

ASYMPTOTIC SERIES ASSOCIATED WITH EPSTEIN ZETA-FUNCTIONS AND THEIR INTEGRAL TRANSFORMS

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1. INTRODUCTION

Throughout the following, $s = \sigma + it$ denotes the complex variable, and $z = x + iy$ the complex parameter in the upper-half plane. The main object of this article is the Epstein zeta-function (attached to the positive-definite quadratic form $|u + vz|^2$) defined by

$$(1.1) \quad \zeta_{Z^2}(s; z) = \sum_{(m,n) \in Z^2 \setminus \{(0,0)\}} |m + nz|^{-2s} \quad (\operatorname{Re} s > 1),$$

and its meromorphic continuation over the whole s -plane (cf. [Si Chap. I]).

Let α, β be complex numbers which will be fixed later, and let $\Gamma(s)$ denote the gamma function. We introduce the Laplace-Mellin and the Riemann-Liouville (or the Erdélyi-Kober) transforms of $\zeta_{Z^2}(s; x + iy)$ (with the normalization multiples) as

$$(1.2) \quad \mathcal{LM}_{y;Y}^\alpha \zeta_{Z^2}(s; x + iy) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \zeta_{Z^2}(s; x + iyY) y^{\alpha-1} e^{-y} dy,$$

$$(1.3) \quad \mathcal{RL}_{y;Y}^{\alpha,\beta} \zeta_{Z^2}(s; x + iy) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \zeta_{Z^2}(s; x + iyY) y^{\alpha-1} (1-y)^{\beta-1} dy$$

for $Y > 0$. These can be regarded as weighted mean values of $\zeta_{Z^2}(s; x + iy)$; the factor $y^{\alpha-1}$ is inserted to secure the convergence of the integrals as $y \rightarrow +0$, while the functions e^{-y} and $(1-y)^{\beta-1}$ have effects to extract the parts corresponding to $y = O(Y)$ from $\zeta_{Z^2}(s; z)$ with their respective weights. Note that the *confluence* operation

$$(1.4) \quad \mathcal{RL}_{y;\beta Y}^{\alpha,\beta} \zeta_{Z^2}(s; x + iy) \xrightarrow{(\beta \rightarrow +\infty)} \mathcal{LM}_{y;Y}^\alpha \zeta_{Z^2}(s; x + iy)$$

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is valid by the definitions (1.2) and (1.3), since $\zeta_{Z^2}(s; x + iy) = O(y^{\max(0, 1-2\sigma)})$ as $y \rightarrow +\infty$ (see Theorem 1 below).

It is of importance from both theoretical and applicational point of view to study asymptotic aspects of $\zeta_{Z^2}(s; z)$ when $y = \text{Im } z$ is large (cf. [CS1–CS2]). We have established in [Ka10] a complete asymptotic expansion of $\zeta_{Z^2}(s; z)$ when $\text{Im } z \rightarrow +\infty$, and that of the Laplace-Mellin transform (1.2) when $Y \rightarrow +\infty$. The subsequent paper [Ka11] proceeds to this direction by showing that a similar asymptotic series still exists for the Riemann-Liouville transform (1.3) when $Y \rightarrow +\infty$. It is the aim of this article to present these asymptotic expansions, together with their several consequences.

We first present a complete asymptotic expansion of $\zeta_{Z^2}(s; z)$ when $\text{Im } z \rightarrow +\infty$ (Theorem 1 below) upon giving an explicit (vertical) t -estimate for the remainder term. This theorem in particular clarifies the key ingredients by which the functional equation of $\zeta_{Z^2}(s; z)$ is to be valid (Corollary 1.1). Moreover, several specific cases of Theorem 1 naturally reduce to the Kronecker limit formula for $\zeta_{Z^2}(s; z)$ when $s \rightarrow 1$, and to its variants for $\zeta_{Z^2}(m; z)$ ($m = 2, 3, \dots$) and $\zeta'_{Z^2}(-n; z)$ ($n = 0, 1, \dots$), where $\zeta'_{Z^2}(s; z) = (\partial/\partial s)\zeta_{Z^2}(s; z)$ (Corollaries 1.2 and 1.3). In connection with Theorem 1, Matsumoto [Ma] obtained asymptotic expansions (with respect to z) of holomorphic Eisenstein series, while Noda [No] derived an asymptotic formula (as $t \rightarrow +\infty$) for the non-holomorphic Eisenstein series on the line $\sigma = 1/2$. We next present complete asymptotic expansions of the Laplace-Mellin transform (1.2) and of the Riemann-Liouville transform (1.3) both when $Y \rightarrow +\infty$ (Theorems 2 and 3 in Section 3). One can observe that the asymptotic expansion of (1.3) precisely reduces to that of (1.2) through the *confluence* operation (1.4). It should be noted that various hypergeometric functions appear and work in the proofs of these expansions; especially their summation and transformation properties play crucial rôles in the analysis of the remainder terms.

