

The Projectivity of Bridgeland Moduli Spaces of del Pezzo Surface of Picard Rank Three

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Main Theorem

S : del-Pezzo Surf of $\rho = 3$

$v \in K_0(S)$,
 σ : Bridgeland stab condi,
 $\mathcal{M}_\sigma(v)$: Moduli sp of σ -ss obj

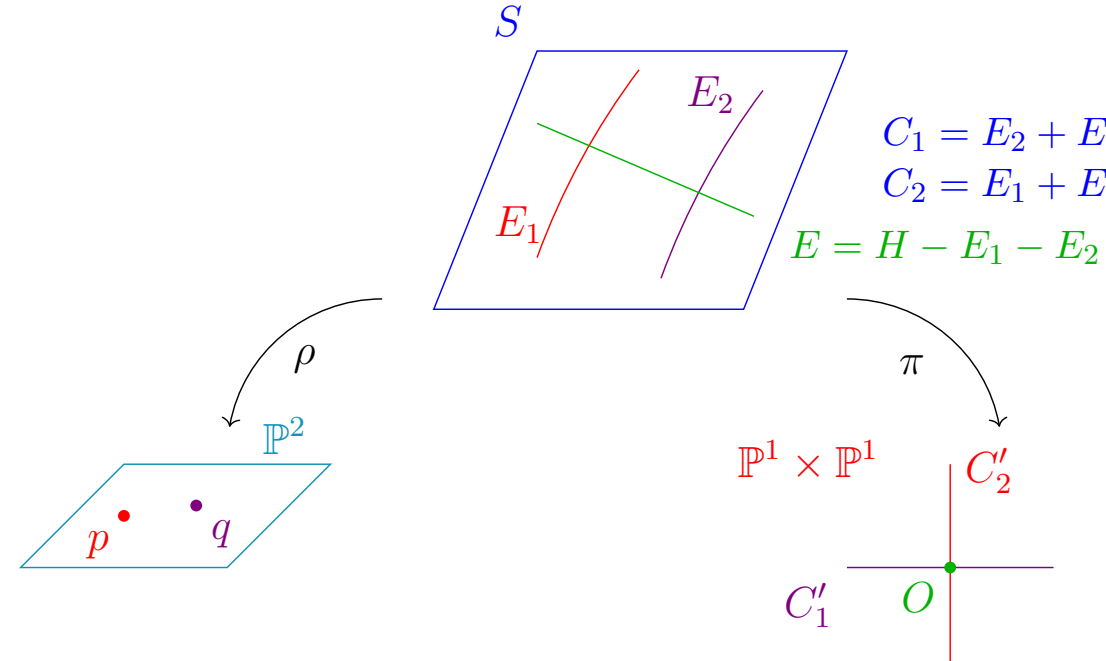


Figure 1: Notations of Divisors

Main Theorem

If $\mathcal{O}_S$ is semi-stable for $\sigma_{D,A}$,

$\mathcal{M}_{\sigma_{D,A}}(v)$ is a Projective Scheme

BSC from Divisors (Geometrical Setting)

$$\begin{aligned} N^1(S) \times \text{Amp}(S) &\xrightarrow{\sim} \text{Stab}_{div}(S) \subset \text{Stab}(S) \\ (D, A) &\longmapsto \sigma_{D,A} = (Z_{D,A}, \mathcal{A}_{D,A}) \\ (Z_{D,A} = -\int_S e^{-(D+iH)} \text{ch}, \mathcal{A}_{D,A} = \langle \mathcal{F}_{D,H}[1], \mathcal{Q}_{D,H} \rangle_{\text{ex}}) \end{aligned}$$

Rmk In general, $\mathcal{M}_{\sigma_{D,A}}(v)$ is only an Alg Stack.

