisphere of a spherical convex body W . The *width* of W with respect to $H(P)$ is defined by ([9])

$$\text{width}_{H(P)}(K) = \min\{\Delta(H(P) \cap H(Q)) \mid W \subset H(Q)\}.$$

The minimum width of W is called *thickness of W* , denoted by ΔW . Following [9], a spherical convex body W is said to be *of constant width*, if all widths of W with respect to any supporting hemispheres $H(P)$ are equal; and a convex body W of $\mathbb{S}^n$ is said to be *reduced* if $\Delta(X) < \Delta(W)$ for every convex body X properly contained in W .

2. SOME RESULTS

2.1. Width of spherical convex bodies.

Lemma 2.1 ([15]). *The subset $\mathcal{P}$ is a spherical polytope if and only if $\mathcal{P}^\circ$ is a spherical polytope.*

Lemma 2.2 ([12]). *Every spherical convex body of constant width smaller than $\pi/2$ on S^n is strictly convex.*

Lemma 2.3 ([8]). *Let C be a spherical convex body in S^n , and $0 < \tau < \pi$. The following two statements are equivalent:*

- (1) *C is of constant width τ ;*
- (2) *C° is of constant width $\pi - \tau$.*

An alternative proof of Lemma 2.3 in the case of $\mathbb{S}^2$ is given in [14].

Theorem 1 ([8, 13]). *Let C be a spherical convex body in S^n , and $0 < \tau < \pi$. The following two statements are equivalent:*

- (1) *C is of constant diameter τ ;*
- (2) *C is of constant width τ .*

Then by Theorem 1 and Lemma 2.3 we have the following:

Corollary 2.1 ([8]). *Let C be a spherical convex body in $\mathbb{S}^n$, and $0 < \tau < \pi$. The following two propositions are equivalent:*

- (1) *C is of constant diameter τ ;*
- (2) *C° is of constant diameter $\pi - \tau$.*

We say a spherical convex body C is *self-dual* if $C = C^\circ$. A characterization of self-dual spherical convex body is given as follows:

Theorem 2 ([6]). *Let C be a spherical convex body. Then the following two statements are equivalent:*

- (1) *C is self dual;*
- (2) *C is of constant width $\pi/2$.*

Let $\mathcal{P}$ be a spherical polytope. By Lemma 2.4, its dual $\mathcal{P}^\circ$ is a spherical polytope. By Lemma 2.3, if $\mathcal{P}$ is of constant width τ if and only if $\mathcal{P}^\circ$ is of constant width $\pi - \tau$ for any $0 < \tau < \pi$. Then by Lemma 2.2, we have the following fact:

Lemma 2.4 ([2]). *Let $\mathcal{P}$ be a hemispherical convex polytope of constant width τ . Then $\tau = \pi/2$.*

Then by Lemma 2.4, Theorem 2 and Theorem 1, we know that the condition $\pi/2$ of Theorem 2 is unnecessary if $\mathcal{P}$ is a convex polytope:

Theorem 3 ([2]). *Let Wulff shape $\mathcal{P}$ be a spherical convex polytope. Then the following three statements are equivalent:*

- (1) $\mathcal{P}$ is self-dual;
- (2) $\mathcal{P}$ is of constant width spherical convex body;
- (3) $\mathcal{P}$ is of constant width spherical convex body.

A Problem 2.7 from [11]: “Do there exist reduced spherical n -dimensional polytopes (possibly some simplices?) on $\mathbb{S}^n$, where $n \geq 3$, different from the $1/2^n$ part of $\mathbb{S}^n$?”.

An affirmative answer (in the case $n = 3$): Let Q be an interior point of the spherical convex hull of $\{P_1, P_2, P_3, -N\}$ of $\mathbb{S}^3$. Set

$$Q_i = \partial H(Q) \cap NP_i, \quad i = 1, 2, 3.$$

Then the spherical polytope

$$\mathcal{P} = \text{s-conv}\{Q_1, Q_2, Q_3, P_1, P_2, P_3, Q\}$$

is of constant width $\pi/2$ (see the proof in general case in [2]).