Prior to the proof of Theorem 1, we have prepared the analytic continuation of $\zeta_{Z^2}(s; z)$ by means of Mellin-Barnes integral transformations (cf. [Ka10, Propositions 1 and 2]). This procedure was recently developed, independently of each other, by Kanemitsu-Tanigawa-Yoshimoto [KTY] (in a more general setting), and by the author [Ka10] for $\zeta_{Z^2}(s; z)$; the procedure, differs slightly from other previously known method of the analytic continuation, gives a new alternative proof of the Fourier expansion of $\zeta_{Z^2}(s; z)$, due to Chowla-Selberg [CS1–CS2]. We remark that Mellin-Barnes transformation technique was extensively utilized by Motohashi to investigate higher power moments of zeta and allied functions (see for e.g., [Mo1–Mo3]). The technique was also applied by the author [Ka1–Ka9] to study certain asymptotic aspects and transformation properties of zeta and theta functions.

2. RESULTS ON $\zeta_{Z^2}(s; z)$

We write $\sigma_w(l) = \sum_{0 < h|l} h^w$, and use the notations $e(z) = e^{2\pi iz}$ and

$$e^*(z) = e(z) + \overline{e(z)} = e(z) + e(-\bar{z}),$$

where $\bar{w}$ denotes the complex conjugate of w . We further introduce the function

$$\Phi_{r,s}^*(e(z)) = \sum_{h,k=1}^{\infty} h^r k^s e^*(h k z) = \sum_{l=1}^{\infty} \sigma_{r-s}(l) l^s e^*(l z),$$

which converges absolutely for all complex r, s if $\text{Im } z > 0$, and for $\text{Re } r < -1, \text{Re } s < -1$ if $\text{Im } z = 0$; in each case it defines a holomorphic function of r and s in the region of absolute convergence.

Let $\zeta(s)$ be the Riemann zeta-function, and $(s)_n = \Gamma(s+n)/\Gamma(s)$ for any integer n Pochhammer's symbol. Further let $U(\lambda; \nu; Z)$ denote the confluent hypergeometric function defined by

$$U(\lambda; \nu; Z) = \frac{1}{\Gamma(\lambda)} \int_0^{\infty} e^{-Zw} w^{\lambda-1} (1+w)^{\nu-\lambda-1} dw$$

for $\text{Re } \lambda > 0$ and $|\arg Z| < \pi/2$ (cf. [Sl, p.5, 1.3]). Then our first main result asserts

Theorem 1. ([Ka10, Theorem 1]). *Let $\zeta_{Z^2}(s; z)$ be defined by (1.1). Then for any complex $z = x + iy$ with $y > 0$ and any integer $N \geq 0$ the formula*

$$\begin{aligned} \zeta_{Z^2}(s; z) &= 2\zeta(2s) + \frac{2\sqrt{\pi}\Gamma(s-1/2)}{\Gamma(s)} \zeta(2s-1) y^{1-2s} \\ &\quad + \frac{2(2\pi)^{2s}}{\Gamma(s)} \{S_N(s, x; y) + R_N(s, x; y)\} \end{aligned}$$

holds in the region $-N < \sigma < 1 + N$ except at $s = 1$. Here

$$S_N(s; z) = \sum_{n=0}^{N-1} \frac{(-1)^n (s)_n (1-s)_n}{n!} \Phi_{s-n-1, -s-n}^*(e(z)) (4\pi y)^{-s-n}$$

is the asymptotic series in the descending order of y , and R_N is the remainder term, which is expressed as