BSC from FEC and Quiver (Algebraic Setting)

$\mathbb{E} = (E_1, \dots, E_n)$: Strong FEC on S

Then, we have

1. $\mathbb{F} := (F_n, \dots, F_1)$: Ext-EC ($F_i := \mathbb{L}_{E_{i-1}} \dots \mathbb{L}_{E_1}(E_i)$)
2. $\exists(Q, \rho)$: Quiver w/ rel &

$$\exists \Phi_{\mathbb{E}} : D^b(S) \xrightarrow{\sim} D^b(\text{Rep}(Q, \rho))$$

$$\mathcal{A}_{\mathbb{F}} := \langle F_n, \dots, F_1 \rangle_{\text{ex}} \xrightarrow{\sim} \text{Rep}(Q, \rho)$$

3. $\exists$ BSC w/ the heart $\mathcal{A}_{\mathbb{F}}$ via $\Phi_{\mathbb{E}}|_{\mathcal{A}_{\mathbb{F}}}$

Important Fact (ABCH, King) :

$\forall \vec{h} \in \mathbb{H}^n_-$, $\exists \sigma_{\mathbb{F}, \vec{h}} = (\mathcal{A}_{\mathbb{F}}, Z_{\vec{h}}) \in \text{Stab}(S)$ &
 $\mathcal{M}_{\sigma_{\mathbb{F}, \vec{h}}}(v)$ is a **Proj Sch**.

Rotations give isoms of Bridgeland moduli spaces

$\exists$ Action $\mathbb{R} \curvearrowright \text{Stab}(S)$: called **Rotation** (Figure 2)

$$\begin{aligned} (\sigma, \varphi) &\mapsto \sigma[\varphi] = (\mathcal{A}[\varphi], Z[\varphi]), \\ (Z[\varphi] = e^{-\varphi\pi i} Z, \mathcal{A}[\varphi] = \langle \mathcal{F}_\varphi[1], \mathcal{Q}_\varphi \rangle_{\text{ex}}) \end{aligned}$$

Importance of the Rotation :

$$\mathcal{M}_\sigma(v) \simeq \mathcal{M}_{\sigma[\varphi]}(v)$$

Esp, $\exists \varphi$ s.t. $\sigma_{D,A}[\varphi] = \sigma_{\mathbb{F}, \vec{h}}$, then $\mathcal{M}_{\sigma_{D,A}}(v)$ is **Proj!**

