

Semipositivity of relative canonical bundles via Kähler-Ricci flows

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Abstract

In this paper, we shall discuss the fact that the fiberwise Kähler-Ricci flow preserves the semipositivity on a smooth projective family. The full accounts will be given in [B-T].

1 Introduction

In [Ka1], Y. Kawamata proved a semipositivity of the direct image of a relative pluricanonical systems. The second author extended the result to the case of logpluricanonical systems in terms of the generalized Kähler-Einstein metric by using the method in [T4] ([T7]).

In February in 2010, the second author attended the talk given by R. Berman in Luminy about [B].

Inspired by this talk the authors began to work on the stability of the semipositivity of the fiberwise Kähler-Ricci flows on a smooth projective family. This enables us to provide the homotopy version of the semipositivity of relative canonical bundles (cf. Theorem 7). This provides us a new tool to explore the projective (or possibly) Kähler families. For example, as a consequence we may give an alternative proof of the quasiprojectivity of the moduli space of polarized varieties with semiample canonical sheaves.

This is a research announcement and the full accounts will be given in [B-T].

1.1 Kähler-Einstein metrics

Let X be a compact Kähler manifold. It is important to construct a canonical Kähler metric on X .

Let (X, ω) be a compact Kähler manifold. (X, ω) is said to be Kähler-Einstein, if there exists a constant c such that

$$\mathrm{Ric}(\omega) = c \cdot \omega$$

holds, where the Ricci tensor: $\mathrm{Ric}(\omega)$ is defined by

$$\mathrm{Ric}(\omega) = -\sqrt{-1} \partial \bar{\partial} \log \det \omega.$$

This means that X admits a Kähler-Einstein metrics, then $c_1(X)$ is positive or negative or 0.

Theorem 1 ([A, Y1]) *Let X be a compact Kähler manifold.*

(1) If $c_1(X) < 0$, then there exists a Kähler-Einstein metric ω such that

$$-\text{Ric}(\omega) = \omega.$$

(2) If $c_1(X)$ is 0, for every Kähler class c , there exists a Ricci flat Kähler metric ω such that $[\omega] = c$ and

$$\text{Ric}(\omega) = 0.$$

□

1.2 Twisted Kähler-Einstein metrics

Let X be a smooth projective variety defined over $\mathbb{C}$ and let (L, h_L) : a (singular) hermitian $\mathbb{Q}$ -line bundle on X with $\sqrt{-1}\Theta_{h_L} \geq 0$.

ω is said to be a twisted Kähler-Einstein metrics associated with (L, h_L) , if

$$-\text{Ric}(\omega) + \sqrt{-1}\Theta_{h_L} = \omega$$

holds in the sense of current.

Theorem 2 ([T7]) *If h_L is C^∞ on a nonempty Zariski open subset and $\mathcal{I}(h_L) \simeq \mathcal{O}_X$. Then there exists a closed positive current ω on X such that*

- (1) *There exists a nonempty Zariski open subset U of X such that $\omega|_U$ is C^∞ ,*
- (2) *$-\text{Ric}(\omega) + \sqrt{-1}\Theta_{h_L} = \omega$ holds on U ,*
- (3) *$(\omega^n)^{-1} \cdot h_L$ is an AZD of $K_X + L$. □*

1.3 Bergman metrics

Let X be a smooth projective variety and let (L, h_L) be a singular hermitian line bundle on X . We set

$$K(X, K_X + L, h_L) := \sum_i |\sigma_i|^2,$$

where $\{\sigma_i\}$ is an orthonormal basis of $H^0(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{I}(h_L))$ with respect to the inner product:

$$(\sigma, \tau) := \int_X \sigma \cdot \bar{\tau} \cdot h_L.$$

We call $K(X, K_X + L, h_L)$ the Bergman kernel of X with respect to (L, h_L) . If $|H^0(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{I}(h_L))|$ is very ample, then the pull back of the Fubini-Study metric

$$\omega := \sqrt{-1}\partial\bar{\partial} \log K(X, K_X + L, h_L)$$

is a Kähler form on X . We call it the Bergman metric on X with respect to (L, h_L) .

1.4 Dynamical construction of K-E-metrics

Let X be a smooth projective n -fold with ample K_X and (A, h_A) be a sufficiently ample line bundle with C^∞ -metric h_A . We set $K_1 = K(X, K_X + A, h_A)$, $h_1 = K_1^{-1}$. And inductively we define

$$K_m = K(X, mK_X + A, h_{m-1}), h_m = K_m^{-1}$$

for $m \geq 2$. Then we have the following rather unexpected result.