$$\begin{aligned} R_N(s; z) &= \frac{(-1)^N (s)_N (1-s)_N}{(N-1)!} \sum_{h,k=1}^{\infty} e^*(h k z) h^{2s-1} \\ &\quad \times \int_0^1 \xi^{-s-N} (1-\xi)^{N-1} U(s+N; 2s; 4\pi h k y / \xi) d\xi \end{aligned}$$

for $N \geq 0$ (the case $N = 0$ should read without the factor $(-1)!$ and the ξ -integration), satisfying the estimate

$$R_N(s; z) = O\{|t| + 1)^{2N} e^{-2\pi y} y^{-\sigma-N}\}$$

for any $y \geq y_0 > 0$, in the region $-N < \sigma < 1 + N$, where the O -constant depends on N , σ and y_0 .

Remark. We see that

$$\Phi_{r,s}^*(e(z)) = e^*(z) + O\left\{\sum_{l=2}^{\infty} l^{\max(\operatorname{Re} r, \operatorname{Re} s)+\varepsilon} |e^*(lz)|\right\} = e^*(z) + O(e^{-4\pi y})$$

as $y \rightarrow +\infty$, and hence

$$\Phi_{r,s}^*(e(z)) \ll e^{-2\pi y} \quad (y \geq y_0 > 0).$$

Therefore the term with the index n in $S_N(s; z)$ is estimated as $\ll (|t|+1)^{2n} e^{-2\pi y} y^{-\sigma-n}$; this shows that the presence of the bound above for $R_N(s; z)$ is reasonable.

Let $\zeta_{Z^2}^*(s; z)$ be defined by

$$\zeta_{Z^2}(s; z) = 2\zeta(2s) + \frac{2\sqrt{\pi}\Gamma(s-1/2)}{\Gamma(s)}\zeta(2s-1)y^{1-2s} + 2\zeta_{Z^2}^*(s; z).$$

Then the proof of Theorem 1 show that the following functional equation of $\zeta_{Z^2}^*(s; z)$ reduces eventually to the simple property

$$\Phi_{r,s}^*(e(z)) = \Phi_{s,r}^*(e(z)).$$

Corollary 1.1. ([Ka10, Corollary 1.1]). *For any real x, y with $y > 0$ the functional equation*

$$(y/\pi)^s \Gamma(s) \zeta_{Z^2}^*(s; z) = (y/\pi)^{1-s} \Gamma(1-s) \zeta_{Z^2}^*(1-s; z)$$

follows, and this with the functional equation of $\zeta(s)$ implies that

$$(y/\pi)^s \Gamma(s) \zeta_{Z^2}(s; z) = (y/\pi)^{1-s} \Gamma(1-s) \zeta_{Z^2}(1-s; z).$$

We next state the Kronecker limit formula for $\zeta_{Z^2}(s; z)$ and its variants. Let $\eta(z) = e(z/24) \prod_{n=1}^{\infty} (1 - e(nz))$ be the Dedekind eta function, $\gamma_0 = -\Gamma'(1)$ Euler's constant, and B_n the n -th Bernoulli number (cf. [Er, p.35, 1.13(1)]). Then

Corollary 1.2. ([Ka10, Corollary 1.2]). *For any complex $z = x + iy$ with $y > 0$ the following formulae hold:*

$$\begin{aligned} \lim_{s \rightarrow 1} \left\{ \zeta_{Z^2}(s; z) - \frac{\pi/y}{s-1} \right\} &= \frac{\pi^2}{3} + \frac{2\pi}{y} \{ \gamma_0 - \log(2y) + \Phi_{0,-1}^*(e(z)) \} \\ &= \frac{2\pi}{y} \{ \gamma_0 - \log(3y|\eta(z)|^2) \}, \end{aligned}$$

and for any integer $m \geq 2$,

$$\begin{aligned} \zeta_{Z^2}(m; z) &= \frac{(-1)^{m+1} (2\pi)^{2m} B_{2m}}{(2m)!} + \frac{2\pi(2m-1)!}{\{2^{m-1}(m-1)!\}^2} \zeta(2m-1) y^{1-2m} \\ &\quad + \frac{(2\pi)^{2m}}{\{(m-1)!\}^2} \sum_{n=0}^{m-1} \binom{m-1}{n} (m+n-1)! \\ &\quad \times \Phi_{m-n-1, -m-n}^*(e(z)) (4\pi y)^{-m-n}. \end{aligned}$$