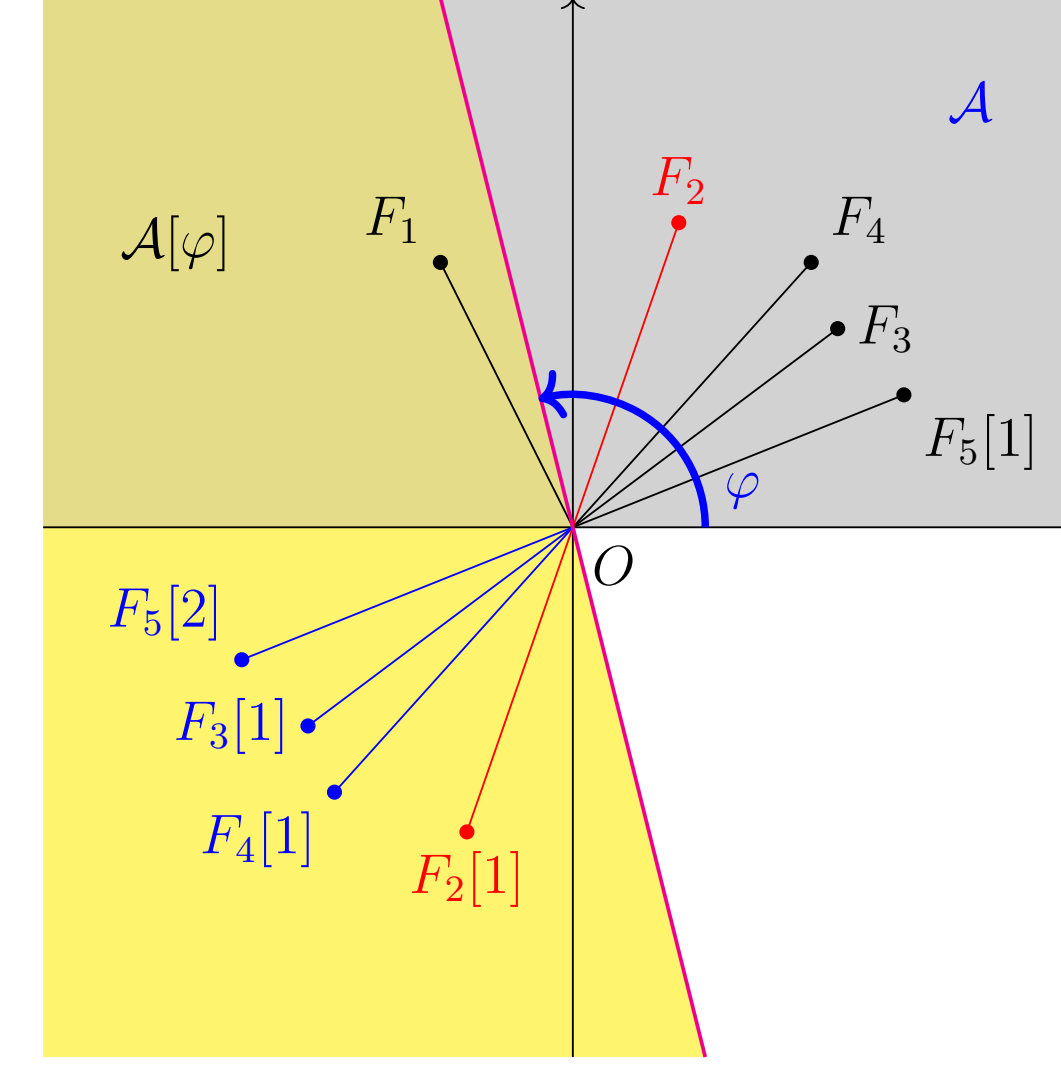


Figure 2: Rotation of BSC

All we need is quiver region

Def (Quiver Region)

$$QR_{\mathbb{F}} = \{\sigma \in \text{Stab}_{div}(S) \mid \exists \varphi \in \mathbb{R}, \text{ s.t. } \sigma[\varphi] = \sigma_{\mathbb{F}, \vec{h}}\}$$

Want : $\exists \mathbb{E}_1, \dots, \mathbb{E}_r$: strong FECs s.t.

$$\text{Stab}_{div}(S) \subset \bigcup_{i=1, \dots, r} QR_{\mathbb{F}_i}$$

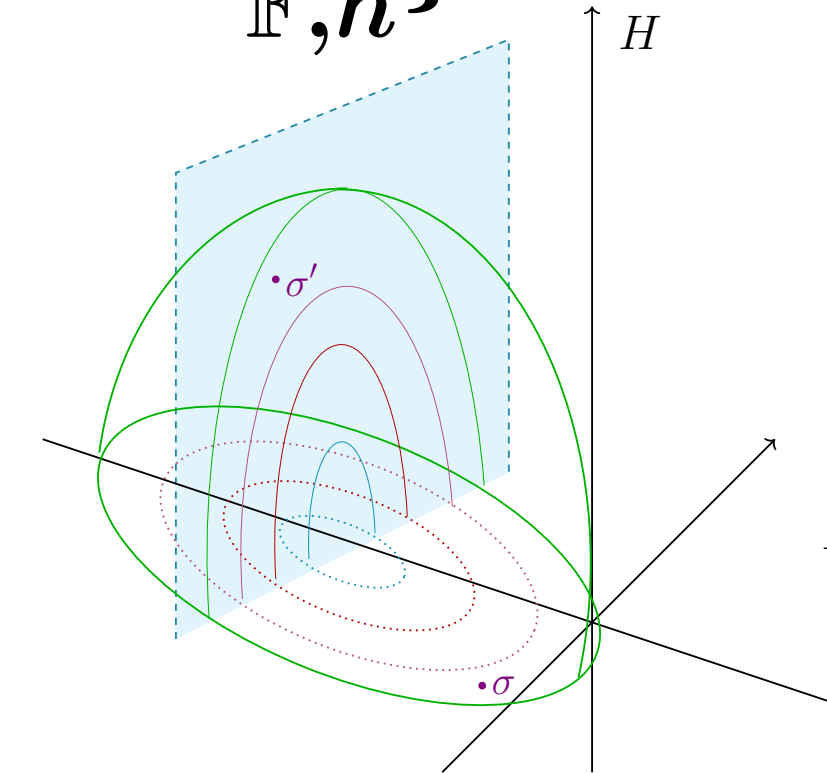


Figure 3: Bertram's Nested Wall Theorem
 σ, σ' are in the same chamber $\mathcal{C}$, $\mathcal{M}_\sigma(v) \simeq \mathcal{M}_{\sigma'}(v)$

$\downarrow$ By Wall & Chamber Structure of $\text{Stab}(S)$
"Bertram's nested wall Theorem" (Figure 3)

Enough To Show : $\forall A \in \text{Amp}(S)$
 $\lim_{\epsilon \rightarrow 0} \left((N^1(S) \times \epsilon A) \cap \bigcup_{i=0}^r QR_{\mathbb{F}_i} \right) \supset \text{An unit cube in } \mathbb{R}^3$

