

グラフの星部分グラフ分解

Star decomposition indexes of graphs

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We deal with finite simple graphs, which have neither multiple edges nor loops. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The arboricity $a(G)$ of G is the minimum integer n for which $E(G)$ can be decomposed into n forests. A formula for the arboricity of a graph was obtained by Nash-William [5],[6]. The formula is the following:

$$a(G) = \max \left\lceil \frac{|E(H)|}{|V(H)| - 1} \right\rceil$$

where the maximum is taken over all subgraphs H of G , and $[x]$ denotes the least integer not less than x . If we impose some conditions to forests, then we obtain new invariants. A graph is called a linear forest if each component of it is a path, and linear arboricity $\ell(G)$ of G is defined to be the minimum n for which $E(G)$ can be decomposed into n linear forests. Some results on linear arboricity can be found in [1],[4]. We call a graph H a star if H is isomorphic to the complete bipartite graph $K_{1,n}$ for some n (Fig. 1). We call a graph G

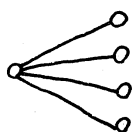


Figure 1. $K_{1,4}$.

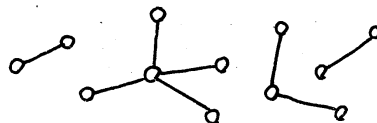


Figure 2. A star-forest.

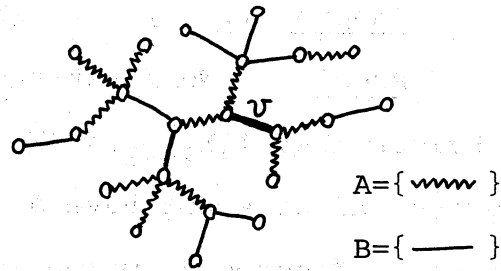
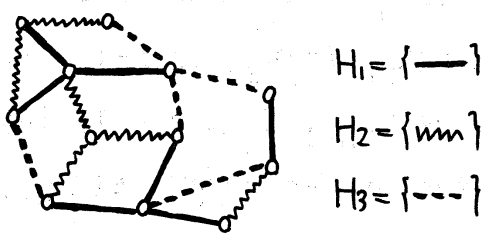


Figure 3. A graph G with $*(G)=3$. Figure 4. A and B .

a star-forest if each component of G is a star (Fig. 2). We define the star decomposition index $*(G)$ of G by the minimum n for which $E(G)$ can be decomposed into n star-forests (Fig. 3). In this paper we shall investigate star decomposition indexes.

We begin with the following easy result.

Proposition 1. Let T be a tree. If T is not a star, then $*(T)=2$.

Proof Let T be a tree that is not a star. Then it is obvious that $*(T) \geq 2$. Let $L(T)$ be the line graph of T (i.e. $V(L(T))=E(T)$ and two vertices of $L(T)$ are adjacent if and only if corresponding edges of T are adjacent.). For two vertices x and y of $L(T)$, we denote by $d(x,y)$ the distance between x and y in $L(T)$. Choose any vertex v of $L(T)$, and set

$$A = \{x \in V(L(T)) \mid d(v,x) \text{ is odd}\} \text{ and}$$

$$B = \{x \in V(L(T)) \mid d(v,x) \text{ is even}\} \ni v \text{ (Fig. 4).}$$

Then A and B are star-forests of T , and thus $*(T) \leq 2$. Therefore $*(T)=2$. $\square$

By K_n and $K_{n,m}$, we denote the complete graph of order n and the complete graph of order $n+m$, respectively. Let A be a graph. Then an A -factor of a graph is its spanning subgraph each component of which is isomorphic to A .

Theorem 1. [2] $*(K_{2n-1}) = *(K_{2n}) = n+1$, where $n \geq 3$.

Proof We first show that $*(K_{2n}) \geq *(K_{2n-1}) \geq n+1$. It is obvious that $*(K_{2n-1}) \leq *(K_{2n})$. Since K_{2n-1} is a $2(n-1)$ -regular graph and does not have a $K_{1,n-1}$ -factor, we obtain $*(K_{2n-1}) \geq n+1$ by Theorem 3, which will be given later.

We next show that $*(K_{2n}) \leq n+1$. Let $V(K_{2n}) = \{v_1, v_2, \dots, v_{2n}\}$ and put

$$F_t = \{v_t v_i \mid t < i' < t+n, i \equiv i' \pmod{2n}\}$$

$$\cup \{v_{n+t} v_j \mid n+t < j' < t+2n, j \equiv j' \pmod{2n}\} \subset E(K_{2n})$$

for $t=1, \dots, n$, and define

$$F_{n+1} = \{v_1 v_{n+1}, v_2 v_{n+2}, \dots, v_n v_{2n}\} \quad (\text{Fig. 5}).$$

Then $K_{2n} = F_1 \cup F_2 \cup \dots \cup F_{n+1}$, and we conclude that $*(K_{2n}) \leq n+1$.