Corollary 1.3. ([Ka10, Corollary 1.3]). Let $\zeta'_{Z^2}(s; z) = (\partial/\partial s)\zeta_{Z^2}(s; z)$. Then for any complex $z = x + iy$ with $y > 0$ the following formulae hold:

$$\zeta'_{Z^2}(0; z) = -2\log 2\pi + \frac{\pi y}{3} + 2\Phi_{-1,0}^*(e(z)) = -2\log(2\pi|\eta(z)|^2),$$

and for any integer $m \geq 1$,

$$\begin{aligned} \zeta'_{Z^2}(-m; z) &= \frac{2(-1)^m(2m)!}{(2\pi)^{2m}}\zeta(2m+1) + \frac{2\pi(2^m m!)^2 B_{2m+2}}{(2m+1)!(m+1)}y^{2m+1} \\ &+ \frac{2(-1)^m}{(2\pi)^{2m}} \sum_{n=0}^m \binom{m}{n} (m+n)! \Phi_{-m-n-1, m-n}^*(e(z)) (4\pi y)^{m-n}. \end{aligned}$$

3. RESULTS ON $\mathcal{LM}_{y,Y}^\alpha \zeta_{Z^2}(s; z)$ AND $\mathcal{RL}_{y,Y}^{\alpha,\beta} \zeta_{Z^2}(s; z)$

We write

$$\Gamma\left(\frac{\alpha_1, \dots, \alpha_m}{\beta_1, \dots, \beta_n}\right) = \frac{\prod_{h=1}^m \Gamma(\alpha_h)}{\prod_{k=1}^n \Gamma(\beta_k)}$$

for complex numbers α_h, β_k ($1 \leq h \leq m; 1 \leq k \leq n$), and denote the generalized hypergeometric function by ${}_mF_n\left(\frac{\alpha_1, \dots, \alpha_m}{\beta_1, \dots, \beta_n}; z\right)$ for $m \leq n+1$. Then our second main result can be stated as

Theorem 2. ([Ka10, Theorem 2]). Let α be fixed with $\operatorname{Re} \alpha > 1$. Then for any integer $N \geq 0$ and any real x, Y with $Y > 0$ the formula

$$\begin{aligned} \mathcal{LM}_{y,Y}^\alpha \zeta_{Z^2}(s; x + iy) &= 2\zeta(2s) + 2\sqrt{\pi}\Gamma\left(\frac{s-1/2, \alpha+1-2s}{s, \alpha}\right)\zeta(2s-1)Y^{1-2s} \\ &+ \frac{2\pi^{2s}}{\Gamma(s)}\{S_{\alpha,N}(s, x; Y) + R_{\alpha,N}(s, x; Y)\} \end{aligned}$$

holds in the region $\sigma < \operatorname{Re} \alpha/2$. Here

$$\begin{aligned} S_{\alpha,N}(s, x; Y) &= \sum_{n=0}^{N-1} \frac{(-1)^n(\alpha)_n}{n!} \Gamma\left(\frac{(\alpha+n+1)/2-s}{(\alpha+n+1)/2}\right) \\ &\times \Phi_{2s-1-\alpha-n, -\alpha-n}^*(e(x))(2\pi Y)^{-\alpha-n} \end{aligned}$$

is the asymptotic series in the descending order of Y , and $R_{\alpha,N}$ is the remainder term, which is expressed as

$$\begin{aligned} R_{\alpha,N}(s, x; Y) &= \frac{(-1)^N}{(N-1)!} \Gamma\left(\frac{\alpha+1-2s}{\alpha+1-s}\right) \sum_{h,k=1}^{\infty} e^*(h k x) h^{2s-1} \\ &\times \int_0^1 \xi^{-\alpha-N} (1-\xi)^{N-1} (1+2\pi h k Y/\xi)^{-\alpha-N} \\ &\times {}_2F_1\left(\frac{\alpha+N, s}{\alpha+N+1-s}; \frac{1-2\pi h k Y/\xi}{1+2\pi h k Y/\xi}\right) d\xi \end{aligned}$$