Novelties of Research

- $\rho(S)$ is **Greater** than any other known cases.
- **More FECs are needed** (In Known cases, at most two FECs were sufficient).
 Length (or $\text{rk}(K_0)$) up $\rightsquigarrow$ Quiver Region small $\rightsquigarrow$ more FECs are needed

Known Cases by the quiver region method

Projective Plane $\mathbb{P}^2$ (ABCH) $\mathbb{P}^1 \times \mathbb{P}^1$ (Arcara-Miles) $\text{Bl}_p \mathbb{P}^2$ (Arcara-Miles)

Calculations

We need 14 strong FECs...

Strong Exceptional Collections	Dual Collection
$(\mathcal{O}, \mathcal{O}(C_1), \mathcal{O}(C_2), \mathcal{O}(H), \mathcal{O}(C_1 + C_2))$	$(\mathcal{O}(-H)[2], \mathcal{O}(E) _E[1], \mathcal{O}(-C_2)[1], \mathcal{O}(-C_1)[1], \mathcal{O})$
$(\mathcal{O}, E_{02}^*, \mathcal{O}(C_1), \mathcal{O}(C_2), \mathcal{O}(-H))$	$(\mathcal{O}(-C_1 - C_2)[2], \mathcal{O}_{E_2}(-1)[1], \mathcal{O}_{E_1}(-1)[1], \mathcal{O}(-H)[1], \mathcal{O})$
$(\mathcal{O}, \mathcal{O}(E_1), \mathcal{O}(E_2), E_{2,1,1,2}(H), \mathcal{O}(H))$	$(\mathcal{O}(-C_1 - C_2)[2], \mathcal{O}(-E)[1], \mathcal{O}_{E_2}(-1), \mathcal{O}_{E_1}(-1), \mathcal{O})$
$(\mathcal{O}, \mathcal{O}(E), \mathcal{O}(C_2), \mathcal{O}(C_1), \mathcal{O}(C_1 + C_2))$	$(\mathcal{O}(-E)[2], \mathcal{O}(-E_1)[1], \mathcal{O}(-E_2)[1], \mathcal{O}_E(-1), \mathcal{O})$
$(\mathcal{O}, \mathcal{O}(E), E_3^*, \mathcal{O}(C_1), \mathcal{O}(C_2))$	$(\mathcal{O}(-C_1 - H)[2], \mathcal{O}(-C_2 - H)[2], \mathcal{O}(-H)[1], \mathcal{O}_E(-1), \mathcal{O})$
$(\mathcal{O}, \mathcal{O}(E), E_3^*, \mathcal{O}(CC_1), \mathcal{O}(C_2))$	$(\mathcal{O}(-C_1 - H)[2], \mathcal{O}(-C_1 - H)[2], \mathcal{O}_E(-1)[1], \mathcal{O}(-C_1 - C_2)[1], \mathcal{O})$
$(\mathcal{O}, \mathcal{O}(C_2 - C_1), G^1, \mathcal{O}(C_2 - E), \mathcal{O}(C_2))$	$(\mathcal{O}(-C_1 - H)[2], \mathcal{O}_E(-1)[1], \mathcal{O}(-C_1)[1], \mathcal{O}(C_2 - C_1), \mathcal{O})$
$(\mathcal{O}, \mathcal{O}(C_1 - C_2), G^2, \mathcal{O}(C_1 - E), \mathcal{O}(C_1))$	$(\mathcal{O}(-C_2 - H)[2], \mathcal{O}_E(-1)[1], \mathcal{O}(-C_2)[1], \mathcal{O}(C_1 - C_2), \mathcal{O})$
$(\mathcal{O}, \mathcal{O}(C_2 - C_1), G^1, F^1, \mathcal{O}(C_2))$	$(\mathcal{O}(-C_1 - H)[2], \mathcal{O}(-C_1 + E)[1], \mathcal{O}_E(-1), \mathcal{O}(C_2 - C_1), \mathcal{O})$
$(\mathcal{O}, \mathcal{O}(C_1 - C_2), G^2, F^2, \mathcal{O}(C_1))$	$(\mathcal{O}(-C_2 - H)[2], \mathcal{O}(-C_2 + E)[1], \mathcal{O}_E(-1), \mathcal{O}(C_1 - C_2), \mathcal{O})$
$(\mathcal{O}, G^3, F^3, I^3, \mathcal{O}(C_1))$	$(\mathcal{O}(-C_2 - H)[2], \mathcal{O}(-2C_1 - C_2)[2], \mathcal{O}_{E_2}(-1)[1], \mathcal{O}(-C_1 - H)[1], \mathcal{O})$
$(\mathcal{O}, G^4, F^4, I^4, \mathcal{O}(C_2))$	$(\mathcal{O}(-C_1 - H)[2], \mathcal{O}(-C_1 - 2C_2)[2], \mathcal{O}_{E_1}(-1)[1], \mathcal{O}(-C_2 - H)[1], \mathcal{O})$
$(\mathcal{O}, \mathcal{O}(E_2), G'^5, F'^5, \mathcal{O}(C_1))$	$(\mathcal{O}(-C_2 - H)[2], \mathcal{O}(-2C_1 - C_2)[2], \mathcal{O}(-C_1 - C_2)[1], \mathcal{O}_{E_2}(-1), \mathcal{O})$
$(\mathcal{O}, \mathcal{O}(E_1), G'^6, F'^6, \mathcal{O}(C_2))$	$(\mathcal{O}(-C_1 - H)[2], \mathcal{O}(-C_1 - 2C_2)[2], \mathcal{O}(-C_1 - C_2)[1], \mathcal{O}_{E_1}(-1), \mathcal{O})$