Theorem 3 (*[T]*) $dV_E = \lim_{m \rightarrow \infty} \sqrt[n]{(m!)^{-n} K_m}$ is the K-E volume form on X , i.e., $\omega_E = -\text{Ric } dV_E$ is K-E-form. $\square$

1.5 Kähler-Ricci flow

Let X be a compact Kähler manifold and let ω_0 : C^∞ -Kähler form on X .

We consider the initial value problem:

$$\frac{\partial}{\partial t} \omega(t) = -\text{Ric}(\omega(t)) - \omega(t) \quad (1)$$

on $X \times [0, T)$,

$$\omega(0) = \omega_0,$$

where $\text{Ric}(\omega(t)) = -\sqrt{-1} \partial \bar{\partial} \log \det \omega(t)$ and T is the maximal existence time for the C^∞ -solution. This type of Kähler-Ricci flow was first considered by the second author in [T1]. Then by taking the exterior derivative of the both sides of (1),

$$[\omega(t)] = (1 - e^{-t}) 2\pi c_1(K_X) + e^{-t} [\omega_0] \in H^{1,1}(X, \mathbb{R})$$

Let $\mathcal{K}(X)$ denote the Kähler cone of X . Then the following holds:

Proposition 1 (*[T1]*)

$$T = \sup\{t \mid [\omega(t)] \in \mathcal{K}(X)\}$$

holds. $\square$

The next question is what happens on $\omega(t)$ after exiting the Kähler cone. Let $PE(X)$ denote the pseudoeffective cone $\subseteq H^{1,1}(X, \mathbb{R})$.

Definition 1 Let T be a closed positive $(1, 1)$ current on X . T is said to be of minimal singularities, if for every closed positive $(1, 1)$ -current T' with $[T'] = [T]$, there exists a L^1 -function φ such that

$$T' = T + \sqrt{-1} \partial \bar{\partial} \varphi$$

and is bounded from above. $\square$

The following proposition is an easy consequence of [Le, p.26, Theorem 5].

Proposition 2 Let $\eta \in PE(X)$ be a pseudoeffective class. Then there exists a closed positive $(1, 1)$ -current $T_{\min}$ with minimal singularities which represents η . $\square$

A closed semipositive current T with $[T] \in PE(X)$ is said to be of almost minimal singularities if we write T as $T = T_{min} + \sqrt{-1}\partial\bar{\partial}\varphi$ for some $\varphi \in L^1(X)$, $e^{-\varphi} \in L^p(X)$ holds for every $p \geq 1$.

For a pseudoeffective $\mathbb{R}$ -line bundle F on a smooth projective manifold M , we say that the decomposition:

$$F = P + N \quad (P, N \in \text{Div}(M) \otimes \mathbb{R})$$

is said to be a Zariski composition, if there exists a closed semipositive $(1,1)$ current T on M such that

- (1) T is a closed semipositive current of almost minimal singularities in $2\pi c_1(F)$,
- (2) $T_{sing} = 2\pi N$ in the sense of currents, where $T = T_{abc} + T_{sing}$ is the Lebesgue decomposition.

Let X be a smooth projective variety with pseudoeffective K_X . Then we have the following lemma by [B-C-H-M].

Lemma 1 *There exists a sequence: $T = T_0 < T_1 < \dots < T_j < \dots$ such that for each j , there exists a modification $\pi_j : X_j \rightarrow X$ such that $\pi_j^*(e^{-t}L + (1 - e^{-t})K_X)$ admits a Zariski decomposition:*

$$\pi_j^*(e^{-t}L + (1 - e^{-t})K_X) = P_t + N_t$$

such that N_t is independent of $t \in [T_j, T_{j+1})$. $\square$

Then we have the following theorem.

Theorem 4 *Let X be a smooth projective variety with pseudoeffective canonical class. Let (L, h_L) be a C^∞ -hermitian line bundle such that $\omega_0 := \sqrt{-1}\Theta_{h_L}$ is a Kähler form on X . Then the initial value problem:*

$$\frac{\partial}{\partial t}\omega(t) = -\text{Ric}(\omega(t)) - \omega(t) \quad \text{on } X \times [0, \infty), \quad (2)$$

$\omega(0) = \omega_0$ has the unique long time soluriton $\omega(t)$ such that

- (1) For $t \in [T_j, T_{j+1})$, $\omega(t)$ is C^∞ on a nonempty Zariski open subset $U(T_j)$ depending on $T_j \in [0, \infty)$ defined as in Lemma 1.
- (2) For $t \in [T_j, T_{j+1})$, $\omega(t)$ satisfies the equation (2) on $U(T_j)$.
- (3) $\omega(t)$ is a closed semipositive current with almost minimal singularity in $(1 - e^{-t})2\pi c_1(K_X) + e^{-t}c_1(L)$. $\square$

2 Proof of Theorem 4

Let X be a smooth projective variety with pseudoeffective canonical class and let (L, h_L) be a C^∞ -hermitian line bundle on X such that $\omega_0 = \sqrt{-1}\Theta_{h_L}$ is a Kähler form.

