

On structure of CAT(0) groups

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We introduce some results on finitely generated groups of isometries of CAT(0) spaces and CAT(0) groups in [15].

Definition and detail of CAT(0) spaces are found in [3] and [9].

Let X be a metric space and let γ be an isometry of X . Then the *translation length* of γ is defined as $|\gamma| = \inf\{d(x, \gamma x) \mid x \in X\}$, and the *minimal set* of γ is defined as $\text{Min}(\gamma) = \{x \in X \mid d(x, \gamma x) = |\gamma|\}$. An isometry γ is said to be *semi-simple* if $\text{Min}(\gamma)$ is non-empty. Also an isometry γ is called

- (1) *elliptic* if γ has a fixed point,
- (2) *hyperbolic* if γ is semi-simple and not elliptic, and
- (3) *parabolic* if γ is not semi-simple.

(cf. [3, Chapter II.6]).

In [15], we obtained the following theorem by observing the proof of [3, Theorem II.6.12].

Theorem 1. *Let X be a CAT(0) space and let Γ be a finitely generated group acting by isometries on X . If the center of Γ contains a hyperbolic isometry γ_0 of X , then there exist a normal subgroup $\Gamma' \subset \Gamma$, an element $\delta_0 \in \Gamma$ and a number $k_0 \in \mathbb{N}$ such that*

- (i) $\Gamma = \Gamma' \rtimes \langle \delta_0 \rangle$,
- (ii) $\Gamma' \rtimes \langle \delta_0^{k_0} \rangle = \Gamma' \times \langle \gamma_0 \rangle$ is a finite-index subgroup of Γ and
- (iii) Γ/Γ' is isomorphic to $\mathbb{Z}$.

A *geometric action* on a CAT(0) space is an action by isometries which is proper ([3, p.131]) and cocompact. A group Γ is called a *CAT(0) group*, if Γ acts geometrically on some CAT(0) space. We note that every CAT(0) space on

which some group acts geometrically is a proper space ([3, p.132]). Also we note that CAT(0) groups are finitely presented (cf. [3, Corollary I.8.11]).

For example, Bieberbach groups ([3, p.246], [4]), crystallographic groups ([4]), Coxeter groups and their torsion-free subgroups of finite-index ([6], [7], [19]) and fundamental groups of compact geodesic spaces of non-positive curvature ([3, p.159, p.237]) are CAT(0) groups. In particular, fundamental groups of Riemannian manifolds of non-positive sectional curvature are CAT(0) groups. Moreover, M. W. Davis [6] has constructed a closed aspherical manifold of dimension $n \geq 5$ whose universal covering is not homeomorphic to $\mathbb{R}^n$ ([6], [8]). The fundamental groups of these exotic manifolds are also CAT(0) groups.

On structure of CAT(0) groups, we obtained the following theorem from Theorem 1 in [15].

Theorem 2. *Let Γ be a CAT(0) group. Then there exist subgroups $\Gamma = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_n$, elements $\delta_{i+1}, \gamma_{i+1} \in \Gamma_i$ and $k_{i+1} \in \mathbb{N}$ for $i = 0, \dots, n-1$ such that*

- (1) γ_{i+1} is an element of the center of Γ_i with the order $o(\gamma_{i+1}) = \infty$ for $i = 0, \dots, n-1$,
- (2) $\Gamma_i = \Gamma_{i+1} \rtimes \langle \delta_{i+1} \rangle$ for $i = 0, \dots, n-1$,
- (3) $\Gamma_{i+1} \rtimes \langle \delta_{i+1}^{k_{i+1}} \rangle = \Gamma_{i+1} \times \langle \gamma_{i+1} \rangle$ is a finite-index subgroup of Γ_i ,
- (4) Γ_i / Γ_{i+1} is isomorphic to $\mathbb{Z}$ for $i = 0, \dots, n-1$,
- (5) $\Gamma = (\cdots ((\Gamma_n \rtimes \langle \delta_n \rangle) \rtimes \langle \delta_{n-1} \rangle) \rtimes \langle \delta_{n-2} \rangle) \cdots) \rtimes \langle \delta_1 \rangle$,
- (6) Γ_n has finite center, and
- (7) $\Gamma_n \times A$ is a finite-index subgroup of Γ where $A = \langle \gamma_1 \rangle \times \cdots \times \langle \gamma_n \rangle$ which is isomorphic to $\mathbb{Z}^n$.

Here, we introduce an easy example of a CAT(0) group.