Consequently, $*(K_{2n-1}) = *(K_{2n}) = n+1$. $\square$

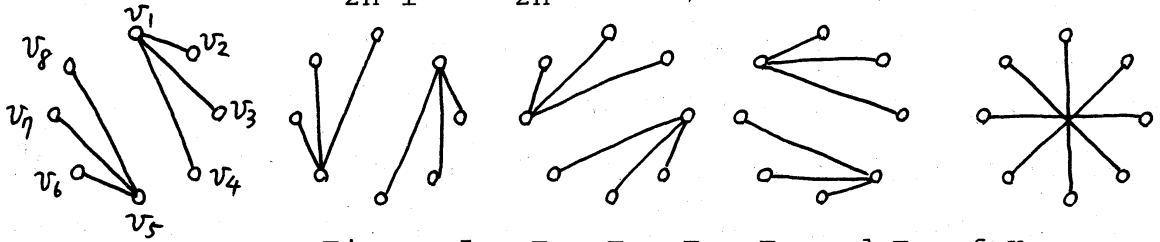


Figure 5. F_1, F_2, F_3, F_4 and F_5 of K_8 .

The star decomposition index of the complete bipartite graph $K_{n,n}$ was determined by Egawa, Fukuda, Nagoya and Urabe [3], and $*(K_{n,m})$ for some classes of n, m are obtained by Enomoto and etc.

Theorem 2. [3] $*(K_{2n,2n}) = *(K_{2n-1,2n-1}) = n+2$, where $n \geq 4$.

Proof We prove only that $*(K_{2n-1,2n-1}) \leq *(K_{2n,2n}) \leq n+2$. For the proof of $*(K_{2n-1,2n-1}) \geq n+2$, the reader should refer to [3]. It is trivial that $*(K_{2n-1,2n-1}) \leq *(K_{2n,2n})$. Let $V(K_{2n,2n}) = \{a_1, \dots, a_n, b_1, \dots, b_n\} \cup \{c_1, \dots, c_n, d_1, \dots, d_n\}$. For every k , $1 \leq k \leq n$, we define

$$F_k = \{a_k c_i, b_k d_i, a_i d_k, b_i c_k \mid 1 \leq i \leq n, i \neq k\},$$

and put

$$F_{n+1} = \{a_i c_i, a_i d_i \mid 1 \leq i \leq n\}, \text{ and}$$

$$F_{n+2} = \{b_i c_i, b_i d_i \mid 1 \leq i \leq n\} \quad (\text{Fig. 6}).$$

Then $K_{2n, 2n} = F_1 \cup F_2 \cup \dots \cup F_{n+2}$. Consequently, $*(K_{2n, 2n}) \leq n+2$.

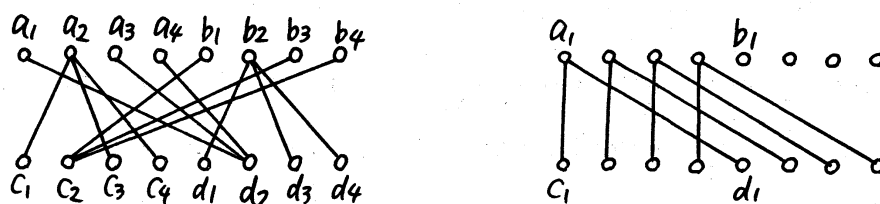


Figure 6. F_2 and F_5 of $K_{8,8}$.

We write $d_G(v)$ for the degree of vertex v in G . A graph G is called an r -regular graph if $d_G(x) = r$ for all vertices x .

Theorem 3. Let G be a $2r$ -regular graph. Then

$$*(G) \geq r+1$$

with the equality if and only if G can be decomposed into $r+1$ edge-disjoint $K_{1,r}$ -factors.

Proof. Since $a(G) \geq |E(G)| / (|V(G)| - 1) > r$, we have $*(G) \geq a(G) \geq r+1$. Suppose $*(G) = r+1$. Then G can be decomposed into $r+1$ star-forests $H_1, H_2, \dots, H_{r+1}$. We denote by $n_i(t)$ the number of components $K_{1,t}$ in H_i . Put $p = |V(G)|$ and $x_i = |V(G)| - |V(H_i)|$. Then we have

$$p = x_k + \sum_{j=1}^{2r} (j+1)n_k(j) \quad \text{for all } k, 1 \leq k \leq r+1 \quad (1)$$

$$|E(H_k)| = \sum_{j \geq 1} j n_k(j) = p - (x_k + \sum_{j \geq 1} n_k(j)), \text{ and}$$

$$\begin{aligned} \sum_{x \in V(G)} d_G(x) &= 2pr = 2|E(G)| = 2 \sum_{k=1}^{r+1} |E(H_k)| \\ &= 2p(r+1) - 2 \sum_{k=1}^{r+1} \{x_k + \sum_{j \geq 1} n_k(j)\} \end{aligned}$$