2.1 Discretization of Kähler-Ricci flows

Let a be a positive integer. We consider the following successive equations:

$$a(\omega_{m,a} - \omega_{m-1,a}) = -\text{Ric}_{\omega_{m,a}} - \omega_{m,a} \quad (3)$$

for $m \geq 1$ under the initial condition $\omega_{0,a} = \omega_0$. We see that the cohomology class $[\omega_{m,a}]$ satisfies the equations:

$$a([\omega_{m,a}] - [\omega_{m-1,a}]) = 2\pi c_1(K_X) - [\omega_{m,a}] \quad (4)$$

Hence we see that

$$[\omega_{m,a}] = \left(1 - \left(1 + \frac{1}{a}\right)^{-m}\right) 2\pi c_1(K_X) + \left(1 + \frac{1}{a}\right)^{-m} [\omega_0] \quad (5)$$

We define the singular hermitian metric

$$h_{m,a} := n!(\omega_{m,a}^n)^{-\frac{1}{a+1}} \cdot h_{m-1}^{\frac{a}{a+1}} \quad (6)$$

on

$$(1 - t_{m,a})L + t_{m,a}K_X, \quad (7)$$

where

$$t_{m,a} = 1 - \left(1 + \frac{1}{a}\right)^{-m}. \quad (8)$$

$$\omega(m, a) := t_{m,a}(-\text{Ric } \Omega) + (1 - t_{m,a})\omega_0 \quad (9)$$

Then the $\{u_{m,a}\}_{m=0}^\infty$ satisfies the successive differential equations:

$$a(u_{m,a} - u_{m-1,a}) = \log \frac{(\omega(m, a) + \sqrt{-1}\partial\bar{\partial}u_{m,a})^n}{\Omega} - u_{m,a}. \quad (10)$$

Now we introduce the following notation:

$$\delta_a u_{m,a} := a(u_{m,a} - u_{m-1,a}), \quad (11)$$

i.e., $\delta_a u_{m,a}$ denotes the (backward) difference at $u_{m,a}$.

Then (10) is denoted as:

$$\delta_a u_{m,a} = \log \frac{(\omega(m, a) + \sqrt{-1}\partial\bar{\partial}u_{m,a})^n}{\Omega} - u_{m,a}. \quad (12)$$

Later we shall see that this equation corresponds to the parabolic Monge-Ampère equation:

$$\frac{\partial u}{\partial t} = \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n}{\Omega} - u, \quad (13)$$

where

$$\omega_t := (1 - e^{-t})(-\text{Ric } \Omega) + e^{-t}\omega_0 \quad (14)$$

with the initial condition: $u = 0$ on $X \times \{0\}$.

And there are correspondences:

$$\frac{m}{a} \leftrightarrow t, u_{m,a} \leftrightarrow u(\cdot, t), \omega(m, a) \leftrightarrow \omega_t$$

and

$$\delta_a u_{m,a} \leftrightarrow \frac{\partial u}{\partial t}.$$

We set

$$T := \sup\{t \in \mathbb{R} | 2\pi(1 - e^{-t})c_1(K_X) + e^{-t}[\omega_0] \in \mathcal{K}\}. \quad (15)$$

Since the Kähler-Ricci flow corresponds to the minimal model with scalings in [B-C-H-M] in an obvious manner, we have the following lemma.

Lemma 2 ([B-C-H-M]) *The followings holds:*

- (1) $e^{-T} \in \mathbb{Q}$,
- (2) $(1 - e^{-T})K_X + e^{-T}L$ is semiample. $\square$

By Lemma 2, there exists a C^∞ -function ϕ such that

$$\omega_{T,\phi} := (1 - e^{-T})(\text{Ric } \Omega + \sqrt{-1}\partial\bar{\partial}\phi) + e^{-T}\omega_0 \quad (16)$$

is a C^∞ -semipositive form on X and is strictly positive on a nonempty Zariski open subset of X . We set

$$\begin{aligned} \omega(m, a)_\phi &:= \left(1 - \left(1 + \frac{1}{a}\right)^{-m}\right) (\text{Ric } \Omega + \sqrt{-1}\partial\bar{\partial}\phi) + \left(1 + \frac{1}{a}\right)^{-m} \omega_0 \\ &= \omega(m, a) + \left(1 - \left(1 + \frac{1}{a}\right)^{-m}\right) \sqrt{-1}\partial\bar{\partial}\phi \end{aligned} \quad (17)$$