Example. Let $\Gamma = \langle a, b \mid ab^2 = b^2a \rangle$ and let $X = \mathbb{R}^2$ the euclidean plane. We consider the action of the group Γ on X defined by

$$a \cdot (x, y) = (x, y + 1)$$

$$b \cdot (x, y) = (x + 1, -y)$$

for any $(x, y) \in \mathbb{R}^2 = X$. Then $D = [0, 1] \times [-\frac{1}{2}, \frac{1}{2}] \subset \mathbb{R}^2$ is a fundamental domain, $\Gamma D = X$ and Γ acts geometrically on X . Here we note that X/Γ is a

Klein bottle and the group Γ is a $\text{CAT}(0)$ group which is the fundamental group of the Klein bottle.

Then $\gamma_0 := b^2$ is a center of the $\text{CAT}(0)$ group Γ and a hyperbolic isometry of X . Here we obtain that

- (i) $\Gamma = \langle a \rangle \rtimes \langle b \rangle$,
- (ii) $\langle a \rangle \rtimes \langle b^2 \rangle = \langle a \rangle \times \langle b^2 \rangle$ is a finite-index subgroup of Γ which is isomorphic to $\mathbb{Z}^2$ and
- (iii) $\Gamma / \langle a \rangle$ is isomorphic to $\mathbb{Z}$.

REFERENCES

- [1] N. Bourbaki, *Groupes et Algèbres de Lie*, Chapters IV-VI, Masson, Paris, 1981.
- [2] P. Bowers and K. Ruane, *Boundaries of nonpositively curved groups of the form $G \times \mathbb{Z}^n$* , Glasgow Math. J. 38 (1996), 177–189.
- [3] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Springer-Verlag, Berlin, 1999.
- [4] L. S. Charlap, *Bieberbach groups and flat manifolds*, Universitext. Springer-Verlag, New York, 1986.
- [5] C. B. Croke and B. Kleiner, *Spaces with nonpositive curvature and their ideal boundaries*, Topology 39 (2000), 549–556.
- [6] M. W. Davis, *Groups generated by reflections and aspherical manifolds not covered by Euclidean space*, Ann. of Math. 117 (1983), 293–324.
- [7] M. W. Davis, *Nonpositive curvature and reflection groups*, in Handbook of geometric topology (Edited by R. J. Daverman and R. B. Sher), pp. 373–422, North-Holland, Amsterdam, 2002.
- [8] M. W. Davis and T. Januszkiewicz, *Hyperbolization of polyhedra*, J. Diff. Geom. 34 (1991), 347–388.
- [9] E. Ghys and P. de la Harpe (ed), *Sur les groupes hyperboliques d'après Mikhael Gromov*, Progr. Math. vol. 83, Birkhäuser, Boston MA, 1990.
- [10] M. Gromov, *Hyperbolic groups*, Essays in group theory (S. M. Gersten, ed.), M.S.R.I. Publ. 8, 1987, pp. 75–264.
- [11] M. Gromov, *Asymptotic invariants for infinite groups*, Geometric Group Theory (G.A. Niblo and M.A. Roller, eds.), LMS Lecture Notes, vol. 182, Cambridge University Press, Cambridge, 1993, pp. 1–295.
- [12] P. de la Harpe, *Topics in geometric group theory*, Chicago Lectures in Math. Ser., The University of Chicago Press.
- [13] T. Hosaka, *Minimality of the boundary of a right-angled Coxeter system*, Proc. Amer. Math. Soc., 137 (2009), 899–910.
- [14] T. Hosaka, *On splitting theorems for $\text{CAT}(0)$ spaces and compact geodesic spaces of non-positive curvature*, Math. Z., 272 (2012), 1037–1050.
- [15] T. Hosaka, *Remarks on structure of $\text{CAT}(0)$ groups*, preprint.
- [16] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge University Press, 1990.
- [17] N. Monod, *Superrigidity for irreducible lattices and geometric splitting*, J. Amer. Math. Soc. 19 (2006), 781–814.
- [18] C. Mooney, *Examples of non-rigid $\text{CAT}(0)$ groups from the category of knot groups*, Algebr. Geom. Topology 8 (2008), 1667–1690.

- [19] G. Moussong, *Hyperbolic Coxeter groups*, Ph.D. thesis, The Ohio State University, 1988.
- [20] J. M. Wilson, *A $CAT(0)$ group with uncountably many distinct boundaries*, J. Group Theory 8 (2005), 229–238.