Therefore

$$p = \sum_{k=1}^{r+1} \{x_k + \sum_{j \geq 1} n_k(j)\}. \quad (2)$$

The vertex of $K_{1,t}$ ($t \geq 2$) with degree t is called the center of $K_{1,t}$. It follows that every vertex v of $K_{1,j}$ ($j < r$) in H_1 must be the center of a component $K_{1,t}$ ($t \geq 2$) in some H_k ($k \geq 2$), since otherwise $2r = d_G(v) = d_{H_1}(v) + \sum_{k \geq 2} d_{H_k}(v) < r+r$, a contradiction.

Similarly, every end vertex of $K_{1,j}$ ($j \geq r$) in H_1 is contained in the center of a component in some H_k ($k \geq 2$). Hence

$$x_1 + \sum_{j=1}^{r-1} (j+1)n_1(j) + \sum_{j=r}^{2r} jn_1(j) \leq \sum_{k=2}^{r+1} \left(\sum_{t \geq 2} n_k(t) \right) \quad (3)$$

By substituting (3) into (2), we obtain

$$\begin{aligned} p &= x_1 + \sum_{j \geq 1} n_1(j) + \sum_{k=2}^{r+1} x_k + \sum_{k=2}^{r+1} \left(\sum_{j \geq 2} n_k(j) \right) + \sum_{k=2}^{r+1} n_k(1) \\ &\geq x_1 + \sum_{j \geq 1} n_1(j) + \sum_{k=2}^{r+1} x_k + x_1 + \sum_{j=1}^{r-1} (j+1)n_1(j) + \sum_{j=r}^{2r} jn_1(j) \\ &\quad + \sum_{k=2}^{r+1} n_k(1) \\ &= x_1 + \sum_{j \geq 1} (j+1)n_1(j) + \sum_{j=1}^{r-1} n_1(j) + \sum_{k=2}^{r+1} x_k + x_1 + \sum_{k=2}^{r+1} n_k(1) \\ &= p + \sum_{j=1}^{r-1} n_1(j) + \sum_{k=2}^{r+1} x_k + x_1 + \sum_{k=2}^{r+1} n_k(1). \quad (\text{by (1)}) \end{aligned}$$

Hence $n_1(j)=0$ for every j , $1 \leq j \leq r-1$, and $x_1 = \dots = x_{r+1} = 0$.

We can similarly show that $n_k(j)=0$ for all k , j ($k \geq 2$ and $j \leq r-1$).

Therefore, each component of H_k is $K_{1,t}$ ($t \geq r$), and H_k is a spanning subgraph of G . If $d_{H_k}(v) \geq r+1$ for some $k \geq 1$ and $v \in V(G)$, then $2r = d_G(v) = \sum_{t \geq 1} d_{H_t}(v) \geq r+1+r = 2r+1$, a contradiction.

Consequently, each H_k has no $K_{1,t}$ for $t \geq r+1$, and we conclude that every H_k is a $K_{1,r}$ -factor of G . Hence the proof is complete. $\square$

The next theorem can be proved by the same argument in the proof of Theorem 3.

Theorem 4. Let G be a $(2r+1)$ -regular graph. Then $\ast(G) \geq r+2$.

By Theorems 3 and 4, we have

r	2	3	4	5	6
$\ast(G)$ of r -regular graph G	2	3	3,4	4,5	4,5,6

Note that the existence of a 5-regular graph G_1 with $\ast(G_1)=5$ and of a 6-regular graph G_2 with $\ast(G_2)=6$ is unknown.

A triangle cluster is a connected graph whose edges partition into disjoint triangles with the property that any two triangles have at most one vertex in common and if such a vertex exists, then it is a cut vertex of the cluster (Fig. 7).

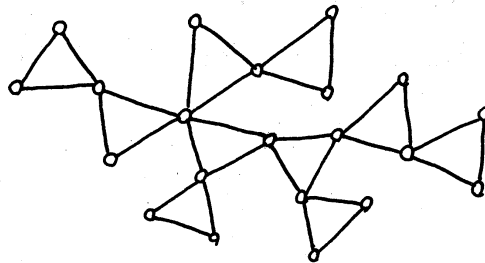


Figure 7. A triangle cluster.

Proposition 2. (Fukuda [5]) Let TC be a triangle cluster. Then

$$\ast(TC) = \begin{cases} 2 & \text{if every triangle has a vertex of degree 2} \\ 3 & \text{otherwise.} \end{cases}$$

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