We set

$$m(a) := \sup \left\{ m \left| \left(1 - \left(1 + \frac{1}{a}\right)^{-m}\right) c_1(K_X) + \left(1 + \frac{1}{a}\right)^{-m} [\omega_0] \in \mathcal{K} \right. \right\}. \quad (18)$$

Then since

$$\omega(m, a)_\phi = \frac{1 - \left(1 + \frac{1}{a}\right)^{-m}}{1 - e^{-T}} \omega_{T,\phi} + \frac{\left(1 + \frac{1}{a}\right)^{-m} - e^{-T}}{1 - e^{-T}} \omega_0. \quad (19)$$

for every $m < m(a)$, $\omega(m, a)_\phi$ is a C^∞ -Kähler form on X and for $m = m(a)$, $\omega(m, a)_\phi = \omega_{T,\phi}$ holds.

Theorem 5 (3) has a smooth solution $\omega_{m,a}$ as long as $[\omega(m, a)] \in \mathcal{K}$. And (10) has C^∞ -solution as $[\omega(m, a)] \in \mathcal{K}$. $\square$

Lemma 3 Suppose that T is finite, then we see that

$$\omega(T) := \lim_{t \uparrow T} \omega(t)$$

exists in C^∞ -topology on $X \setminus E$ and is well defined as a limit of closed positive current on X . $\square$

2.2 Beyond the Kähler cone

After exiting the Kähler cone, the singular solution of the Kähler-Ricci flow can be constructed as follows.

Theorem 6 *There exists a sequence of closed semipositive currents $\{\omega_{m,a}\}_{m=0}^\infty$ such that*

- (1) *For every $m \geq 0$, $\omega_{m,a}$ is a closed semipositive current on X ,*
- (2) *There exists a nonempty Zariski open subset U_m of X such that $h_{m,a}|_{U_m}$ is C^∞ ,*
- (3) *$h_{m,a}$ is an AZD of the $\mathbb{Q}$ -line bundle $(1 - t_{m,a})L + t_{m,a}K_X$,*
- (4) *$\omega_{m,a} = \sqrt{-1}\Theta_{h_{m,a}}$ is a well defined closed semipositive current on X ,*
- (5) *$\{\omega_{m,a}\}_{m=0}^\infty$ satisfies the equations (3) on U_m . $\square$*

The following lemma is a slight refinement of Lemma 1.

Lemma 4 *There exists a sequence of positive number $T = T_0 < T_1 < \dots < T_j < \dots$ such that for every $t \in [T_j, T_{j+1})$*

- (1) *There exists a modification $\pi_j : X_j \rightarrow X$ such that $\pi_j^*(e^{-t}L + (1 - e^{-t})K_X)$ admits a Zariski decomposition:*

$$\pi_j^*(e^{-t}L + (1 - e^{-t})K_X) = P_t + N_t (P_t, N_t \in \text{Div}(X_j) \otimes \mathbb{R}),$$

where P_t is nef and N_t is effective and

$$H^0(X_j, \mathcal{O}_{X_j}(\lfloor mP_j \rfloor)) \simeq H^0(X_j, \mathcal{O}_{X_j}(m\pi_j^*(e^{-t}L + (1 - e^{-t})K_X)))$$

holds for every m such that $me^{-t} \in \mathbb{Z}$.

- (2) *N_t is independent of $t \in [T_j, T_{j+1})$,*

- (3) *If $e^{-t} \in \mathbb{Q}$, then P_t is semiample. $\square$*

We set $N_j := N_t (t \in [T_j, T_{j+1}))$. Let τ_j be the multivalued holomorphic section of N_j with divisor N_j . Then there exists a C^∞ -hermitian metric $\|\cdot\|$ such that $\omega_{T_j} + \sqrt{-1}\partial\bar{\partial} \log \|\tau_j\|^2$ is a closed semipositive current. We set

$$\phi_j := \log \|\tau_j\|^2. \quad (20)$$

Suppose that we have already defined $u_{0,a}(\phi_j)$ such that for every $\varepsilon > 0$, there exists a constant $C(\varepsilon)$

$$u_{0,a}(\phi_j) \geq \varepsilon \phi_j + C(\varepsilon) \quad (21)$$

holds. We set

$$\omega_j(m, a) := \left(1 - e^{-T_j} \left(1 + \frac{1}{a}\right)^{-m}\right) (-\text{Ric } \Omega) + e^{-T_j} \left(1 + \frac{1}{a}\right)^{-m} \omega_0. \quad (22)$$

We consider the Ricci iteration:

$$\delta_a u_{m,a}(\phi_j) = \log \frac{(\omega(m, a)_{\phi_j} + \sqrt{-1}\partial\bar{\partial} u_{m,a}(\phi_j))^n}{\Omega \cdot e^{-\phi_j}} - u_{m,a}(\phi_j). \quad (23)$$

The rest of the proof is similar to the case $t \in [0, T)$.