Table 4: Strong exceptional collection and its dual collection.

Conditions determining the Quiver Region of $\mathbb{F}'$

$$\mathbb{F}' = (\mathcal{O}(-C_1 - C_2)[2], \mathcal{O}_{E_2}(-1)[1], \mathcal{O}_{E_1}(-1)[1], \mathcal{O}(-H)[1], \mathcal{O})$$

1. $\mathcal{O}, \mathcal{O}(-C_1 - C_2)[1] \in \mathcal{A}$
2. $\beta(\mathcal{O}(-H)[1]) > \beta(\mathcal{O}_{E_1}(-1)), \beta(\mathcal{O}(-C_1 - C_2)[1])$
3. $\beta(\mathcal{O}) > \beta(\mathcal{O}(-H))$ (if $\mathcal{O}(-H) \in \mathcal{A}$)
4. $\beta(\mathcal{O}) > \beta(\mathcal{O}_{E_1}(-1)), \beta(\mathcal{O}_{E_2}(-1)), \beta(\mathcal{O}(-C_1 - C_2)[1])$ (if $\mathcal{O}(-H)[1] \in \mathcal{A}$)

The longer the FEC, the greater the number of conditionals. As a result, the quiver region becomes smaller, and more FECs are needed.

Figures of Quiver Regions

$$\lim_{\epsilon \rightarrow 0} \left(\{ \sigma_{xC_1 + yC_2 + zE, \epsilon H} \mid x, y, z \in \mathbb{R} \} \cap \bigcup_{i=0}^r QR_{\mathbb{F}_i} \right)$$