for $N \geq 0$ (the case $N = 0$ should read without the factor $(-1)!$ and the ξ -integration), satisfying the estimate

$$R_{\alpha,N}(s, x; Y) = O(Y^{-\operatorname{Re} \alpha - N})$$

for any $Y \geq Y_0 > 0$, in the region $\sigma < \operatorname{Re} \alpha/2$, where the O -constant depends at most on α, N, σ, t and Y_0 . In particular when $\alpha \in \mathbb{R}$, more explicitly

$$R_{\alpha,N}(s, x; Y) = O\{e^{-\pi|t|/2}(|t| + 1)^{(\alpha+N)/2-\sigma} Y^{-\alpha-N}\}$$

for any $Y \geq Y_0 > 0$ in the region $\sigma < \alpha/2$, where the O -constant depends on α, N and σ .

Remark 2.1. The condition $\operatorname{Re} \alpha > 1$ is crucial for the convergence of $\mathcal{LM}_{y;Y}^\alpha \zeta_{\mathbb{Z}^2}(s; x + iy)$, especially for that of $\mathcal{LM}_{y;Y}^\alpha \zeta_{\mathbb{Z}^2}^*(s; x + iy)$.

Remark 2.2. It is seen that

$$|\Phi_{r,s}^*(e(x))| \leq 2\zeta(-\operatorname{Re} r)\zeta(-\operatorname{Re} s) < +\infty$$

for $\operatorname{Re} r < -1$, $\operatorname{Re} s < -1$, and hence when $\alpha \in \mathbb{R}$ the term with the index n in $S_{\alpha,N}(s, x; Y)$ is estimated as $\ll e^{-\pi|t|/2}(|t| + 1)^{(\alpha+n)/2-\sigma} Y^{-\alpha-n}$; this shows that the presence of the bound for $R_{\alpha,N}(s, x; Y)$ above is reasonable.

It is in fact shown that $\lim_{N \rightarrow \infty} R_{\alpha,N}(s, x; Y) = 0$ for $\sigma < \operatorname{Re} \alpha/2$ and $Y > 1/2\pi$. The limiting case $N \rightarrow \infty$ of Theorem 2 therefore gives

Corollary 2.1. ([Ka10, Corollary 2.1]). *For any real x, Y with $Y > 1/2\pi$ the formula*

$$\begin{aligned} \mathcal{LM}_{y;Y}^\alpha \zeta_{\mathbb{Z}^2}(s; x + iy) &= 2\zeta(2s) + 2\sqrt{\pi} \Gamma\left(\begin{matrix} s - 1/2, \alpha + 1 - 2s \\ s, \alpha \end{matrix}\right) \zeta(2s - 1) Y^{1-2s} \\ &\quad + \frac{2\pi^{2s}}{\Gamma(s)} S_\alpha^*(s, x; Y) \end{aligned}$$

holds in the region $\sigma < \operatorname{Re} \alpha/2$, where

$$\begin{aligned} S_\alpha^*(s, x; Y) &= \sum_{n=0}^{\infty} \frac{(-1)^n (\alpha)_n}{n!} \Gamma\left(\begin{matrix} \alpha + n + 1/2 - s \\ (\alpha + n + 1)/2 \end{matrix}\right) \\ &\quad \times \Phi_{2s-1-\alpha-n, -\alpha-n}^*(e(x)) (2\pi Y)^{-\alpha-n}. \end{aligned}$$

We next proceed to state our third main result.

Theorem 3. ([Ka11, Theorem 2]). *Let α, β be fixed with $\operatorname{Re} \alpha > 1$, $\operatorname{Re} \beta > 1$. Then for any integer $N \geq 0$ and any real x, Y with $Y > 0$ the formula*

$$\begin{aligned} \mathcal{RL}_{y;Y}^{\alpha,\beta} \zeta_{\mathbb{Z}^2}(s; x + iy) &= 2\zeta(2s) + 2\sqrt{\pi} \Gamma\left(\begin{matrix} s - 1/2, \alpha + \beta, \alpha + 1 - 2s \\ s, \alpha, \alpha + \beta + 1 - 2s \end{matrix}\right) \zeta(2s - 1) Y^{1-2s} \\ &\quad + 2\pi^s \Gamma\left(\begin{matrix} \alpha + \beta \\ s \end{matrix}\right) \{S_{\alpha,\beta,N}(s, x; Y) + R_{\alpha,\beta,N}(s, x; Y)\} \end{aligned}$$