3 Semipositivity of a Kähler-Ricci flow

In this section we shall sketch the proof of the fact that the relative Kähler-Ricci flows preserve the semipositivity in the horizontal direction on projective families.

3.1 Main results

Let $f : X \rightarrow S$ be a smooth projective family and let ω be a relative Kähler form on X . We set $n := \dim X - \dim S$ and $k := \dim S$. We define the relative Ricci form $\text{Ric}_{X/S, \omega}$ of ω by

$$\text{Ric}_{X/S, \omega} = -\sqrt{-1} \partial \bar{\partial} \log (\omega^n \wedge f^* |ds_1 \wedge \cdots \wedge ds_k|^2), \quad (24)$$

where $(s_1, \dots, s_k)$ is a local coordinate on S . Then it is easy to see that $\text{Ric}_{X/S, \omega}$ is independent of the choice of the local coordinate $(s_1, \dots, s_k)$. The Kähler-Ricci flow preserves the semipositivity in the following sense.

Theorem 7 *Let $f : X \rightarrow S$ be a smooth projective family of varieties with pseudoeffective canonical bundles. Let L be an ample line bundle on X and let h_L be a C^∞ -hermitian metric on L with strictly positive curvature. Suppose that there exists a C^∞ -relative volume form Ω on $f : X \rightarrow S$ such that $\text{Ric} \Omega + \sqrt{-1} \Theta_{h_L}$ is also a Kähler form on X . We set $\omega_0 := \sqrt{-1} \Theta_{h_L}$. We consider the normalized Kähler-Ricci flow:*

$$\frac{\partial}{\partial t} \omega(t) = -\text{Ric}_{X/S, \omega(t)} - \omega(t)$$

on X with the initial condition $\omega(0) = \omega_0$, where $\text{Ric}_{X/S, \omega(t)}$ denotes the relative Ricci form of $\omega(t)$ on X

Then $\omega(t)$ is a closed semipositive current on X for every $t \in [0, \infty)$. $\square$

In Theorem 7, the semipositivity of $\omega(t)$ corresponds to the pseudoeffectivity of $(1 - e^{-t})K_{X/S} + e^{-t}L$. And as t goes to infinity, we observe that the relative canonical bundle $K_{X/S}$ is pseudoeffective.

Similarly we have the following theorem.

Theorem 8 *Let $f : X \rightarrow S$ be a smooth projective family of varieties with pseudoeffective canonical bundles. Let L be an ample line bundle on X and let h_L be a C^∞ -hermitian metric on L with strictly positive curvature. Let K be a closed semipositive current on X such that K is C^∞ on a nonempty Zariski open subset of X and $[K] \in 2\pi c_1(K_{X/S})$. We set $\omega_0 := \sqrt{-1} \Theta_{h_L}$. We consider the Kähler-Ricci flow:*

$$\frac{\partial}{\partial t} \omega(t) = -\text{Ric}_{X/S, \omega(t)} - K$$

on X with the initial condition $\omega(0) = \omega_0$, where $\text{Ric}_{X/S, \omega(t)}$ denotes the relative Ricci form of $\omega(t)$ on X

Then $\omega(t)$ is a closed semipositive current on X for every $t \in [0, \infty)$. Moreover as t goes to infinity, $\omega(t)$ converges to a current solution of $-\text{Ric}_{X/S, \omega(t)} = K$. $\square$

3.2 Some conjecture for the Kähler case

We expect that the similar statement holds even in the case that $f : X \rightarrow S$ is a smooth Kähler fibration.

Conjecture 1 *Let X be a compact Kähler manifold with pseudoeffective canonical bundle. And let ω_0 be a C^∞ -Kähler form on X . Suppose that there exists a C^∞ -volume form Ω such that*

$$\text{Ric } \Omega + \omega_0$$

is also a Kähler form on X . Then there exists a family of closed semipositive current $\omega(t)$ on X such that

- (1) $\omega(0) = \omega_0$,
- (2) *For every $T > 0$, there exists a nonempty Zariski open subset $U(T)$ depending on T such that $\omega(t)$ is Kähler form on $U(T) \times [0, T)$,*
- (3) $[\omega(t)] = 2\pi(e^{-t}[\omega_0] + (1 - e^{-t})c_1(K_X))$ holds for every $t \in [0, \infty)$,
- (4) *On $U(t) \times [0, T)$ $\omega(t)$ satisfies the differential equation:*

$$\frac{\partial \omega(t)}{\partial t} = -\text{Ric}_{\omega(t)} - \omega(t).$$

□

Conjecture 2 *Let $f : X \rightarrow S$ be a smooth Kähler family with pseudoeffective canonical bundles. Let ω_0 be a C^∞ -Kähler form on X . Suppose that there exists a C^∞ -relative volume form Ω on $f : X \rightarrow S$ such that $\text{Ric } \Omega + \omega_0$ is also a Kähler form on X . We consider the normalized Kähler-Ricci flow:*

$$\frac{\partial}{\partial t} \omega(t) = -\text{Ric}_{X/S, \omega(t)} - \omega(t)$$

on X with the initial condition: $\omega(0) = \omega_0$, where $\text{Ric}_{X/S, \omega(t)}$ denotes the relative Ricci form of $\omega(t)$ on X