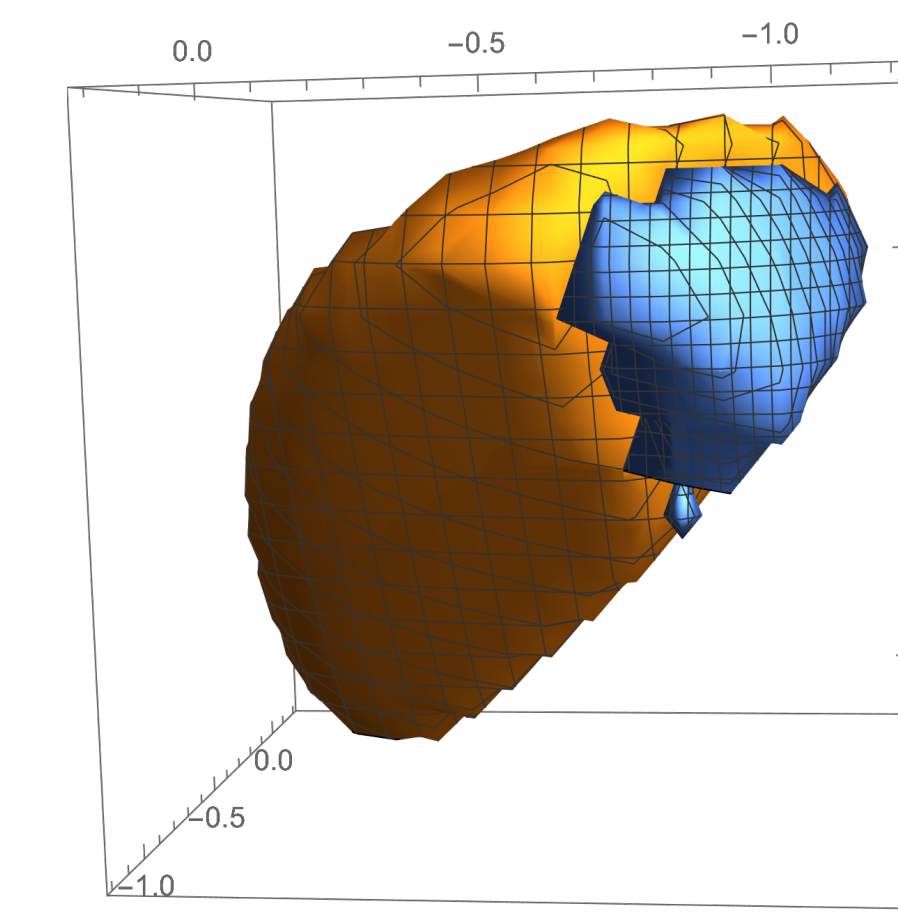


Figure 5: Quiver region of $\mathbb{F}'$ in (x, y, z) -space

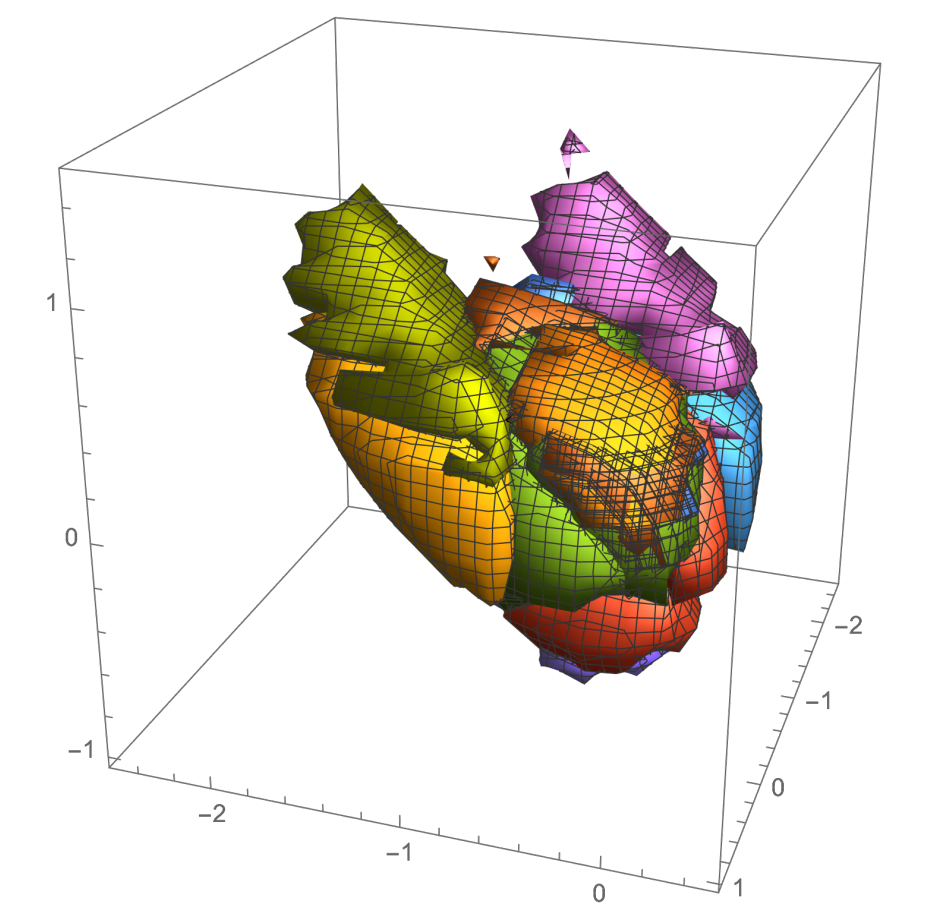


Figure 6: Quiver region that is combined of all in Table 4.