2.2. Approximation of spherical convex bodies of constant width.

Theorem 4 ([10]). *For any spherical convex body $C \subset S^2$ of constant width $\tau < \pi/2$, and for any $\varepsilon > 0$ there exists a body C_ε of constant width $\Delta(C) = \Delta(C_\varepsilon)$ whose boundary consists only of arcs of circles of radius $\Delta(C)$ such that*

$$h(C, C_\varepsilon) \leq \varepsilon,$$

where $h(C_1, C_2)$ means the Hausdorff distance between C_1 and C_2 .

In [4], applying the fact that spherical dual transform is an isometry ([7]) and the relationships between boundary of C and its dual obtains a counterpart result of Theorem 4 as follows:

Theorem 5 ([4]). *For any spherical convex body $\tilde{C} \subset S^2$ of constant width $\tau > \pi/2$, and for any $\varepsilon > 0$ there exists a body $\tilde{C}_\varepsilon$ of constant width τ whose boundary consists only of arcs of circles of radius $\tau - \frac{\pi}{2}$ and great circle arcs, such that*

$$h(\tilde{C}, \tilde{C}_\varepsilon) \leq \varepsilon,$$

where h is the Hausdorff distance.

Notice that the spherical convex body of C in Theorem 4 is of constant width $\tau < \pi/2$ and the spherical convex body of C in Theorem 5 is of constant width $\tau > \pi/2$. For the remaining case (the spherical convex body C of $\pi/2$), we have a conjecture as follows.

Conjecture: Any spherical convex body of constant width $\pi/2$ can be approximated by a sequence of spherical convex polytopes of constant width $\pi/2$ ([4]).

Since a polytope $\mathcal{P}$ is of constant width τ , then $\tau = \pi/2$ (Lemma 2.4). This means any spherical convex body of constant width $\tau \neq \pi/2$ can not be approximated by a sequence of spherical convex polytopes of constant width τ .

3. APPLICATIONS TO WULFF SHAPES

Let $\gamma : \mathbb{S}^n \rightarrow \mathbb{R}_+$ be a continuous function, where $\mathbb{R}_+$ is the set consisting of positive real numbers. Then the *Wulff shape* associated with the function γ , denoted by $\mathcal{W}_\gamma$, is defined by

$$\bigcap_{\theta \in \mathbb{S}^n} \Gamma_{\gamma, \theta}.$$

Here $\Gamma_{\gamma, \theta}$ is the half space determined by the given continuous function γ and $\theta \in \mathbb{S}^n$,

$$\Gamma_{\gamma, \theta} = \{x \in \mathbb{R}^{n+1} \mid x \cdot \theta \leq \gamma(\theta)\}.$$

By definition, Wulff shape is a convex body, namely, convex, compact and contains the origin of $\mathbb{R}^{n+1}$ as an interior point. Conversely, for any convex body W contains the origin of $\mathbb{R}^{n+1}$ as an interior point, there exists a continuous function $\gamma : \mathbb{S}^n \rightarrow \mathbb{R}_+$ such that $\mathcal{W}_\gamma = W$. For more details in Wulff shapes, see for instance [1, 3, 5]. Let $Id : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1} \times \{1\} \subset \mathbb{R}^{n+2}$ be the mapping defined by

$$Id(x) = (x, 1).$$

Let $N = (0, \dots, 0, 1) \in \mathbb{R}^{n+2}$ be the north pole of $\mathbb{S}^{n+1}$, and let $\mathbb{S}_{N,+}^{n+1}$ denote the north open hemisphere of $\mathbb{S}^{n+1}$,

$$\mathbb{S}_{N,+}^{n+1} = \mathbb{S}^{n+1} \setminus H(-N) = \{Q \in \mathbb{S}^{n+1} \mid N \cdot Q > 0\}.$$