Then $\omega(t)$ is a closed semipositive current on X for every $t \in [0, \infty)$. □

This conjecture will lead us to the invariance of plurigenera in the Kähler case.

4 Proof of Theorem 7

The essential technical difficulty here is the fact that we cannot apply the direct calculation of the variation, since the Kähler-Ricci flow in Theorem 4 has singularities. We overcome this difficulty by using the dynamical construction of the solution of the Ricci iterations as in [LC]

4.1 The relative Ricci iterations to the relative Kähler-Ricci flow

Let $f : X \rightarrow S$ be a smooth projective family of varieties with pseudoeffective canonical bundles. Let L be an ample line bundle on X and let h_L be a C^∞ -hermitian metric on L with strictly positive curvature. Suppose that there exists a C^∞ -relative volume form Ω on $f : X \rightarrow S$ such that $\text{Ric } \Omega + \sqrt{-1} \Theta_{h_L}$ is also a Kähler form on X . We set $\omega_0 := \sqrt{-1} \Theta_{h_L}$. We consider the normalized Kähler-Ricci flow:

$$\frac{\partial \omega(t)}{\partial t} = -\text{Ric}_{X/S, \omega(t)} - \omega(t) \quad (25)$$

on X with the initial condition $\omega(0) = \omega_0$, where $\text{Ric}_{\omega(t)}$ denotes the relative Ricci form on X .

For every $s \in S$, we consider Lemma 1. Then by the invariance of the twisted plurigenra, we see that for every $C > 0$ the sequence

$$T = T_0 < T_1 < \dots < T_j < \dots < C \quad (26)$$

in Lemma 1 are constant on a nonempty Zariski open subset $S(C)$ of S .

Suppose that we have already proven the (logarithmic) plurisubharmonic variation property of the solution $\omega(t)$ of (25) for every $t < C$ on $f^{-1}(S(C))$. Then the removable singularity theorem for plurisubharmonic function implies the logarithmic plurisubharmonic variation property of the solution $\omega(t)$ over the whole X .

Hence we may and do assume that the sequence $T_0 < \dots < T_j < \dots$ are constant over the whole S without loss of generality. Moreover since the assertion of Theorem 7 is local in S , we may and do assume that S is the unit open polydisk Δ^k in $\mathbb{C}^k$.

The plurisubharmonic variation property of the Ricci iteration is proven by the parallel argument as follows.

We set

$$m(a) := \sup \left\{ m \left| \left(1 + \frac{1}{a} \right)^{-m} > e^{-T_0} \right. \right\}. \quad (27)$$

First we shall consider the relative Ricci iteration:

$$\delta_a \omega_{m,a} = -\text{Ric}_{\omega_{m,a}/S} - \omega_{m,a}, \omega_{0,a} = \omega_0 \quad (28)$$

on X for $0 \leq m < m(a)$. This is equivalent to the fiberwise Ricci iteration:

$$\delta_a \omega_{m,a,z} = -\text{Ric}_{\omega_{m,a}/S,s} - \omega_{m,a,s}, \omega_{0,a} = \omega_0|_{X_s}, \quad (29)$$

on X_s for $0 \leq m < m(a)$. Then by the proof of Theorem 4, letting a tends to infinity, we may construct the solution of the relative Kähler-Ricci flow:

$$\frac{\partial \omega(t)}{\partial t} = -\text{Ric}_{X/S, \omega(t)} - \omega(t) \quad (30)$$

on $X \times [0, T_0)$.

Then as in the previous section, we may continue this process beyond the critical time T_0 and we obtain the long time existence of the current solution of the relative Kähler-Ricci flow on X .

4.2 Auxiliary Ricci iterations

We prove Theorem 7 by decomposing the Ricci iterations by a dynamical system of Bergman kernels and apply the plurisubharmonic variation properties of Bergman kernels due to Berndtsson. The main difficulty is to deal with $\mathbb{Q}$ -line bundles. We deal with $\mathbb{Q}$ -line bundles in terms of the auxiliary Ricci iterations.