holds in the region $\sigma < \operatorname{Re} \alpha/2$. Here

$$S_{\alpha,\beta,N}(s,x;Y) = \sum_{n=0}^{N-1} \frac{(-1)^n (\alpha)_n}{n!} \Gamma\left(\frac{(\alpha+n+1)/2-s}{(\alpha+n+1)/2, \beta-n}\right) \\ \times \Phi_{2s-1-\alpha-n, -\alpha-n}^*(e(x))(2\pi Y)^{-\alpha-n}$$

is the asymptotic series in the descending order of Y , and $R_{\alpha,\beta,N}$ is the remainder term, which is expressed as

$$R_{\alpha,\beta,N}(s,x;Y) = \frac{2^{2s}(-1)^N(\alpha)_N}{(N-1)!} \sum_{h,k=1}^{\infty} e^*(h k x) h^{2s-1} \\ \times \int_0^1 \xi^{-\alpha-N} (1-\xi)^{N-1} F_{\alpha+N,\beta-N}(s; 2\pi h k Y/\xi) d\xi$$

for any $N \geq 0$ (the case $N = 0$ should read without the factor $(-1)!$ and the ξ -integration), where

$$F_{\alpha,\beta}(s;Z) = \Gamma\left(\frac{1-2s}{1-s, \alpha+\beta}\right) {}_2F_3\left(\frac{\alpha/2, (\alpha+1)/2}{(\alpha+\beta)/2, (\alpha+\beta+1)/2, s+1/2}; Z^2/4\right) \\ + \Gamma\left(\frac{2s-1, \alpha+1-2s}{s, \alpha, \alpha+\beta+1-2s}\right) (2Z)^{1-2s} \\ \times {}_2F_3\left(\frac{(\alpha+1)/2-s, \alpha/2+1-s}{(\alpha+\beta+1)/2-s, (\alpha+\beta)/2+1-s, 3/2-s}; Z^2/4\right)$$

with α, β replaced by $\alpha+n, \beta-N$, and it satisfies

$$R_{\alpha,\beta,N}(s,x;Y) = O(Y^{-\operatorname{Re} \alpha - N})$$

for any $Y \geq Y_0 > 0$ in the region $\sigma < \operatorname{Re} \alpha/2$, where the O -constant depends on $\alpha, \beta, N, \sigma, t$ and Y_0 . In particular when $\alpha, \beta \in \mathbb{R}$, more explicitly

$$R_{\alpha,\beta,N}(s,x;Y) = O\{e^{-\pi|t|/2}(|t|+1)^{(\alpha+N)/2-\sigma} Y^{-\alpha-N}\}$$

for any $Y \geq Y_0 > 0$ in the region $\sigma < \alpha/2$, where the O -constant depends on α, β, N, σ and Y_0 .

Corollary 3.1. ([Ka11, Corollary 2.1]). *The asymptotic expansion in Theorem 3 for $\mathcal{RL}_{y;Y}^{\alpha,\beta}\zeta_{Z^2}(s;x+iy)$ precisely reduces to that in Theorem 2 for $\mathcal{LM}_{y;Y}^{\alpha}\zeta_{Z^2}(s;x+iy)$ through the confluence operation (1.4).*

Remark 3.1. The conditions $\operatorname{Re} \alpha > 1$ and $\operatorname{Re} \beta > 1$ are crucial for the convergence of $\mathcal{RL}_{y;Y}^{\alpha,\beta}\zeta_{Z^2}(s;x+iy)$, especially for that of $\mathcal{RL}_{y;Y}^{\alpha,\beta}\zeta_{Z^2}^*(s;x+iy)$.

Remark 3.2. Similarly to Remark 2.2, when $\alpha, \beta \in \mathbb{R}$ the term with the index n in $S_{\alpha,\beta,N}$ is estimated as $\ll e^{-\pi|t|/2}(|t|+1)^{(\alpha+n)/2-\sigma} Y^{-\alpha-n}$; this shows that the presence of the bound for $R_{\alpha,\beta,N}(s,x;Y)$ above is reasonable.

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