Let $\alpha_N : \mathbb{S}_{N,+}^{n+1} \rightarrow \mathbb{R}^{n+1} \times \{1\}$ be the central projection relative to N , defined by

$$\alpha_N(P_1, \dots, P_{n+1}, P_{n+2}) = \left(\frac{P_1}{P_{n+2}}, \dots, \frac{P_{n+1}}{P_{n+2}}, 1 \right).$$

We call the spherical convex body $\widetilde{\mathcal{W}}_\gamma = \alpha_N^{-1}(Id(\mathcal{W}_\gamma))$ is the *spherical Wulff shape* of $\mathcal{W}_\gamma$. The Wulff shape

$$Id^{-1} \circ \alpha_N((\alpha_N^{-1} \circ Id(\mathcal{W}_\gamma))^{\circ}).$$

is called *dual Wulff shape* of $\mathcal{W}_\gamma$, denoted by $\mathcal{DW}_\gamma$. We call a Wulff shape $\mathcal{W}$ is a *self-dual* if $\mathcal{W} = \mathcal{DW}$, namely, $\mathcal{W}$ and its dual Wulff shape $\mathcal{DW}$ are exactly the same convex body. By Theorem 1, Lemma 2.3 and Corollary 2.1, we have the following.

Corollary 3.1 ([8]). *Let $\gamma : \mathbb{S}^n \rightarrow \mathbb{R}_+$ be a continuous function. Suppose that the spherical Wulff shape $\widetilde{\mathcal{W}}_\gamma = \alpha_N^{-1} \circ Id(\mathcal{W}_\gamma)$ of $\mathcal{W}_\gamma$ is of constant width. Then*

- (1) $\Delta(\widetilde{\mathcal{W}}_\gamma) + \text{diam}(\widetilde{\mathcal{W}}_\gamma^{\circ}) = \pi$,
- (2) $\Delta(\widetilde{\mathcal{W}}_\gamma) + \Delta(\widetilde{\mathcal{W}}_\gamma^{\circ}) = \pi$,
- (3) $\text{diam}(\widetilde{\mathcal{W}}_\gamma) + \Delta(\widetilde{\mathcal{W}}_\gamma^{\circ}) = \pi$,
- (4) $\text{diam}(\widetilde{\mathcal{W}}_\gamma) + \text{diam}(\widetilde{\mathcal{W}}_\gamma^{\circ}) = \pi$,

where $\Delta(C)$ and $\text{diam}(C)$ are the width and the diameter of spherical convex body C in $\mathbb{S}^n$, respectively.

A characterization of self-dual Wulff shape is given as follows.

Proposition 3.1 ([6]). *Let $\gamma : \mathbb{S}^n \rightarrow \mathbb{R}_+$ be a continuous function. Then $\mathcal{W}_\gamma$ is a self-dual Wulff shape if and only if its spherical Wulff shape is of constant width $\pi/2$, namely, the spherical convex body $\alpha_N^{-1} \circ Id(\mathcal{W}_\gamma)$ is of constant width $\pi/2$.*

By Theorem 1, we have the following:

Corollary 3.2 ([8]). *Let $\gamma : \mathbb{S}^n \rightarrow \mathbb{R}_+$ be a continuous function. Then $\mathcal{W}_\gamma$ is a self-dual Wulff shape if and only if its spherical Wulff shape is of constant diameter $\pi/2$, namely, the spherical convex body $\alpha_N^{-1} \circ \text{Id}(\mathcal{W}_\gamma)$ is of constant diameter $\pi/2$.*

By Theorem 3, we have the following:

Theorem 6 ([2]). *Let Wulff shape W_γ be a convex polytope. Then the following statements are equivalent:*

- (1) *the Wulff shape W_γ is self-dual;*
- (2) *the spherical Wulff shape of W_γ is constant width;*
- (3) *the spherical Wulff shape of W_γ is constant diameter.*

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