Lemma 5 *For every $0 \leq m \leq m(a)$, $\omega_{m,a}$ is semipositive on X . $\square$*

We prove Lemma 5 by induction on m .

For $m = 0$ $\omega_{0,a} = \omega_0$ is a Kähler form on X by the assumption. Hence Lemma 5 holds for $m = 0$. Suppose that $\omega_{m,a}$ is semipositive on X . We shall prove that $\omega_{m+1,a}$ is also semipositive on X .

To prove this assertion, we consider the auxiliary Ricci iteration which connects $\omega_{m,a}$ and $\omega_{m+1,a}$.

First we define the $\mathbb{Q}$ -line bundle L_m by

$$L_m := \left(1 - \left(1 + \frac{1}{a}\right)^{-m}\right) K_{X/S} + \left(1 + \frac{1}{a}\right)^{-m} L. \quad (31)$$

Let $q = q(m+1)$ be a positive integer such that qL_{m+1} is a genuine line bundle on X . Since

$$L_{m+1} = \left(1 - \left(1 + \frac{1}{a}\right)^{-(m+1)}\right) K_{X/S} + \left(1 + \frac{1}{a}\right)^{-(m+1)} L$$

is of the form $\beta(K_{X/S} + \alpha L)$ for some positive rational numbers α and β . By B-C-H-M, we have that the relative logcanonical ring:

$$R(X, K_{X/S} + \alpha L) = \bigoplus_{\nu=0}^{\infty} f_* \mathcal{O}_X(\lfloor \nu(K_{X/S} + \alpha L) \rfloor)$$

is a finitely generated algebra over $\mathcal{O}_S$. By the invariance of twisted plurigeera, we see that each $f_* \mathcal{O}_X(\lfloor \nu(K_{X/S} + \alpha L) \rfloor)$ is a vector bundle over S which is biholomorphic to the unit open polydisk Δ^k . We take a sufficiently large positive integer ν_0 and take a set of generators $\{\sigma_i\}$ of $f_* \mathcal{O}_X(\nu_0!(K_{X/S} + \alpha L))$ (In this case $K_{X/S} + \alpha L$ is relatively ample. But later we also consider the case $K_{X/S} + \alpha L$ is big, but not relatively ample). Then we set

$$h_{m,a,0} := \left(\sum_i |\sigma_i|^2 \right)^{-\frac{\beta}{\nu_0!}} \quad (32)$$

and

$$\omega_{m,a,0} := \sqrt{-1} \Theta_{h_{m,a,0}}. \quad (33)$$

Then $h_{m,a,0}$ is a hermitian metric of $L_{m+1} = \beta(K_{X/S} + \alpha L)$ with semipositive curvature on X . Now we shall consider the following Ricci iteration:

$$-\text{Ric}_{\omega_{m,a,\ell}} + (q - a - 1)\omega_{m,a,\ell-1} + a\omega_{m,a} = q\omega_{m,a,\ell} \quad (34)$$

for $\ell \geq 1$. The following lemma follows entirely the same way as the dynamical construction of Kähler-Einstein metrics.

Lemma 6 $\lim_{\ell \rightarrow \infty} \omega_{m,a,\ell}$ exists in C^∞ -topology on X . And

$$\lim_{\ell \rightarrow \infty} \omega_{m,a,\ell} = \omega_{m+1,a} \quad (35)$$

holds. $\square$

We use this auxiliary Ricci iteration to connect $\omega_{m,a}$ and $\omega_{m+1,a}$ by a dynamical system of Bergman kernels. This method is exactly the same one in [T7].

4.3 Dynamical systems of Bergman kernels

To prove the semipositivity of $\omega(t)$ on X for $t \in [0, T_0]$, it is enough to prove the following lemma.

Lemma 7 $h_{m,a}$ has semipositive curvature on X . $\square$

We now use the strategy as in [T7]. We shall prove Lemma 7 by induction on m . Since h_L has positive curvature, $h_{0,a} = h_L$ has semipositive curvature.

Suppose that we have already proven that $h_{m-1,a}$ has semipositive curvature.

Let A be a sufficiently ample line bundle on X and let h_A be a C^∞ -hermitian metric on X with strictly positive curvature.

Now we shall define the metric on L_{m+1} by

$$h_{m,a,\ell}|_{X_s} = h_{m,a,\ell,s}(s \in S). \quad (36)$$

By induction on ℓ , we shall prove the following lemma.

Lemma 8 $h_{m,a,\ell}$ has semipositive curvature on X for every $\ell \geq 0$. $\square$

Proof of Lemma 8. By the construction (cf. (32)), $h_{m,a,0}$ has semipositive curvature.

Suppose that we have already proven that $h_{m,a,\ell-1}$ is a hermitian metric with semipositive curvature on X . For every $s \in S$, we shall consider the dynamical system of Bergman kernels as follows. We set

$$K_{1,s} := K \left(X_s, A + K_{X_s} + (q - a - 1)L_{m+1} + aL_m|_{X_s}, h_A \cdot h_{m,\ell-1}^{q-a-1} \cdot h_{m,a}^a|_{X_s} \right) \quad (37)$$

and

$$h_{1,s} := K_{1,s}^{-1}. \quad (38)$$

Suppose that we have already constructed $K_{p-1,s}$ and $h_{p-1,s}$ for some $p \geq 2$. Then we define $K_{p,s}$ and $h_{p,s}$ by

$$K_{p,s} := K \left(X_s, A + p(K_{X_s} + (q - a - 1)L_{m+1} + aL_m|_{X_s}), h_{m,\ell-1}^{q-a-1} \cdot h_{m,a}^a \cdot h_{p-1}|_{X_s} \right) \quad (39)$$

and

$$h_{p,s} := \frac{1}{K_{p,s}}. \quad (40)$$

Similarly as in [T4, T7] we have the following lemma.

Lemma 9

$$K_{\infty,s} := \limsup_{p \rightarrow \infty} ((p!)^{-n} h_A \cdot K_{p,s})^{\frac{1}{pq}} \quad (41)$$

exists in L^1 -topology and

$$h_{m,a,\ell,s} := K_{\infty,s}^{-1} \quad (42)$$

is a C^∞ -hermitian metric on $L_{m+1}|X_s$. And the curvature

$$\omega_{m,a,\ell,s} := \sqrt{-1}\Theta_{h_{m,a,\ell,s}} \quad (43)$$

satisfies the differential equation:

$$-\text{Ric}_{\omega_{m,a,\ell,s}} + (q - a - 1)\omega_{m,a,\ell-1,s} + a\omega_{m,a,s} = q\omega_{m,a,\ell,s} \quad (44)$$

on X . $\square$

We define the relative Bergman kernel K_p on X by

$$K_p|X_s = K_{p,s}.$$

Then $h_p = K_p^{-1}$ is a hermitian metric with semipositive curvature on $A + p(K_{X/S} + (q - a - 1)L_{m+1} + aL_m)$ by induction on p by the following theorem mainly due to B. Berndtsson.

Theorem 9 ([B1, B2, B3, B-P]) *Let $f : X \rightarrow S$ be a projective family of projective varieties over a complex manifold S . Let S° be the maximal nonempty Zariski open subset such that f is smooth over S° .*

Let (L, h_L) be a pseudo-effective singular hermitian line bundle on X . Let $K_s := K(X_s, K_X + L|_{X_s}, h|_{X_s})$ be the Bergman kernel of $K_{X_s} + (L|_{X_s})$ with respect to $h|_{X_s}$ for $s \in S^\circ$. Then the singular hermitian metric h of $K_{X/S} + L|_{f^{-1}(S^\circ)}$ defined by

$$h|_{X_s} := K_s^{-1} (s \in S^\circ)$$

has semipositive curvature on $f^{-1}(S^\circ)$ and extends to X as a singular hermitian metric on $K_{X/S} + L$ with semipositive curvature in the sense current. $\square$

Now we prove the semipositivity of $\sqrt{-1}\Theta_{h_p}$ by induction on p . First the semipositivity of $\sqrt{-1}\Theta_{h_1}$ follows from Theorem 9 by the assumption that $\sqrt{-1}\Theta_{h_{m,a,\ell-1}}$ and $\sqrt{-1}\Theta_{h_{m-1,a}}$ are semipositive. Suppose that we have already proven the semipositivity of h_{p-1} for some $p \geq 2$. We note that $h_{p-1}, h_{m,a,\ell-1}$ and $h_{m,a}$ has semipositive curvature on X by the induction assumption. Then by the inductive definition of h_p (cf. (39) and (40)) and Theorem 9, we see that $\sqrt{-1}\Theta_{h_p}$ is also semipositive.

Hence by induction, we see that $\{h_p\}_{p=1}^\infty$ has semipositive curvature on X . Then by Lemma 9, we see that $h_{m,a,\ell}$ has semipositive curvature. This completes the proof of Lemma 8. $\square$

By Lemmas 6 and 8, we see that h_{m+1} is a metric on L_{m+1} with semipositive curvature. Hence by induction on m , we complete the proof of Lemma 7. $\square$

Now by Lemma 7 and the proof of Theorem 1, we see that $\omega(t)$ is semipositive on X for $t \in [0, T_0]$.

Now we complete the proof of Theorem 7 by repeating the similar argument inductively for $t \in [T_j, T_{j+1}] (j \geq 0)$. This completes the proof of Theorem 7. $\square$

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